7.2. (a) $\displaystyle\int_{y=-\sqrt{3}}^{\sqrt{3}} \int_{x=1-\sqrt{4-y^2}}^{-1+\sqrt{4-y^2}} x^2y^2\,dx\,dy$

(b) The region D is symmetric with respect to the y-axis and the integrand is unchanged when x is replaced by $-x$. So let D^+ be the half of D in the first and 4th quadrants. Then

$$\int_D x^2y^2\,dA = 2\int_{D^+} x^2y^2\,dA = 2\int_{x=0}^{1}\int_{y=-\sqrt{4-(x+1)^2}}^{y=\sqrt{4-(x+1)^2}} x^2y^2\,dy\,dx$$

(c) $\displaystyle 2\int_{x=0}^{1} x^2\frac{y^3}{3}\Bigg|_{y=-\sqrt{4-(x+1)^2}}^{y=\sqrt{4-(x+1)^2}} dx = 2\int_{x=0}^{1} 2x^2\sqrt{4-(x+1)^2}^{\,3}\,dx$

Let $x+1 = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $\sqrt{4-(x+1)^2} = 2\cos\theta$. The integral equals

$$4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin\theta-1)^2\cdot 8\cos^3\theta\cdot 2\cos\theta\,d\theta = 64\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (4\sin^2\theta - 4\sin\theta+1)\cos^4\theta\,d\theta \text{ etc.}$$

7.4. Draw the region D. It is a triangle with vertices $(0,0)$, $(1.5,1.5)$, and $(3,9)$.

(a) The line $y = 1.5$ separates D into two triangular regions E_1 and E_2, each with a horizontal side. The lower triangle E_1 is bounded on the left by $y = 3x$ and on the right by $y = x$; the triangle E_2 is bounded on the left by $y = 3x$ and on the right by $y = 5x - 6$. Thus,

$$\int_D xy\,dA = \int_{E_1} xy\,dA + \int_{E_2} xy\,dA = \int_{y=0}^{1.5}\int_{x=y/3}^{y} xy\,dx\,dy + \int_{y=1.5}^{9}\int_{x=y/3}^{(y+6)/5} xy\,dx\,dy$$

(b) The line $x = 1.5$ separates D into two triangular regions D_1 and D_2, each with a vertical side. The left triangle D_1 is bounded below by $y = x$ and above by $y = 3x$; the triangle D_2 is bounded below by $y = 5x - 6$ and above by $y = 3x$. Thus,

$$\int_D xy\,dA = \int_{D_1} xy\,dA + \int_{D_2} xy\,dA = \int_{x=0}^{1.5}\int_{y=x}^{3x} xy\,dy\,dx + \int_{x=1.5}^{3}\int_{y=5x-6}^{3x} xy\,dy\,dx$$

(c) Using (b),

$$\begin{aligned}\int_D xy\,dA &= \int_{x=0}^{1.5}\left(x\frac{(3x)^2}{2} - x\frac{x^2}{2}\right)dx + \int_{x=1.5}^{3}\left(x\frac{(3x)^2}{2} - x\frac{(5x-6)^2}{2}\right)dx \\ &= \int_0^{1.5} 4x^3\,dx + \int_{1.5}^{3} -8x^3+30x^2-18x\,dx \\ &= x^4\Big|_0^{3/2} + (-2x^4+10x^3-9x^2)\Big|_{3/2}^{3} = \frac{81}{16} - 162 + 270 - 81 + \frac{162}{16} - \frac{270}{8} + \frac{81}{4} = \frac{459}{16}\end{aligned}$$

7.8. Let the change of variable $\mathbf{u} = \mathbf{u}(\mathbf{x})$ be defined by $u = xy$ and $v = y/x$. Let $\bar{D}$ be the region in the u, v-plane defined by $1 \le u \le 2$ and $1 \le v \le 2$. Then $\bar{D}$ corresponds to D under the correspondence $\mathbf{u}(\mathbf{x})$.

Solving $u = xy$ and $v = y/x$ for x and y gives the inverse correspondence $\mathbf{x}(\mathbf{u})$, namely $uv = y^2$, $y = \sqrt{uv}$, $x = u/y = \sqrt{u/v}$.

