Test I Solutions, Math 291 Fall 2017

October 27, 2017

1: Consider the plane passing through the three points

$$\mathbf{p}_1 = (-2, 0, 2) \quad \mathbf{p}_2 = (1, -2, 2) \qquad \text{and} \qquad \mathbf{p}_3 = (3, -1, -2)$$

and the line passing through

$$\mathbf{z}_0 = (1, 4, -2) \qquad \text{and} \qquad \mathbf{z}_1 = (0, -3, 1)$$

(a) Find a parametric representation $\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$ for the plane.

(b) Find a parametric representation $\mathbf{z}(u) = \mathbf{z}_0 + u\mathbf{w}$ for the line.

(c) Find an equation for the plane.

(d) Find a system of equations for the line.

(e) Find the points, if any, where the line intersects the plane.

(f) Find the distance from $\mathbf{p}_1$ to the line.

(g) Find the distance from $\mathbf{z}_0$ to the plane.

SOLUTION (a). We compute $\mathbf{v}_1 := \mathbf{p}_2 - \mathbf{p}_1 = (3, -2, 0)$ and $\mathbf{v}_2 := \mathbf{p}_3 - \mathbf{p}_1 = (5, -1, -4)$. Some people used $\mathbf{p}_3 - \mathbf{p}_2 = (2, 1, -4)$ in place of either $\mathbf{v}_1$ or $\mathbf{v}_2$, which is also correct. Then the parameterization (for this choice of $\mathbf{v}_1$ and $\mathbf{v}_2$) is

$$\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2 = \big(-2 + 3s + 5t, -2s - t, 2 - 4t\big) \ .$$

(b). Define $\mathbf{w} = \mathbf{z}_1 - \mathbf{z}_0 = \big((-1, -7.3\big)$. Then the line is parameterized by

$$\mathbf{x}(u) = \mathbf{z}_0 + u\mathbf{w} = \big(1, 4, -2\big) + u\big(-1, -7, 3\big) \ .$$

(c). Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel to the plane, the normal line to the plane is the line along

$$\mathbf{a} := \mathbf{v}_1 \times \mathbf{v}_2 = \big(8, 12, -7\big) \ . \tag{0.1}$$

We compute $\mathbf{a} \cdot \mathbf{p}_1 = -2$, and hence $\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}_1) = 0$ is the same as

$$8x + 12y + 7z = -2 \ .$$

this is the equation of the plane. notice this is an equation in the three variables x, y, z, nat is is satisfied if and only if (x, y, z) in on the plane.

(d). We seek an equation in the variables x, y, z such that the equation is satisfied by x, y, z if and only if (x, y, z) belong to the line. Such an equation is given by

$$\mathbf{w} \times (\mathbf{x} - \mathbf{z}_0) = \mathbf{0} \ .$$

This is the same as $\mathbf{w} \times \mathbf{x} = \mathbf{w} \times \mathbf{z}_0$. Computing we fine

$$\mathbf{w} \times \mathbf{x} = (\,-3y, 7z, 3x + z, 7x - 7y) \qquad \text{and} \qquad \mathbf{w} \times \mathbf{z}_0 = (2, 1, 3) \ .$$

Hence (x, y, z) is on the line if and only if

$$\begin{aligned} -3y - 7z &= 2 \\ 3x + z &= 1 \\ 7x - 7y &= 3 \ . \end{aligned}$$

Each of these equations describes a plane. any two of these planes intersect in our line. Hence any two of the equations may be takes as the system of equations for the line: Each individual equation describes a plane, and the intersection of the two planes is the line.

