**Note:** In these solutions we sometimes write $f_x$, $f_{xx}$, $f_{xy}$ for $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial^2 f}{\partial x^2}$, and $\dfrac{\partial^2 f}{\partial y \partial x}$, etc. We also write $A^t$ for the transpose of a matrix $A$. Finally, contour maps are not drawn, but are described in detail.

**6.2.** $[Hess_f] = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$ so $[Hess_f(\mathbf{x}_0)] = \begin{bmatrix} 0 & -4 \\ -4 & 12 \end{bmatrix}$. Then

$$\left.\frac{d^2}{dt^2} f(\mathbf{x}_0 + t\mathbf{v})\right|_{t=0} = \mathbf{v}[Hess_f(\mathbf{x}_0)]\mathbf{v}^t = (1, -1) \begin{bmatrix} 0 & -4 \\ -4 & 12 \end{bmatrix} \bullet (1, -1) = 20.$$

**6.4.** $[Hess_f] = \begin{bmatrix} 0 & z-1 & y+1 \\ z-1 & 0 & x \\ y+1 & x & 0 \end{bmatrix}$ so $[Hess_f(\mathbf{x}_0)] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. Then

$$\left.\frac{d^2}{dt^2} f(\mathbf{x}_0 + t\mathbf{v})\right|_{t=0} = \mathbf{v}[Hess_f(\mathbf{x}_0)]\mathbf{v}^t = (1, 2, 1) \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \bullet (1, 2, 1) = 8.$$

**6.8.** $\det \begin{bmatrix} 1-t & 2 \\ 2 & 5-t \end{bmatrix} = t^2 - 6t + 1$, eigenvalues are $t = (6 \pm \sqrt{32})/2 = 3 \pm \sqrt{8}$.

For the eigenvalue $t_1 = 3 + \sqrt{8}$, an eigenvector must be perpendicular to the rows of $A - t_1 I = \begin{bmatrix} -2-\sqrt{8} & 2 \\ 2 & 2-\sqrt{8} \end{bmatrix}$. Such an eigenvector is $(2, 2+\sqrt{8})$, which when normalized becomes

$$\mathbf{u}_1 = \frac{1}{\sqrt{4+2\sqrt{2}}}(1, 1+\sqrt{2}).$$

A second unit eigenvector, corresponding to the eigenvector $t_2 = 3 - \sqrt{8}$, must be orthogonal to this, such as

$$\mathbf{u}_2 = \frac{1}{\sqrt{4+2\sqrt{2}}}(1+\sqrt{2}, -1).$$

**6.10.** (a) $f_x = 2xy + y^2 - y$, $f_y = x^2 + 2xy - x$. To find the critical points we must solve

$$y(2x + y - 1) = 0, x(x + 2y - 1) = 0.$$

$x = y = 0$ gives the critical point $(0, 0)$, $y = x + 2y - 1 = 0$ gives $(1, 0)$. $x = 2x + y - 1 = x = 0$ gives $(0, 1)$, and $2x + y - 1 = x + 2y - 1 = 0$ gives $(1/3, 1/3)$. There are four critical points in all.

$[Hess_f] = \begin{bmatrix} 2y & 2x+2y-1 \\ 2x+2y-1 & 2x \end{bmatrix}$.

$[Hess_f(0,0)] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, with characteristic equation $t^2 - 1 = 0$ and eigenvalues $\pm 1$. So $f$ has a saddle point at $(0,0)$. Just by observing that the constant term of the characteristic equation is negative, and that that constant term is the product of the eigenvalues, shows that there is one positive and one negative eigenvalue, so $f$ has a saddle point. This trick works for $2 \times 2$ matrices but not for $3 \times 3$.

$[Hess_f(1,0)] = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, with characteristic equation $t^2 - 2t - 4 = 0$. The determinant is negative so $f$ has a saddle point at $(1,0)$.

$[Hess_f(0,1)] = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, with the same negative determinant as at $(1,0)$. So $f$ has a saddle point at $(0,1)$.

$[Hess_f(1/3,1/3)] = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$, with eigenvalues $\mu_1 = 1/3$, $\mu_2 = 1$. $(1/3,1/3)$ is a local minimum of $f$.

(b) The eigenvalues of $A = [Hess_f(1/3,1/3)] = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$ have been found above. Corresponding to $\mu_1$ is the eigenvector $\mathbf{u}_1 = 2^{-1/2}(1,-1)$, and corresponding to $\mu_2$ is the eigenvector $\mathbf{u}_2 = 2^{-1/2}(1,1)$.

Let $u = (\mathbf{x} - (1/3,1/3)) \bullet \mathbf{u}_1 = (1/\sqrt{2})(x - (1/3) + y - (1/3))$ and

$v = (\mathbf{x} - (1/3,1/3)) \bullet \mathbf{u}_2 = (1/\sqrt{2})(-x + (1/3) + y - (1/3))$. Then the best quadratic approximation near $(1/3,1/3)$ is

$$f(x,y) \approx f(1/3,1/3) + \frac{1}{2}\left[(1/3)u^2 + v^2\right] \text{ as } (x,y) \to (1/3,1/3)$$

The level curves of $f$ near $(1/3,1/3)$ are approximately the graphs of

$$\frac{1}{3}u^2 + v^2 = K$$

for various constants $K$. These are ellipses centered on (1/3,1/3) with the major axis parallel to the line $y = -x$, and the major axis $\sqrt{3}$ tijmes as long as the minor axis.