The Jacobian of the inverse correspondence and its determinant are

$$D_{\mathbf{x}}(\mathbf{u}) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{uv}} & -\frac{\sqrt{u}}{\sqrt{v^3}} \\ \frac{\sqrt{v}}{\sqrt{u}} & \frac{\sqrt{u}}{\sqrt{v}} \end{bmatrix}, \quad \det D_{\mathbf{x}}(\mathbf{u}) = \frac{1}{4}\frac{2}{v} = \frac{1}{2v}$$

Finally

$$\begin{aligned} \int_D xy\, d^2\mathbf{x} &= \int_{\bar{D}} u \cdot |\det D_{\mathbf{x}}(\mathbf{u})|\, d^2\mathbf{u} = \int_{u=1}^{2}\int_{v=1}^{2} \frac{u}{2v}\, dv\, du \\ &= \int_{u=1}^{2} \frac{u \ln 2}{2}\, du = \frac{3\ln 2}{4}. \end{aligned}$$

7.10. The curve is $r^2 = (r^2 - r\cos\theta)^2 = r^2(r - \cos\theta)^2$. Apart from the point at the origin, this is equivalent to $(r-\cos\theta)^2 = 1$ or $r - \cos\theta = \pm 1$. Since r is always positive except at the origin, and since $\cos\theta \le 1$ for all θ, this is equivalent to $r - \cos\theta = 1$, or $r = 1 + \cos\theta$.

The curve looks like a heart resting on its left side. The enclosed area D is described in polar coordinates by $0 \le \theta \le 2\pi$, $0 \le r \le 1 + \cos\theta$. The area of D is

$$\int_D d^2\mathbf{x} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1+\cos\theta} r\, dr\, d\theta = \int_{\theta=0}^{2\pi} \frac{1}{2}(1+\cos\theta)^2\, d\theta = \frac{3}{2}\pi$$

7.12. As θ goes from 0 to 2π, this equation traces out a figure eight, going from NorthEast (the point $(\sqrt{2}, \sqrt{2})$ at $\theta = \pi/4$) to SouthWest (the point $(-\sqrt{2}, -\sqrt{2})$ at $\theta = 5\pi/4$). The area of the enclosed region D is

$$\int_D d^2\mathbf{x} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1+\sin 2\theta} r\, dr\, d\theta = \int_{\theta=0}^{2\pi} \frac{1}{2}(1+\sin 2\theta)^2\, d\theta = \frac{3}{2}\pi.$$

7.14. Let $\mathbf{u}(\mathbf{x})$ be defined by $u = xy$ and $v = x^2 y$. Let $\bar{D}$ be the region in the u, v-plane defined by $1 \le u \le 2$ and $3 \le v \le 4$. Then $\bar{D}$ corresponds to D under the correspondence $\mathbf{u}(\mathbf{x})$.

Solving $u = xy$ and $v = x^2 y$ for x and y gives the inverse correspondence $\mathbf{x}(\mathbf{u})$, namely $x = v/u$, $y = u^2/v$.

The Jacobian of the inverse correspondence and its determinant are

$$D_{\mathbf{x}}(\mathbf{u}) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} -v/u^2 & 1/u \\ 2u/v & -u^2/v^2 \end{bmatrix}, \quad \det D_{\mathbf{x}}(\mathbf{u}) = -\frac{1}{v}.$$

Finally

$$\int_D xy\, d^2\mathbf{x} = \int_{\bar{D}} u \cdot |\det D_{\mathbf{x}}(\mathbf{u})|\, d^2\mathbf{u} = \int_{u=1}^{2} \int_{v=3}^{4} \frac{u}{v}\, dv\, du = \frac{3}{2} \ln\left(\frac{4}{3}\right).$$

7.16. The region D has two symmetric pieces, D_1 in the first quadrant and D_2 in the third quadrant. The integrand is unchanged by the double sign change $x \to -x$ and $y \to -y$, so

$$\int_D (x^2+y^2)\, d^2\mathbf{x} = 2\int_{D_1} x^2 + y^2 d^2\mathbf{x}$$

Define $\mathbf{u}(\mathbf{x})$ by $u = x^2 - y^2$ and $v = xy$ and let $\bar{D}_1$ be the region $0 \le u \le 4$, $1 \le v \le 2$ in the u, v-plane.

