Solutions for Selected Exercises from Chapter 2

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2.2. Let $v(x) = \tan(x)$, which is continuous on $-\pi/2 < x < \pi/2$. For all x_0 in this interval, find all solutions of

$$x'(t) = v(x(t)) \ , \qquad x(0) = x_0 \ .$$

For which values of t is each solution defined?

SOLUTION: By Barrows formula,

$$t(x) = \int_{x_0}^{x} \cot(z)\mathrm{d}z = \ln(\sin x - \ln(\sin x_0) = \ln\left(\frac{\sin x}{\sin x_0}\right) \ .$$

Therefore $\sin(x(t)) = e^t \sin x_0$, and then

$$x(t) = \arcsin(e^t \sin x_0) \ .$$

The only solution of $v(x) = 0$ in $(-\pi/2, \pi.2)$ is $x = 0$. Hence $(-\pi/2, \pi.2)$ divides into two maximal intervals, $(-\pi/2, 0)$ and $(0, \pi.2)$. Since $v'(x) = 1 + \tan^2 x$, which is bounded on a neighborhood of 0, v is Lipschitz near $x = 0$, and the equilibrium solutions $x(t) = 0$ for all t is the only solution with $x(0) = 0$. Obviously, this solution exists for all time. For $x_0 \in (0, \pi.2)$, running time backwards, the solution never his $x = 0$ since v is Lipschitz there. But at $t = -\ln(\sin x_0)$, it reaches $\pi/2$, where v is undefined. So the interval of existence for such x is $(-\infty, -\ln(\sin x_0))$. Similar reasoning shows that it $x_0 \in (-\pi/2, 0)$, the interval of existence is $(\ln(\sin(-x_0)), \infty)$.

2.3. **(a)** Let $v(x) = (1 - x^4)^{1/2}$ Consider the solution of $x'(t) = v(x(t))$ with $x(0) = 0$. Does this solution exist for all t and remain within the interval $(-1, 1)$ for all t? Justify your answer.

(b) Let $v(x) = (1 - x^4)^2$ Consider the solution of $x'(t) = v(x(t))$ with $x(0) = 0$, Does this solution exist for all t and remain within the interval $(-1, 1)$ for all t? Justify your answer.

SOLUTION: **(a)** Note that $(1 - x^4) = (1 + x^2)(1 - x^2) = (1 + x^2)(1 - x)(1 + x)$ and so $v(x) = (1 + x^2)^{1/2}(1 - x)^{1/2}(1 + x)^{1/2}$. This is not Lipschitz because the derivative is unbounded at $x = -1$ and $x = 1$. This suggests that the solution may reach both $x = -1$ and $x = 1$ in a finite time. To prove it, we note that for $x \geq 1/2$,

$$\frac{1}{v(x)} = (1 + x^2)^{-1/2}(1 - x)^{-1/2}(1 + x)^{-1/2} \leq (15/16)^{-1/2}(1 - x)^{-1/2}$$

and the right hand side has a finite integral over $(1/2, 1)$. Hence the solution will reach 1 from $1/2$ in a finite time, and since it will reach $1/2$ from any point in $(-1, 1/2)$ in a finite time, the solution will reach 1 in a finite time starting from any $x_0 \in (-1, 1)$. Since v is non-negative, solutions can never approach $x = -1$, and we need not analyze this.

(b) $v(x) = (1 - x^4)^2$ is Lipschitz on $[-1, 1]$ and so the solution exists and remains for all time in $(-1, 1)$.

0.1 Remark. Note the asymmetry between **(a)** and **(b)**. The Lipschitz condition is necessary, but not sufficient, for solutions to exist for all time. when it applies, we simply invoke the theorem on existence for Lipschitz vector fields. When the Lipschitz condition is not satisfied, we must examine Barrows formula for convergence or divergence of the relevant integrals.

10. Consider the equation

$$x''(t) = F(x(t)) \qquad \text{where} \qquad F(x) = -\frac{\mathrm{d}}{\mathrm{d}x}V(x) \tag{0.1}$$

for some continuously differentiable function V.

(a) Define the function $H(x, y)$ by

$$H(x, y) = \frac{1}{2}y^2 + V(x)\ . \tag{0.2}$$

Show that if $x(t)$ is any solution of (0.1) defined on some open interval containing t_0, then

$$H(x'(t), x(t)) = H(x'(t_0), x(t_0))$$

for all t in the interval. Therefore, to solve (0.1) with $x(t_0) = x_0$ and $x'(t_0) = v_0$, we need only solve

$$x' = \pm\sqrt{H(v_0, x_0)) - V(x)}\ . \tag{0.3}$$

(b) Let $V(x) = \frac{1}{2}x^2$, and take $x_0 = 1$ and $v_0 = 0$. There will be infinitely many solutions of (0.3). Describe all of them (The description will involve arbitrary "rest periods" at equilibrium points.). Of these solutions, how many are twice continuously differentiable?

(c) How many solutions of

$$(x'(t))^2 + (x(t))^4 = 1$$

are there with $x(0) = 1$ and $x'(0) = 0$? How many of these are twice continuously differentiable?

SOLUTION: (a) Applying the standard "reduction of order" method, we define $\mathbf{x}(t) := (x(t), y(t))$ where $y(t) = x'(t)$. Then the equation can be written as

$$\mathbf{x}'(t) = (y(y), -V'(x(t)) = \nabla H(\mathbf{x}(t))^\perp\ .$$

Then by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}H(\mathbf{x}(t)) = \nabla H(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = \nabla H(\mathbf{x}(t)) \cdot \nabla H(\mathbf{x}(t))^\perp = 0\ .$$

Hence H is constant along solutions.

(b) For $V(x) = \frac{1}{2}x^2$, and take $x_0 = 1$ and $v_0 = 0$, $H(x_0, v_0) = \frac{1}{2}$, and so

$$x'(t) = \pm\frac{1}{\sqrt{2}}\sqrt{1 - x^2}\ .$$

Since $x_0 = 1$ is a steady-state solution of our first order equation. one solution is to remain at $x = 1$ up to some time T_1, and then to move to the left –the only option as $\sqrt{1 - x^2}$ is not real for $x > 1$ – along the solutions of

$$x' = -\frac{1}{\sqrt{2}}\sqrt{1 - x^2}\ .$$

These can be written down in therms of the arcsin function using Barrow's formula. The solution reaches $x = -1$ in a finite time. It can then wait there an arbitrary amount of time before leaving again, moving to the right.

If the solution is to be twice continuously differentiable, it must never rest. Indeed, suppose the solution takes an initial rest, so that $x(t) = 1$ for $0 \leq t \leq T_1$, and then moves to the left. As soon as it starts moving to the left, we must have $x'' = -x$, which is close to -1 for t close to T_1. But since $x(t)$ is constant on $[0, T_1]$, $x''(t) = 0$ for $t \in (0, T_1)$. Hence x'' is not continuous if $T_1 > 0$. The same reasoning applies each time an a steady state solution of the first order equation in reached.

(c) The reasoning used in **(b)** applies even when we cannot do the integrals to explicitly evaluate Barrow's formula. It is clear that again in this case,

$$\frac{1}{1 - x^4}$$

is integrable on $[-1, 1]$ – see Exercise 3 from this Chapter. Therefore there will be infinitely many solutions, with arbitrary rest periods each time $x = 1$ or $x = -1$ is reached. But only the solution that never rests is twice continuously differentiable.