Homework Set 1, Math 292 Spring 2013

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These exercises are due Thursday, January 31.

1. This problem illustrates the use of reduction of order. Consider the second order equation
   \[ tx'' - x' = 3t^2. \]
   Introduce a new variable \( y(t) = x'(t) \), and notice that the equation becomes a linear first order equation in \( y \). Use this fact to find \( y(t) \), and then \( x(t) \), given that \( x(1) = 1 \) and \( x'(1) = 2 \).

2. Use the method of the previous exercise to find the solution of
   \[ x'' = -4x, \]
   with \( x(0) = x_0 \) and \( x'(0) = y_0 \). (Your answer will be a function of \( t, x_0 \) and \( y_0 \).)

3. (a) For \( 0 < p < 1 \), and \( x_0 > 0 \), consider the equation
   \[ x' = x(1-x) - px \quad x(0) = x_0, \]
   which is a variant of the logistic equation. Note that the equation is \( x' = v(x) \) with \( v(x) = x(1-x) - px \). This is a quadratic polynomial, and so it has two roots, namely \( x = 0 \) and \( x = 1-p \). Factor it, and solve the equation.
   
   (a) Is \( v(x) \) Lipschitz continuous on \((1, 1-p)\)? justify your answer. For \( x_0 \in (0, 1-p) \), determine \( \lim_{t \to \pm \infty} x(t) \). Justify your answer.
   
   (b) Solve the equation for \( x_0 > 1-p \). Compute \( \lim_{t \to \infty} x(t) \), but show that for each \( x_0 > 1-p \), there is a value \( t_* < 0 \) so that \( \lim_{t \to t_*} x(t) = \infty \); i.e., the graph of \( x(t) \) has a vertical asymptote at \( t = t_* \). Compute \( t_* \) as a function of \( x_0 \).

4. (a) Find the solution of the equation
   \[ x' = 2x^2 + tx^2, \quad x(0) = 1. \]
   
   (b) On which interval in \( t \) around the origin in the solution defined?

   (c) What is the minimum value of the solution?

5. We have seen that in solving \( x'(t) = v(x(t)) \), it is useful to find the inverse function \( t(x) \) first, and then invert this to obtain \( x(t) \).
This approach is also useful for other sorts of equations. Recalling that \( t'(x) = 1/x'(t(x)) \), or more briefly, \( t' = 1/x' \), convert
\[
(e^x - 2tx)x' = x^2 , \quad x(1) = 2
\]
into a differential equation for \( t(x) \), and solve this differential equation. In this case, inverting to find \( x \) as a function of \( t \) involves solving a transcendental equation, so you cannot do it explicitly for all \( t \), but you can use Newton’s method to evaluate \( x(t) \) for any \( t \). Use this approach to compute \( x(2) \) accurately to 10 decimal places.

6. Use the change of variables \( z = x/t \) to find the general solution of
\[
tx' = x + 2te^{-x/t} .
\]