# Test I, Math 291 Fall 2015

October 15, 2015

**NAME:** ______________________________________________

**INSTRUCTIONS:** Non-graphing calculators are allowed, and a one page, one sided formula sheet. Give all answers in *exact* form. In particular, if an answer is $\sqrt{2}$, write $\sqrt{2}$ and *not* a decimal approximation such as 1.414213562.

**1:** Let

$$\mathbf{a} = (1,8,4)\ , \qquad \mathbf{b} = (4,-4,7) \quad \text{and} \quad \mathbf{c} = (2,2,1)$$

**(a)** Consider the two equations

$$\mathbf{a} \times \mathbf{x} = \mathbf{b} \qquad \text{and} \qquad \mathbf{a} \times \mathbf{x} = \mathbf{c}\ .$$

One of the two equations has no solutions, and for the other one, the solution set is a line. Which one is which? For the equation that has no solutions, explain how you know this. For the one that has a line of solutions, find a parameterization of the line.

**(b)** Apply the Gram-Schmidt algorithm to $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ (in this order), to produce an orthonormal set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. (The result will be an orthonormal set.) Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ right handed or left handed?

**(c)** What is the distance between $\mathbf{c}$ and the plane containing $\mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$? Also what is the distance between $\mathbf{c}$ and the line through $\mathbf{0}$ that is orthogonal to this plane? Justify your answer.

**2:** Consider the two curves

$$\mathbf{x}(t) = (\cos^2 t, \sin^2 t, \cos(2t)) \quad \text{and} \quad \mathbf{y}(t) = (\sin t + 8\cos t \;\; 8\sin t + \cos t \;,\;\; 4\sin t - 4\cos t \;)\;.$$

**(a)** Compute the curvature for both curves. Justify your answers, showing your computations.

**(b)** Both of these curves are planar curves, but one is even more special – it is a *linear curve*, meaning that there is a line in $\mathbb{R}^3$ that contains the entire curve. Which of these curves is it? Find an equation for the line containing the linear curve, and an equation for the plane containing the other curve.

**(c)** Find the arc length along $\mathbf{x}(t)$ between $t = 0$ and $t = \pi$.

**3:** Define functions $f$ and $g$ on $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} \dfrac{x^2 \sin(xy)}{x^6 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0)\ . \end{cases}$$

and

$$g(x,y) = \begin{cases} \dfrac{xy \sin(xy)}{x^6 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0)\ . \end{cases}$$

**(a)** Is $f$ continuous at $(0,0)$? Justify your answer for credit.

**(b)** Is $g$ continuous at $(0,0)$? Justify your answer for credit.

**4:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = x^4 + y^4 - 2(x^2 - y^2) .$$

**(a)** Find all points at which the tangent plane to the graph of $f$ is horizontal.

**(b)** Let $\mathbf{y}(t) = (\sin t + 8\cos t \;\; 8\sin t + \cos t \;,\;\; 4\sin t - 4\cos t\; )$, as in Problem 2.

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{y}(t))\bigg|_{t=0} .$$

**(c)** Show that on the set of points $(x, y)$ with $\sqrt{x^2 + y^2} \geq 4$, $f(x, y) \geq 96$. Do this using the inequality $s^2 + b^2 \geq (a + b)^2/2$ to relate $x^4 + y^4$ to $(x^2 + y^2)$, or by using polar coordinates. If you get a different number than 96 that is fine as long as you correctly justify your reasoning.

**(d)** Does $f$ have a minimizer on the $x, y$ plane? If so, what is it, and what is the minimum value of $f$ on this set? Justify your answer.

**(e)** Does $f$ have a maximizer on the $x, y$ plane? If so, what is it, and what is the maximum value of $f$ on this set? Justify your answer.

**Extra Credit:** Let $f(x, y)$ be as in Problem 4 of this test. Find all points $(x, y)$ where the tangent plane to the graph of $z = f(x, y)$ is parallel of the tangent plane of this graph at the point $(1, 2)$.