The inverse transformation is messy but $D_{\mathbf{u}}(\mathbf{x})$ is easy to compute; it's

$$D_{\mathbf{u}}(\mathbf{x}) = \begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix}, \qquad \det D_{\mathbf{u}}(\mathbf{x}) = 2(x^2+y^2)$$

Therefore by the change of variable formula in reverse (tricky!)

$$\int_D (x^2+y^2)\, d^2\mathbf{x} = 2\int_{D_1} (x^2+y^2)\, d^2\mathbf{x} = \int_{D_1} |\det D_{\mathbf{u}}(\mathbf{x})|\, d^2\mathbf{x} = \int_{\bar{D}_1} 1\, d^2\mathbf{u} = \text{ area of } \bar{D}_1 = 4\cdot 1 = 4.$$

7.18. $\mathcal{V}$ is the union of 8 congruent pieces, one in each octant, whose overlaps have zero volume. Let $\mathcal{V}_1$ be the piece in the first octant, that is,

$$x^2+y^2 \le 1,\ x^2+z^2 \le 1,\ y^2+z^2 \le 1,\ x \ge 0,\ y \ge 0,\ z \ge 0.$$

Thus $\mathcal{V}_1$ has a base in the x, y-plane which is the quarter-disk $x^2+y^2 \le 1$ in the first quadrant of the x, y plane. Further, $\mathcal{V}_1$ has vertical sides and a "roof" that is made up of the surfaces $z = \sqrt{1-x^2}$ and $z = \sqrt{1-y^2}$, whichever is lower. Thus the roof is the surface $z = \sqrt{1-x^2}$ where $x \ge y$, and the surface $z = \sqrt{1-y^2}$ where $x \le y$. Let D_1 be the eighth-disk $x^2+y^2 \le 1$, $0 \le y \le x$, and let D_2 be the eighth-disk $x^2+y^2 \le 1$, $0 \le x \le y$. Therefore

$$\text{vol. of } \mathcal{V}_1 = \int_{D_1} \sqrt{1-x^2}\, dA + \int_{D_2} \sqrt{1-y^2}\, dA.$$

The two integrals are done similarly; in fact they are equal, by symmetry. Let's evaluate the first integral. Dividing D_1 into two pieces with the line $x = \sqrt{2}/2$, we get

$$\begin{aligned}
\int_{D_1} \sqrt{1-x^2}\, dA &= \int_{x=0}^{\sqrt{2}/2} \int_{y=0}^{x} \sqrt{1-x^2}\, dy\, dx + \int_{x=\sqrt{2}/2}^{1} \int_{y=0}^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dy\, dx \\
&= \int_0^{\sqrt{2}/2} x\sqrt{1-x^2}\, dx + \int_{\sqrt{2}/2}^{1} (1-x^2)\, dx \\
&= -\frac{1}{3}(1-x^2)^{3/2}\Big|_0^{\sqrt{2}/2} + \left(x - \frac{x^3}{3}\right)\Big|_{\sqrt{2}/2}^{1} \\
&= -\frac{1}{3}\frac{1}{2\sqrt{2}} + \frac{1}{3} + \frac{2}{3} - \frac{1}{\sqrt{2}} + \frac{1}{3}\frac{1}{2\sqrt{2}} = \frac{1}{2}(2-\sqrt{2})
\end{aligned}$$

Finally, vol. of $\mathcal{V} = 16\int_{D_1} \sqrt{1-x^2}\, dA = 8(2-\sqrt{2})$

7.20. (b) In cylindrical coordinates $\mathcal{V}$ is described by $0 \le \theta \le 2\pi$, $z^2 \le r \le 8 - z^2$, $-2 \le z \le 2$. The volume is

$$\int_{\theta=0}^{2\pi} \int_{z=-2}^{2} \int_{r=z^2}^{8-z^2} r\, dr\, dz\, d\theta = 2\pi \int_{z=-2}^{2} \frac{1}{2}\left[(8-z^2)^2 - (z^2)^2\right] dz = 2\pi \int_{-2}^{2} (32 - 8z^2)\, dz = \frac{512\pi}{3}.$$

(c) Use cylindrical coordinates. The two parts of the surface are defined by $0 \le r \le 4$ and $4 \le r \le 8$, respectively. In each part, z runs from -2 to 2.