(e). The general point on the line is given by put parameterization $\mathbf{x}(u) = \mathbf{z}_0 + u\mathbf{w}$. The point $\mathbf{x}(u)$ lies in the plane if and only if it satisfies the equation derived in part **(c)**. That is $\mathbf{x}(u)$ lies in the plane if and only if

$$(8, 12, 7) \cdot \mathbf{x}(u) = -2 \ .$$

From the formula in **(b)**, $u = 44/71$, and hence the unique point of intersection is

$$\mathbf{x}(44/71) = \frac{1}{71}(27, -24, -10) \ .$$

(f). The distance from any point $\mathbf{x}$ to the line is $\dfrac{1}{\|\mathbf{w}\|}|\mathbf{w} \times (\mathbf{x} - \mathbf{z}_0)|$. Hence the distance from $\mathbf{p}_1$ to the line is

$$\frac{1}{\|\mathbf{w}\|}|\mathbf{w} \times (\mathbf{p}_1 - \mathbf{z}_0) = \sqrt{\frac{570}{59}} \ .$$

(g). The distance from any point $\mathbf{x}$ to the planes is $\dfrac{1}{\|\mathbf{a}\|}|\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}_1)|$. Taking $\mathbf{x} = \mathbf{z}_0$, we find that with $\mathbf{x} = \mathbf{p}_1$, the distance is

$$\frac{44}{\sqrt{257}} \ .$$

2: Let $\mathbf{x}(t) = (3t - t^3, 3t^2, 3t + t^3)$ where $r > 0$.

(a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Compute $\mathbf{N}(t)$ and $\mathbf{B}(t)$, as well as the curvature $\kappa(t)$ and the torsion $\tau(t)$

(d) Find the tangent line to this curve at $t = 1$, and the equation of the osculating plane to the curve at $t = 1$.

SOLUTION (a) Differentiating, we find

$$\mathbf{v}(t) = 3(1 - t^2, 2t, 1 + t^2)$$

and

$$\mathbf{a}(t) = (\,-6t, 6, 6t)\ .$$

(b) $v(t) = \|\mathbf{v}(t)\| = 3\sqrt{2}(t^2+1)$. Then

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = \frac{1}{\sqrt{2}(t^2+1)}(1-t^2, 2t, 1+t^2)\ .$$

(c) Recall that $\mathbf{N}(t) = \|\mathbf{a}_\perp(t)\|^{-1}\mathbf{a}_\perp(t)$ and $\kappa(t) = \dfrac{\|\mathbf{a}_\perp(t)\|}{v(t)^2}$. Therefore, to answer this question, we compute

$$\mathbf{a}_\perp(t) = \mathbf{a}(t) - (\mathbf{a}(t)\cdot\mathbf{T}(t))\mathbf{T}(t) = \frac{6}{t^2+1}(\,-2t, 1-t^2, 0)\ .$$

It follows that

$$\mathbf{N}(t) = \frac{1}{t^2+1}(\,-2t, 1-t^2, 0) \quad \text{and} \quad \kappa(t) = \frac{1}{3(t^2+1)^2}\ .$$

Next, we compute

$$\mathbf{v}(t)\times\mathbf{a}(t) = 18(t^2-1, -2, t^2+1)$$

and so

$$\mathbf{B}(t) = \frac{1}{\|\mathbf{v}(t)\times\mathbf{a}(t)\|}\mathbf{v}(t)\times\mathbf{a}(t) = \frac{1}{\sqrt{2}}\left(\frac{t^2-1}{t^2+1}, \frac{-2}{t^2+1}, 1\right)\ .$$

Differentiating,

$$\mathbf{B}'(t) = \frac{\sqrt{2}}{(t^2+1)^2}(2t, t^2-1, 0)) = -\frac{\sqrt{2}}{t^2+1}\mathbf{N}(t)\ .$$

Then from the identity $\mathbf{B}'(t) = -v(t)\tau(t)\mathbf{N}(t)$, we conclude that

$$\tau(t) = \frac{1}{3(t^2+1)^2}\ .$$

Notice that this is a curve for which $\tau(t) = \kappa(t)$.

(d) We compute $\mathbf{x}(1) = (2.3.4)$ and $\mathbf{v}(1) = (0, 6, 6)$. Hence the tangent line is parameterized by $\mathbf{x}(u) = (2, 3, 4) + u(0, 1, 1)$. Next,

$$\mathbf{B}(1) = \frac{1}{\sqrt{2}}(0, -1, , 1)\ .$$

The equation of the osculating plane then is $\mathbf{B}(1)\cdot(\mathbf{x} - \mathbf{x}(1)) = 0$, which reduces to $z - y = 1$.

