Challenge Problem Set 3, Math 292 Spring 2018

Eric A. Carlen[1]
Rutgers University

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This challenge problem set concerns the same simple linear second order system that we studied in the first challenge problem set,

$$\begin{aligned} x'' &= -x \\ y'' &= -y \ . \end{aligned}$$

This time, however, we will study the *phase curves* of the corresponding first order system. Introduce new variables

$$a = x \qquad b = x' \qquad c = y \qquad d = y' \ .$$

Define the 4-dimensional vector $\mathbf{a}(t) = (a(t), b(t), c(t), d(t))$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(a,b,c,d) = (b, -a, d, -c) \ . \tag{0.1}$$

This is a first order differential equation in a four dimensional phase space. In the exercises that follow, Let $(a(t), b(t), c(t), d(t))$ be any solution of (0.1) with $(a(0), b(0), c(0), d(0)) = (a_0, b_0, c_0, d_0) =: \mathbf{a}_0$. It turns out that there are many simple constants of the motion; i.e., functions of (a, b, c, d) that are constant along the phase curves. This will enable to identify the phase curves.

Exercise 1. Define the functions

$$r(a,b,c,d) := a^2 + b^2 \ , \qquad s(a,b,c,d) = c^2 + d^2 \quad \text{and} \quad q(a,b,c,d) = ac + bd \ .$$

Show that $r(\mathbf{a}(t))$, $s(\mathbf{a}(t))$ and $q(\mathbf{a}(t))$ are all independent of t.

Exercise 2. Define the vectors $\mathbf{v}(t) = (a(t), b(t))$ and $\mathbf{w}(t) = (c(t), d(t))$. Assume that $r_0 := \|\mathbf{v}(0)\| > 0$ and $s_0 := \|\mathbf{w}(0)\| > 0$. Show that the angle between $\mathbf{v}(t)$ and $\mathbf{w}(t)$ is independent of t, and then show that there is some angle θ so that

$$s_0\mathbf{v}(t) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} r_0\mathbf{w}(t)$$

for all t. Geometrically, this is because the motion in the a, b plane and the motion in the c, d plane are both circular motion with the same frequency; the two motions“stay in phase”.

Exercise 3. For the angle θ found in Exercise 2, and $r = \sqrt{r_0^2 + s_0^2}$, define the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ by

$$\begin{aligned}
\mathbf{u}_1 &= \frac{1}{r}(s_0, 0, -r_0\cos\theta, r_0\sin\theta) \\
\mathbf{u}_2 &= \frac{1}{r}(0, s_0, -r_0\sin\theta, -r_0\cos\theta) \\
\mathbf{u}_3 &= \frac{1}{r}(r_0, 0, s_0\cos\theta, -s_0\sin\theta) \\
\mathbf{u}_3 &= \frac{1}{r}(0, r_0, -s_0\sin\theta, -s_0\cos\theta)
\end{aligned}$$

Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is orthonormal in $\mathbb{R}^4$ and that

$$\mathbf{a}(t)\cdot\mathbf{u}_1 = \mathbf{a}(t)\cdot\mathbf{u}_2 = 0$$

for all t.

Use this to show that for some continuously differentiable functions $\alpha(t)$ and $\beta(t)$,

$$\mathbf{a}(t) = \alpha(t)\mathbf{u}_3 + \beta(t)\mathbf{u}_4 \ ,$$

and that $\alpha^2(t) + \beta^2(t) = r^2$ for all t.

Finally, show that

$$\begin{aligned}
\alpha'(t) &= \beta(t) \\
\beta'(t) &= -\alpha(t) \ .
\end{aligned}$$

Exercise 4. Explicitly solve for the phase curve $\mathbf{a}(t)$ when

$$\mathbf{a}(0) = (1, 0, 1, \sqrt{3}) \ .$$

Show that $(a(t), c(t)) = (x(t), y(t))$ traces out the ellipse that you found in the first challenge problem set for the corresponding initial data.