On the inner part $\mathcal{S}_1$, $r = z^2$ so z and θ are convenient parameters.

$$\mathbf{X} = (x, y, z) = (r\cos\theta, r\sin\theta, z), \quad \mathbf{X}(z,\theta) = (z^2\cos\theta, z^2\sin\theta, z).$$

Then $\frac{\partial \mathbf{X}}{\partial z} = \dfrac{\partial \mathbf{X}}{\partial z} = (2z\cos\theta, 2z\sin\theta, 1)$ and $\frac{\partial \mathbf{X}}{\partial \theta} = \dfrac{\partial \mathbf{X}}{\partial \theta} = (-z^2\sin\theta, z^2\cos\theta, 0)$ and

$$\begin{aligned}
\frac{\partial \mathbf{X}}{\partial z} \times \frac{\partial \mathbf{X}}{\partial \theta} &= (2z\cos\theta, 2z\sin\theta, 1) \times (-z^2\sin\theta, z^2\cos\theta, 0) = (-z^2\cos\theta, -z^2\sin\theta, 2z^3) \\
\left\| \frac{\partial \mathbf{X}}{\partial z} \times \frac{\partial \mathbf{X}}{\partial \theta} \right\| &= z^2\sqrt{1+4z^2}.
\end{aligned}$$

So surf. area of $\mathcal{S}_1 = \displaystyle\int_{\theta=0}^{2\pi} \int_{-2}^{2} z^2\sqrt{1+4z^2\, dz}$

Similarly on the other piece $\mathbf{X}(z,\theta) = ((8-z^2)\cos\theta, (8-z^2)\sin\theta, z)$,

$$\begin{aligned}
\frac{\partial \mathbf{X}}{\partial z} \times \frac{\partial \mathbf{X}}{\partial \theta} &= (-2z\cos\theta, -2z\sin\theta, 1) \times (-(8-z^2)\sin\theta, (8-z^2)\cos\theta, 0) \\
&= (-(8-z^2)\cos\theta, -(8-z^2)\sin\theta, -2z(8-z^2)) \\
\left\| \frac{\partial \mathbf{X}}{\partial z} \times \frac{\partial \mathbf{X}}{\partial \theta} \right\| &= (8-z^2)\sqrt{1+4z^2}.
\end{aligned}$$

So surf. area of $\mathcal{S}_2 = \displaystyle\int_{\theta=0}^{2\pi} \int_{-2}^{2} (8-z^2)\sqrt{1+4z^2}\, dz$

Adding the areas of $\mathcal{S}_1$ and $\mathcal{S}_2$, and substituting $z = (1/2)\tan u$, $dz = (1/2)\sec^2 u\,du$, gives the total surface area:

$$\int_{\theta=0}^{2\pi}\int_{-2}^{2}\sqrt{1+4z^2}\,dz = 4\pi\int_0^2\sqrt{1+4z^2}\,dz = 2\pi\int_0^{\arctan 4}\sec^3 u\,du = \frac{1}{2}\left(4\sqrt{17}+\ln(4+\sqrt{17})\right)$$

7.22. The base of $\mathcal{V}$ is in the plane $z = 0$; the vertical sides are $x = 0$ (the y,z-plane) and $2x = y$. The "roof" is $y + z = 2$, which cuts the x,y-plane in the line $y = 2$. Thus the base of $\mathcal{V}$ is triangular, with sides $y = 2$, $y = 2x$, and $x = 0$. Then

$$\int_{\mathcal{V}} xe^z\,dV = \int_{x=0}^{1}\int_{y=2x}^{2}\int_{z=0}^{2-y} xe^z\,dz\,dy\,dx = \int_{x=0}^{1}\int_{y=2x}^{2} x(e^{2-y}-1)\,dy\,dx$$

$$= \int_{x=0}^{1} x(-e^{2-y}-y)\Big|_{2x}^{2} = \int_{x=0}^{1} x(-3+e^{2-2x}+2x)\,dx = -\frac{5}{6}-\frac{3}{4}+\frac{e^2}{4} = \frac{3e^2-19}{12}$$

