Solutions for Test II, Math 291 Fall 2017

December 5, 2017

1. Let $f(x,y) = \dfrac{x^3}{3} + 2x^2 + y^2 - 2xy$.

(a) Find all of the critical points of f. Evaluate the Hessian matrix of f at each of these critical points, and determine where each is a local maximum, a local minimum, a saddle, or undecidable from the Hessian.

(b) Sketch a contour plot of f in the vicinity of each critical point. **Especially here, show the computations that lead to the plot to get credit; in drawing the sketches, bear in mind that $\sqrt{5}$ is a bit less than $9/4$. Hyperbolas and ellipses must have the right proportions.**

(c) Find all unit vectors $\mathbf{u}$ in $\mathbb{R}^2$ that minimize **or** maximize

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f((-2,-2) + t\mathbf{u})\bigg|_{t=0} .$$

SOLUTION (a) We compute $\nabla f(x,y) = (x^2 + 4x - 2y, 2y - 2x)$. Then (x,y) is crtical if and only if $x^2 + 4x - 2y = 0$ and $y - x = 0$. Eliminating y, we get $x(x+2) = 0$, hence $x = 0$ or $x = -2$. Thus we have two critical points

$$\mathbf{x}_1 := (0,0) \quad \text{and} \quad \mathbf{x}_2 := (-2,-2) .$$

The Hessian is

$$[\mathrm{Hess}_f(x,y)] = \begin{bmatrix} 2x+4 & -2 \\ -2 & 2 \end{bmatrix} .$$

Evaluating,

$$[\mathrm{Hess}_f(\mathbf{x}_1)] = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad [\mathrm{Hess}_f(\mathbf{x}_2)] = \begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix} .$$

Since $\det([\mathrm{Hess}_f(\mathbf{x}_1)]) = 4 > 0$, and the upper left entry of $[\mathrm{Hess}_f(\mathbf{x}_1)]$ is positive, by Sylvester's criterion, $\det([\mathrm{Hess}_f(\mathbf{x}_1)])$ is a positive matrix, and hence $\mathbf{x}_1$ is a local minimum.

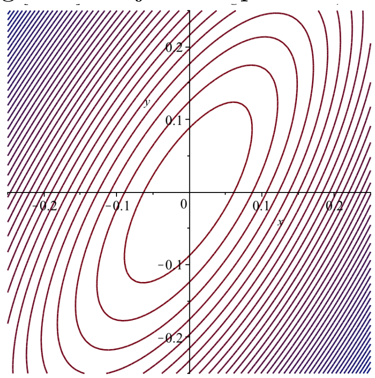
Since $\det([\mathrm{Hess}_f(\mathbf{x}_2)]) = -4 < 0$, $\mathbf{x}_2$ is a saddle point.

(b) We meed the eigenvalues and eigenvectors of $A := \det([\mathrm{Hess}_f(\mathbf{x}_1)]$ and of $B := \det([\mathrm{Hess}_f(\mathbf{x}_2)]$.

We compute $\det(A - tI_{2\times 2}) = t^2 - 6t + 4$. The eigenvalues are $\mu_1 = 3 + \sqrt{5}$ and $\mu_2 = 3 - \sqrt{5}$. An eigenvector corresponding to μ_1 is given by $\mathbf{v}_1 = (2, 1 - \sqrt{5})$, and therefore an eigenvector corresponding to μ_2 is $\mathbf{v}_2 = \mathbf{v}_1^\perp = (\sqrt{5}-1, 2)$. The contour curves of the best quadratic approximation near $\mathbf{x}_1$ will be ellipses. The ratio of the lengths of the major and minor axes is

$$\sqrt{\mu_1/\mu_2} = \frac{2}{3-\sqrt{5}} \ ,$$

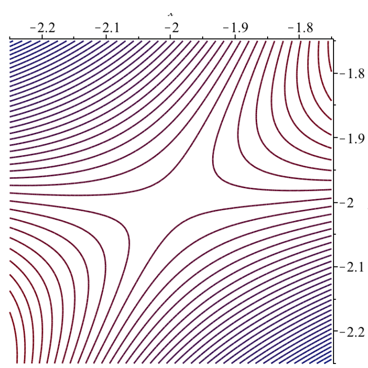
and so two and a half times as long. The major axis points along $\mathbf{v}_2$.