The set of vectors $\mathbf{a} \in \mathbb{R}^4$ satisfying $\mathbf{a}\cdot\mathbf{u}_1 = 0$ and $\mathbf{a}\cdot\mathbf{u}_2 = 0$ is the two dimensional plane spanned by $\mathbf{u}_3$ and $\mathbf{u}_4$. What we have proved so far shows that the phase curve is a circle produced by slicing the sphere of radius r in $\mathbb{R}^4$ by a two dimensional plane through the origin; this is a so-called great circle, and we have seen in 291 that these are the geodesics on the sphere in $\mathbb{R}^4$.

Let us specialize to the case $r = 1$ so that our phase curves are great circles on S^3, the three dimensional unit sphere in $\mathbb{R}^4$. We can now draw some interesting geometric conclusions. First of all, given any point (a_0, b_0, c_0, d_0) on S^3, there is a unique phase curve through this point, and as we have seen, that phase curve is a great circle. Now, two phase curves cannot intersect: Through each point in the phase space there is exactly one phase curve. Hence S^3 is the *disjoint union of these great circles.*

Such a circumstance is certainly not the case for S^2, the familiar unit sphere in $\mathbb{R}^3$: Take out any great circle – the equator, for example – and what remains is two disconnected hemispheres, neither of which contains any great circles at all.

The great circles that arise a phase curves for our second order system are rather special; not every great circle on S^3 is such a phase curve. Indeed, there are infinitely many great circles passing through any point on $\mathbf{p} \in S^3$. Let $\mathbf{q}$ be any unit vector orthogonal to $\mathbf{p}$, and consider the two dimensional plane spanned by $\mathbf{p}$ and $\mathbf{q}$. This slices S^3 in a great circle that passes though $\mathbf{p}$. In fact, it is not hard to see that all great circles passing through $\mathbf{p}$ arise this way. However, there is only one phase curve passing though any point in phase space, and so *exactly one of these* will be a phase curve.

This decomposition of S^3 into a *disjoint union* of great circles is behind an important geometric construction known as the *Hopf fibration.* (It has an interesting Wikipedia page.) Like the decomposition itself, this arises naturally from the study of our second order system.

Exercise 5. In Exercise 1, we found three functions r, s and q that were "constants of the motions". There is a fourth. Define $p(a,b,c,d)$ by

$$p(a,b,c,d) = ad - bc \ .$$

Show that for any phase curve $\mathbf{a}(t)$ of our system, $p(\mathbf{a}(t))$ is independent of t.

Show also that the four constants of the motion are not independent of one another, but are related by

$$(2p(\mathbf{a}))^2 + (2q(\mathbf{a}))^2 + (r(\mathbf{a}) - s(\mathbf{a}))^2 = 1$$

for all $\mathbf{a} \in S^3$.

The *Hopf map* is the function H on S^3 given by

$$H(\mathbf{a}) = (2p(\mathbf{a}), 2q(\mathbf{a}), r(\mathbf{a}) - s(\mathbf{a})) \ .$$

Exercise 5 shows that for all $\mathbf{a} \in S^3$, $H(\mathbf{a})$ is a unit vector in $\mathbb{R}^3$. In other words, $H(\mathbf{a})$ belongs to S^2, the unit sphere in $\mathbb{R}^3$. Thus, the Hopf map is a function from S^3 to S^2.

Going on from here, it is not hard to show that H transforms S^3 onto S^2, and that for any $\mathbf{u} \in S^2$, $H^{-1}(\mathbf{u})$, the inverse image of $\mathbf{u}$ under the Hopf map, is a great circle on S^3, and moreover, is one of our phase curves. This is the so-called "Hopf fibration", which has interesting topological features. It is not hard to show for example, that each pair of the phases curves is linked exactly once. More information can be found on the Wikipedia page. Our point is to show how the Hopf fibration arises very naturally from the consideration of a very simple second order system of differential equations.