7.24. $\mathcal{V}$ is the region defined by $x^2+y^2 \le 9$ and $0 \le z \le y$. The base D of this region is the half-disk $y \ge 0$ inside the disk $x^2+y^2 \le 9$. The top of the region is $z = y$. Therefore the volume is the double integral

$$\int_D y\,dA = \int_{\theta=0}^{\pi}\int_{r=0}^{3} r\sin\theta\cdot r\,dr\,d\theta = \int_0^{\pi} 9\sin\theta\,d\theta = 18.$$

7.26. (a) The plane containing the three points has equation $2x + y + 2z = 6$. A parametrization is $\mathbf{X}(x,y) = (x, y, 3 - x - (y/2))$. The region of the parameter plane (x,y-plane) corresponding to $\mathcal{S}$ is the triangle D with vertices $(0,4)$, $(1,0)$, and $(0,0)$.

From the parametrization $\mathbf{X}$ we obtain

$$\frac{\partial\mathbf{X}}{\partial x}\times\frac{\partial\mathbf{X}}{\partial y} = (1,0,-1)\times(0,1,-1/2) = (1,1/2,1)$$

so $\left\|\frac{\partial\mathbf{X}}{\partial x}\times\frac{\partial\mathbf{X}}{\partial y}\right\| = \|(1,1/2,1)\| = 3/2$ is constant. Then

$$\int_{\mathcal{S}} xyz\,dS = \int_D xy(3-x-(y/2))(3/2)\,dA = \int_{x=0}^{1}\int_0^{4-4x}\frac{18xy-6x^2-3xy^2}{4}\,dy\,dx \text{ etc.}$$

7.28. The circle $(x-b)^2+z^2 = a^2$ is revolved about the z-axis to form the torus, which therefore has the equation $(r-b)^2+z^2 = a^2$ in cylindrical coordinates. Using θ and r as parameters, $b-a \le r \le b+a$, $0 \le \theta \le 2\pi$, the top half of $\mathcal{S}$ is parametrized by

$$\mathbf{X}(r,\theta) = (x,y,z) = (r\cos\theta, r\sin\theta, g(r)) \text{ where } g(r) = \sqrt{a^2-(r-b)^2}.$$

Then noting that $g'(r) = -(r-b)/\sqrt{a^2-(r-b)^2}$ and eventually substituting $u = r - b$, $r = u + b$, $dr = du$,

$$\frac{\partial \mathbf{X}}{\partial r} = (\cos\theta, \sin\theta, g'(r)), \ \frac{\partial \mathbf{X}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0),$$

$$\frac{\partial \mathbf{X}}{\partial r} \times \frac{\partial \mathbf{X}}{\partial \theta} = (-rg'(r)\cos\theta, -rg'(r)\sin\theta, r), \left\| \frac{\partial \mathbf{X}}{\partial r} \times \frac{\partial \mathbf{X}}{\partial \theta} \right\| = r\sqrt{(g'(r))^2+1} = ra/\sqrt{a^2-(r-b)^2},$$

$$\text{so area of } \mathcal{S} = 2 \cdot \text{area of top half } = 2\int_{\theta=0}^{2\pi}\int_{r=b-a}^{b+a} \frac{ra}{\sqrt{a^2-(r-b)^2}}\, dr\, d\theta = 4\pi \int_{u=-a}^{a} \frac{ua+ba}{\sqrt{a^2-u^2}}\, du$$

$$= 4\pi \int_{-a}^{a} \frac{au}{\sqrt{a^2-u^2}}\, du + 4\pi \int_{-a}^{a} \frac{ab}{\sqrt{a^2-u^2}}\, du = 0 + 4\pi ab \arcsin(u/a)\Big|_{-a}^{a} = 4\pi^2 ab$$