3: Define functions f and g on $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} \dfrac{\ln(1+x^2y^2)}{x^6+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0)\ . \end{cases}$$

and

$$g(x,y) = \begin{cases} \dfrac{\ln(1+x^2y^2)}{x^4+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0)\ . \end{cases}$$

(a) Is f continuous at $(0,0)$? Justify your answer for credit.

(b) Is g continuous at $(0,0)$? Justify your answer for credit.

SOLUTION (a) If the function is discontinuous, we are likely to see trouble by making the terms in the denominator equal. So we take $y = x^3$, and compute

$$\lim_{x\to 0} f(x,x^3) = \lim_{x\to 0} \frac{\ln(1+x^8)}{2x^6} \ .$$

Since $\ln(1+t) \le t$,

$$0 \le \frac{\ln(1+x^8)}{2x^6} \le \frac{x^8}{2x^6} = \frac{x^2}{2} \ ,$$

$$\lim_{x\to 0} f(x,x^3) = 0 \ .$$

Since there is no trouble revealed here, we try to prove that the function is in fact continuous using the Squeeze Theorem.

For small t, $\ln(1+t) \approx t$, and so when $\|\mathbf{x}\|$ is small, $\ln(1+xy) \approx xy$. Hence we may as well consider the functions defined by $f(x,y) = \dfrac{x^2y^2}{x^6+y^2}$ for $\mathbf{x} \neq \mathbf{0}$. Now note that

$$0 \le \frac{x^2y^2}{x^6+y^2} \le \frac{x^2y^2}{y^2} = x^2 \le \|\mathbf{x}\|^2 \ .$$

by the Squeeze Theorem,

$$\lim_{\|\mathbf{x}\|\to \mathbf{0}} \frac{x^2y^2}{x^6+y^2} = 0 \ .$$

Going back to the original function, note that since $\ln(1+t) \le t$,

$$0 \le \frac{\ln(1+x^2y^2)}{x^6+y^2} \le \frac{x^2y^2}{x^6+y^2}$$

so the same argument applies to this function too. Hence f is continuous.

(b) If the function is discontinuous, we are likely to see trouble by making the terms in the denominator equal. So we take $y = x$, and compute

$$\lim_{x\to 0} g(x,x) = \lim_{x\to 0} \frac{\ln(1+x^4)}{2x^4} = \frac{1}{2} \ .$$

Hence g is discontinuous at $\mathbf{0}$.

4: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = 4xy - x^4 - y^4$.

(a) Let $\mathbf{x}(t)$ be given by $\mathbf{x}(t) = (t+t^2, t^2+t^3)$ Compute $\left.\dfrac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\right|_{t=1}$.

(b) Find all of the critical points of f, and find the value of f at each of the critical points.

(c) Does f have a maximum value? Explain why or why not. If it does, find all points at which the value of f is maximal; i.e, find all maximizers.

(d) Does f have a minimum value? Explain why or why not. If it does, find all points at which the value of f is minimal; i.e, find all minimizers.

SOLUTION (a) We compute

$$\nabla f(x,y) = 4(\,-x^3 + y, -y^3 + x)\ .$$

Also, $\mathbf{x}'(t) = (1 + 2t, 2t + 3t^2))$ so that $\mathbf{x}'(t) = (3,5)$ and $\mathbf{x}(1) = (2,2)$. Then since $\nabla f(2,2) = -24(1,1)$, the chain rule gives us

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\bigg|_{t=1} = -24(1,1)\cdot(3,5) = -192\ .$$