Next, we compute $\det(B - tI_{2\times 2}) = t^2 - 2t - 4$. The eigenvalues are $\mu_1 = 1+\sqrt{5}$ and $\mu_2 = 1-\sqrt{5}$. An eigenvector corresponding to μ_1 is given by $\mathbf{v}_1 = (-2, \sqrt{5}+1)$, and therefore an eigenvector corresponding to μ_2 is $\mathbf{v}_2 = \mathbf{v}_1^\perp = -(1+\sqrt{5}, 2)$. The contour curves of the best quadratic approximation near $\mathbf{x}_2$ will be hyperbolas. In terms of the u, v variables, the assymptotes of the ellipses are given by $\mu_1 u^2 + \mu_2 v^2 = 0$, or

$$v = \pm\sqrt{(1+\sqrt{5})/(\sqrt{5}-1)}u = \pm\frac{2}{\sqrt{5}-1}u \ .$$

Therefore, the assymptotes make an acute angle with respect to the v axis, which runs along the direction of $\mathbf{v}_2$.



(c) Since

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f((-2,-2) + t\mathbf{u})\bigg|_{t=0} = \mathbf{u}\cdot[\mathrm{Hess}_f(-2,-2)]\mathbf{u} \ ,$$

the maximizers are the two unit vectors that are eigenvalues of $[\mathrm{Hess}_f(-2,-2)]$ with the maximal eigenvalue. We have found these above, apart from normalizing, and so the maximizers are

$$\mathbf{u} = \pm\frac{1}{\sqrt{10+2\sqrt{5}}}(-2, \sqrt{5}+1) \ .$$

The unit vectors that give the minimum will then by $\pm\mathbf{u}^\perp$.

2. Use the method of Lagrange multipliers in both parts of this problem.

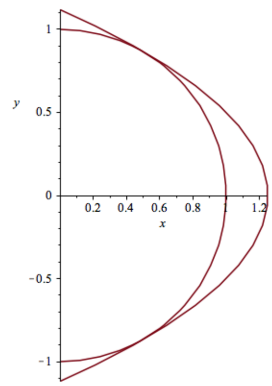
(a) Let D be the region in the plane that is to the right of the unit circle, and to the left of the parabola $x = 5/4 - y^2$. Let $f(x,y) = x + y$. Find the minimum and maximum values of f on D, and find all minimizers and maximizers.

(b) Define

$$g_1(x,y,z) := z + x^2 - 1 \qquad \text{and} \qquad g_2(x,y) := z - y^2 \ .$$

The equations $g_1(x,y,z) = 0$ and $g_2(x,y,z) = 0$ each specify a surface, and the two surfaces intersect in a curve $\mathcal{C}$. Let $f(x,y,z) = x + z$. Find the minimum and maximum values of f on $\mathcal{C}$, and find all minimizers and maximizers.

SOLUTION (a). The region D is shown below.



To find the points where the parabola and circle meet, we note that at these points, $y^2 = 1 - x^2$ and $y^2 = 5/4 - x$. Therefore, $x^2 - x = -1/4$ so that $x = 1/2$ and then $y = \pm\sqrt{3}/2$. Hence the ponts of intersection are $(1/2, \sqrt{3}/2)$ and $(1/2, -\sqrt{3}/2)$. We include these points in our list of suspects.

Next, we find soutions of Lagrange's equations on the circle. For this take $g(x,y) = x^2 + y^2 - 1$. $\nabla f(x,y) = (1,1)$, and $\nabla g(x,y) = 2(x,y)$. Clearly $\nabla f(x,y) = \lambda \nabla g(x,y)$ means $y = x$, and using this in $g(x,y) = 0$, we find $x^2 = 1/2$, so $(1/\sqrt{2}, \pm 1/\sqrt{2})$ are the two solutions that are on the part of the circle that concerns us.

Next, we find soutions of Lagrange's equations on the circle. For this take $g(x,y) = x + y^2 - 5/4$. $\nabla f(x,y) = (1,1)$, and $\nabla g(x,y) = (1, 2y)$. Clearly $\nabla f(x,y) = \lambda \nabla g(x,y)$ means $y = 1/2$, and using this in $g(x,y) = 0$, we find $x = 1$, so $(1, 1/2)$ is the only solution on the parabola.

We thus have 5 points to consider:

$$(1/2, \sqrt{3}/2)\ , \quad (1/2, -\sqrt{3}/2)\ , \quad (1/\sqrt{2}, 1/\sqrt{2})\ , \quad (1/\sqrt{2}, -1/\sqrt{2})\ , \quad (1, 1/2)\ .$$

Evidently, the maximizer is $(1, 1/2)$, the maximum value is $3/2$, the minimizer is $(1/2, -\sqrt{3}/2)$, and the minimum value is $(1 - \sqrt{3})/2$.

