Challenge Problem Set 5, Math 292 Spring 2018

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This challenge problem set concerns zeros of u' and of u for eigenvalues of Sturm-Liouville operators with Neumann boundary conditions.

Let $[a, b]$ be a bounded interval. Let $p(x)$ be strictly positive and continuously differentiable on $[a, b]$. Consider

$$\mathcal{L}u(x) = (p(x)u'(x))' , \tag{0.1}$$

and the eigenfunctions $\mathcal{L}u(x) = \lambda u(x)$ with Neumann boundary conditions:

$$u'(a) = u'(b) = 0 . \tag{0.2}$$

Recall that an eigenfunction, by definition is not the trivial function $u(x) = 0$. Recall also that the function $u_0(x) = 1$ for all x is an eigenfunction with the eigenvalue $\lambda_0 = 0$, and that all other eigenvalues are strictly negative.

We can write the eigenvalue equation in the form

$$u'' + \frac{p'}{p}u' - \frac{\lambda}{p}u = 0 . \tag{0.3}$$

(1) Let u be an eigenfunction for $\mathcal{L}$ with Neumann boundary conditions on $[a, b]$. A critical point of u is a point x where $u'(x) = 0$. By definition a and b are critical points. Suppose the eigenvalue λ for u is not zero. Show that at any critical point x of u, $u''(x) \neq 0$, $u(x) \neq 0$, and the sign of $u''(x)$ is opposite to that of $u(x)$.

(2) Continuing with Exercise (1), show that there can only be finitely many critical points of u in $[a, b]$. Let $x_1, x_2 \in (a, b)$ be successive critical points of u. Show that one is a local maximum and the other is a local minimum. That is, local maxima and local minima alternate.

(3) Continuing with Exercise (2), let $x_1, x_2 \in (a, b)$ be successive critical points of u. Show that the restriction of u to $[x_1, x_2]$ is an eigenfunction of $\mathcal{L}$ with Neumann boundary conditions on $[x_1, x_2]$, and

$$\int_{x_1}^{x_2} u(x)\mathrm{d}x = 0 .$$

Hint: Think about the orthogonality of eigenfunctions and the fact that the constant function is an eigenfunction. Use this to show that between every critical point of u, there is a zero of u. Show also the between each zero there is a critical point.

(4) Continuing with Exercise (3), Show that the critical points and zero of u interlace, and the the critical points alternate between being strict local minima and strict local maxima. Show that for all $n \geq 1$, the nth eigenfunction has exactly n zeros interlacing between $n+1$ critical points (counting the 2 at a and b. Note the condition $n \geq 1$; the statement does not apply to u_0.) In particular, u_2 has two zeros $z < x_1 < x_2 < b$. Show that u_2 restricted to $[x_1, x_2]$ is the first eigenfunction for $\mathcal{L}$ satisfying Dirichlet boundary conditions on $[x_1, x_2]$, and the eigenvalue is λ_2, the second Neumann eigenvalue on $[a, b]$.

(5) Continuing with Exercise (4). Use what you know about estimating Dirichlet eigenvalues to give upper and lower bounds on λ_2. Generalize the argument to give upper and lower bounds on λ_k for all $k \geq 2$. What can you say about λ_1? Make you bounds explicit for

$$\mathcal{L}u(x) = ((1 + x)u'(x))'$$

on $[0, L]$.