(b) We must solve $\nabla f(x,y) = (0,0)$ to find the critical points (x,y). This amounts to the system of equations

$$\begin{aligned} y &= x^3 \\ x &= y^3\ . \end{aligned}$$

substituting $y = x^3$ into the second equation, $x - x^9 = 0$. That is, $x(1-x^8) = 0$. The solutions are $x = 0$ or $x = \pm 14$. Then from $y = x^3$, we have that there are 3 critical points, namely

$$(0,0)\ ,\quad (1,1)\quad \text{and}\quad (\,-1,-1)\ .$$

(c) and **(d)** The fourth powers, which have negative coefficients, are dominant for large $\|\mathbf{x}\|$, and so

$$\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = -\infty\ .$$

Therefore, there is no minimum, but there is a maximum, and the maximizers will be critical points. Evaluating, we find

$$f(0,0) = 0 \quad \text{and} \quad f(1,1) = f(-1,-1) = 2\ .$$

Hence the maximizers are the points $(1,1)$ and $(\,-1,-1)$.

Full credit was given, naturally, for such an argument that no minimizers exist, but the maximizers do exist. A more explicit argument can be given using the arithmetic-geometric mean inequality $ab \le (a^2+b^2)/2$. Using this twice

$$4xy - x^4 - y^4 \le 2(x^2+y^2) - \frac{1}{2}(x^4 + y^2 + 2x^2y^2) = 2\|\mathbf{x}\|^2 - \frac{1}{2}\|\mathbf{x}\|^4 = -\frac{1}{2}\|\mathbf{x}\|^2(\|\mathbf{x}\|^2 - 4)\ .$$

Therefore, for $\mathbf{x}$ such that $\|\mathbf{x}\| \ge 3$, $f(\mathbf{x}) \le -\frac{5}{2}\|\mathbf{x}\|^2 \le -\frac{45}{2}$. Since $f(\mathbf{x})$ is larger than this at the critical points, we need only look for the maximizers in the compact set $\{\mathbf{x}\ :\ \|\mathbf{x}\| \le 3\}$. A maximizer exists in this set, and it cannot be on the boundary, so it occurs at a critical point.

Extra Credit: Let $f(x,y)$ be as in Problem 4 of this test. Find all points (x,y) where the tangent plane to the graph of $z = f(x,y)$ is parallel of the tangent plane of this graph at the point $(1,2)$.

SOLUTION We compute $\nabla f(1,2) = (4,-28)$. Then the "standard" normal to the tangent plane at $(1,2)$ is $(4,-28,-1)$. The "standard" normal to the tangent plane at (x,y) is $(\partial f(x,y)/\partial x, \partial f(x,y)/\partial y, -1)$. The two tangent planes will be parallel if and only if $(4,-28,-1)$ and $(\partial f(x,y)/\partial x, \partial f(x,y)/\partial y, -1)$ are proportional, and since the final entries are both -1, this means that

$$\nabla f(x,y) = (4,-28)\ .$$

This gives us the system of equations

$$\begin{aligned} y - x^3 &= 1 \\ x - y^3 &= -7 \ . \end{aligned}$$

We already know one solution, namely $x = 1$ and $y = 2$. There are no other solutions of this system, so there are no points other than $(1, 2)$ at which the tangent plane is parallel to the tangent plane at $(1, 2)$.

To see there are no other solutions, substitute $y = x^3 + 1$ from the first equation into the second, obtaining

$$p(x) := -x^9 - 3x^6 - 3x^3 + x - 6 = 0 \ .$$

We know that $x = 1$ is a root of the polynomial $p(x)$. A simple plot shows that $p(x) < 0$ for $x > 1$ and $p(x) > 0$ for $x < 1$. Hence this is the only real rooy, and then $y = x^3 + 1$ tells us $y = 2$.

To see that $p(x) < 0$ for $x > 0$, we note that for $x > 1$, $x < x^3$,

$$p(x) \leq -x^9 - 3x^6 - 3x^3 + x^3 - 6 = -x^9 - 3x^6 - 2x^3 - 6 = 0 \ .$$

Now note that

$$-x^9 - 3x^6 - 2x^3 = -x^3(x^6 + 3x^3 + 2) = -x^3(x^3 + 1)(x^3 + 2) < -6$$

for $x > 1$.

Similar reasoning for $x \in [0, 1]$ and $x < -1$ completes the proof the $p(x) > 0$ for all $x < 1$. This much detial, however, was not expected.