(b) Since $\nabla f(x,y,z) = (1,0,1)$, $\nabla g_1(x,y,z) = (2x,0,1)$, and $\nabla g_2(x,y,z) = (0,-2y,1)$. the equation $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ yields

$$0 = \det\left(\begin{bmatrix} 1 & 0 & 1 \\ 2x & 0 & 1 \\ 0 & -2y & 1 \end{bmatrix}\right) = 2y(1 - 2x)\ .$$

The two solutions are $y = 0$ and $x = 1/2$.

For $y = 0$, $g_2 = 0$ gives $z = 0$, and then $g_1 = 0$ gives $x = \pm 1$. So this gives the two solutions $(\pm 1, 0, 0)$. For $x = 1/2$, $g_1 = 0$ gives $z = 3/4$, and then $g_2 = 0$ gives $y = \pm\sqrt{3}/2$. So this gives the two solutions $(1/2, \pm\sqrt{3}/2, 3/4)$. Evidently, $(1/2, \pm\sqrt{3}/2, 3/4)$ are both maximizers, the maximum value is $5/4$, and $(-1, 0, 0,)$ is the minimizer, and the minimum value is -1.

3. (a) Let $f(x, y) = x^2y^2$, and let D be the region that lies inside both of the circles $(x-1)^2+y^2 = 4$ and $(x + 1)^2 + y^2 = 4$. Compute $\int_D f(x, y)\mathrm{d}A$.

(b) Let $f(x, y) = xy$, and let D be the region in the positive quadrant of $\mathbb{R}^2$ bounded by $xy = 1$, $xy = 2$, $y/x = 1$ and $y/x = 2$. Compute $\int_D f(x, y)\mathrm{d}A$.

SOLUTION (a). For (x, y) on both circles, $(x - 1)^2 + y^2 = (x + 1)^2 + y^2 = 4$ and hence $x = 0$, and $y = \pm sqrt3$. If we integrate first in x, our limits of integration are given by

$$1 - \sqrt{4 - y^2} \leq x \leq \sqrt{4 - y^2} - 1 \qquad \text{and} \qquad -\sqrt{3} \leq y \leq \sqrt{3}\ .$$

Therefore,

$$\int_D f(x, y)\mathrm{d}A = \int_{-\sqrt{3}}^{\sqrt{3}} \left(\int_{1-\sqrt{4-y^2}}^{\sqrt{4-y^2}-1} x^2 \mathrm{d}x \right) y^2 \mathrm{d}y = \frac{2}{3} \int_{-\sqrt{3}}^{\sqrt{3}} y^2(\sqrt{4 - y^2} - 1)^3 \mathrm{d}y\ .$$

(b) Let the change of variable $\mathbf{u} = \mathbf{u}(\mathbf{x})$ be defined by $u = xy$ and $v = y/x$. Let $\bar{D}$ be the region in the u, v-plane defined by $1 \leq u \leq 2$ and $1 \leq v \leq 2$. Then $\bar{D}$ corresponds to D under the correspondence $\mathbf{u}(\mathbf{x})$.

Solving $u = xy$ and $v = y/x$ for x and y gives the inverse correspondence $\mathbf{x}(\mathbf{u})$, namely $uv = y^2$, $y = \sqrt{uv}$, $x = u/y = \sqrt{u/v}$. We then have

$$[D_{\mathbf{x}}(u, v)] = \frac{1}{2} \begin{bmatrix} u^{-1/2}v^{-1/2} & -u^{1/2}v^{-3/2} \\ u^{-1/2}v^{1/2} & u^{1/2}v^{-1/2} \end{bmatrix} .$$

Then

$$\det([D_{\mathbf{x}}(u, v)]) = \frac{1}{4}\frac{2}{v} = \frac{1}{2v}\ .$$

Finally,

$$\int_D f(x, y)\mathrm{d}A = \int_1^2 \left(\int_1^2 \frac{u}{2v} \mathrm{d}v \right) \mathrm{d}u = \frac{\ln 2}{2} \int_1^2 u\mathrm{d}u = \frac{3 \ln 2}{4}\ .$$

4. Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that is bounded by the surfaces

$$\begin{aligned} \sqrt{x^2 + y^2} &= z^2 \\ \sqrt{x^2 + y^2} &= 8 - z^2 \end{aligned}$$

. Let $\mathcal{S}$ denote the (total) bounding surface.

(a) Compute the volume of $\mathcal{V}$. Give a numerical value for this one.

(b) Compute the surface area of $\mathcal{S}$. You may leave this in the form of an single variable integral with explicit limits for full credit. (The integral, at least what you get from one natural way to proceed, can be done by a trigonometric substitution.)

SOLUTION The equations for the two bounding surfaces can be written in cylindrical coordinated as

$$r = z^2 \quad \text{and} \quad r = 8 - z^2$$

These meet when $z^2 = 8 - z^2$, or $z^2 = 4$, which has the solutions $z = \pm 2$. Let's do the computations for the part of the region with $z > 0$, and then double the results in the end.

Then

$$\sqrt{r} \le z \le \sqrt{8-r}\ , \quad 0 \le r \le 4\ , \quad 0 \le \theta \le 2\pi\ .$$

Then

$$\mathrm{vol}(\mathcal{V}) = 2\pi \int_0^4 \left(\int_{\sqrt{r}}^{\sqrt{8-r}} \mathrm{d}z \right) r\mathrm{d}r = 2\pi \int_0^4 (r\sqrt{8-r} - r^{3/2})\mathrm{d}r = \frac{512}{15}\sqrt{2} - \frac{128}{3}\ .$$

For the upper part of the surface we have

$$\mathbf{X}(\theta, z) = \big((8-z^2)\cos\theta, (8-z^2)\sin\theta, z\big)\ , \quad 2 \le z \le 2\sqrt{2}\ , 0 \le \theta \le 2\pi\ .$$

We compute

$$\mathbf{X}_z \times \mathbf{X}_\theta = (8-z^2)\big(\cos\theta, \sin\theta, 2z)\big)$$

and hence $\mathrm{d}S = (8-z^2)\sqrt{4z+1}$. The surface area of the upper part is

$$2\pi \int_2^{2^{2/3}} (8-z^2)\sqrt{4z^2+1}\mathrm{d}z\ .$$

For the lower part of the surface we have

$$\mathbf{X}(\theta, z) = \big(z^2\cos\theta, z^2\sin\theta, z\big)\ , \quad 0 \le z \le 0\ , 0 \le \theta \le 2\pi\ .$$

We compute

$$\mathbf{X}_z \times \mathbf{X}_\theta = -z^2\big(\cos\theta, \sin\theta, -2z)\big)$$

and hence $\mathrm{d}S = z^2\sqrt{4z^2+1}$. The surface area of the upper part is

$$2\pi \int_0^2 z^2\sqrt{4z^2+1}\mathrm{d}z\ .$$

The final answer for the surface area is twoce the sum of the contributio0ns form the upper and lower parts in the region $z > 0$.

Extra Credit: Let D be the set in $\mathbb{R}^2$ that is given by

$$0 \le x^2 - y^2 \le 4 \qquad \text{and} \qquad 1 \le xy \le 2\ .$$

Let $f(x,y) = x^2 + y^2$. Compute $\int_D f(x,y)\mathrm{d}A$.

SOLUTION The region D has two symmetric pieces, D_1 in the first quadrant and D_2 in the third quadrant. The integrand is unchanged by the double sign change $x \to -x$ and $y \to -y$, so

$$\int_D (x^2+y^2)\,\mathrm{d}A = 2\int_{D_1} x^2 + y^2\mathrm{d}A$$

Define $\mathbf{u}(\mathbf{x})$ by $u = x^2 - y^2$ and $v = xy$ and let $\overline{D_1}$ be the region $0 \le u \le 4$, $1 \le v \le 2$ in the u, v-plane.

The inverse transformation is messy but $D_{\mathbf{u}}(\mathbf{x})$ is easy to compute; it is

$$[D_{\mathbf{u}}(\mathbf{x})] = \begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix}, \qquad \det([D_{\mathbf{u}}(\mathbf{x})]) = 2(x^2 + y^2)$$

Therefore by the change of variable formula in reverse

$$\int_D (x^2 + y^2)\,\mathrm{d}x\mathrm{d}y = 2\int_{D_1} (x^2 + y^2)\,\mathrm{d}x\mathrm{d}y = \int_{D_1} |\det D_{\mathbf{u}}(\mathbf{x})|\,\mathrm{d}x\mathrm{d}y = \int_{\overline{D_1}} 1\mathrm{d}u\mathrm{d}v = \text{area of} \overline{D_1} = 4.$$