**6.12.** (a) $f_x = 9x^2 + 5y + 10x$, $f_y = 5x - 10y$. At critical points $x = 2y$ so $36y^2 + 25y = 0$, $(x,y) = (0,0)$ or $(-50/36, -25/36)$.

$[Hess_f] = \begin{bmatrix} 18x + 10 & 5 \\ 5 & -10 \end{bmatrix}$.

$[Hess_f(0,0)] = \begin{bmatrix} 10 & 5 \\ 5 & -10 \end{bmatrix}$, negative determinant, therefore $f$ has a saddle point at $(0,0)$.

$[Hess_f(-50/36, -25/36) = \begin{bmatrix} -15 & 5 \\ 5 & -10 \end{bmatrix}$, characteristic equation $t^2 + 25t + 125 = 0$, eigenvalues $t = (-25 \pm \sqrt{125})/2$ are both negative, $f$ has a local maximum at $(-50/36, -25/36)$.

(b) The eigenvalues of $H = [Hess_f(0,0)$ are roots of $t^2 - 125 = 0$, i.e., $t = \pm\sqrt{125}$. An eigenvector $\mathbf{u}_1$ for $t = \sqrt{125}$ is orthogonal to the rows of $H - \sqrt{125}I = \begin{bmatrix} 10 - \sqrt{125} & 5 \\ 5 & -10 - \sqrt{125} \end{bmatrix}$, and after normalizing this becomes

$$\mathbf{u}_1 = \frac{1}{\sqrt{10 + 4\sqrt{5}}}(2 + \sqrt{5}, 1),$$

so $\mathbf{u}_2 = \mathbf{u}_1^{\perp} = \frac{1}{\sqrt{10+4\sqrt{5}}}(-1, 2 + \sqrt{5})$.

Let $u = \mathbf{x} \bullet \mathbf{u}_1 = \frac{1}{\sqrt{10+4\sqrt{5}}}((2 + \sqrt{5})x + y)$ and $v = \mathbf{x} \bullet \mathbf{u}_2 = \frac{1}{\sqrt{10+4\sqrt{5}}}(-x + (2 + \sqrt{5})y)$. Then near $\mathbf{x}_0 = (0, 0)$

$$f(x, y) \approx_2 f(0, 0) + \frac{\sqrt{125}}{2}(u^2 - v^2) \text{ as } (x, y) \to (0, 0),$$

so the level curves near $(0,0)$ are approximately the level curves $\sqrt{125}(u^2 - v^2) = K$ or $u^2 - v^2 = K^*$. These are hyperbolas whose asymptotes are the lines $u^2 - v^2 = 0$, i.e., $u + v = 0$ and $u - v = 0$. The $u$ and $v$-axes are the lines $v = 0$ and $u = 0$ respectively, and these are the lines $x = (2 + \sqrt{5})y$ and $y = -(2 + \sqrt{5})x$, respectively.

**6.14.** $\nabla f = (4y - 4x^3, 4x - 4y^3)$.

(a) $\left.\frac{d}{dt}f(\mathbf{x}(t))\right|_{t=1} = \nabla f(\mathbf{x}(1)) \bullet \mathbf{x}'(1) = \nabla f(2, 2) \bullet x'(1) = (-24, -24) \bullet (3, 5) = -192.$

(b) $4y - 4x^3 = 0 = 4x - 4y^3$ leads to $y = x^3$, $x = y^3$, critical points $(1, 1)$, $(0, 0)$, and $(-1, -1)$. $f(1, 1) = f(-1, -1) = 2$, $f(0, 0) = 0$.

(c) $[Hess_f] = \begin{bmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{bmatrix}$.

$[Hess_f(0, 0) = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$, characteristic polynomial $t^2 - 16$, eigenvalues $\pm 4$, saddle point at $(0, 0)$.

$[Hess_f(1, 1)] = [Hess_f(-1, -1)] = \begin{bmatrix} -12 & 4 \\ 4 & -12 \end{bmatrix}$, characteristic equation $t^2 + 24t + 128 = 0$, eigenvalues $-16$ and $-8$. Local maxima at $(1, 1)$ and $(-1, -1)$.

(d) Yes, $(1, 1)$ and $(-1, -1)$ are maximizers, but this is not simply a consequence of the fact that they are local maxima. Let $C$ be a large closed square centered at the origin. Then on the exterior of $C$ and on the boundary of $C$, $f(x, y) = 4xy - x^4 - y^4$ is negative. On $C$ itself, which is compact, there must exist a maximizer (for $C$). Moreover maximizers relative to $C$ must occur at critical points $(1, 1)$, $(-1, -1)$, where $f(x, y) = 2$, or on the boundary, where $f(x, y) < 0$. So the boundary is an impossible location for maximizers. Then $(1,1)$ and $(-1,-1)$ are maximizers with respect to $C$. Since $f$ has a positive value at them, but negative values off $C$, they are maximizers relative to the entire plane.

(e) No, $f(x,y)$ takes on arbitrarily "large negative" values, for example on the $x$-axis.

(f) For $H = [Hess_f(0,0)]$, we compute that $\mathbf{u}_1 = (1/\sqrt{2})(1,1)$ is a unit eigenvector for $H$ corresponding to the eigenvalue 4, and $\mathbf{u}_2 = (1/\sqrt{2})(1,-1)$ is an eigenvector for $H$ corresponding to the eigenvalue $-4$. Then letting $u = (\mathbf{x}-(0,0))\bullet\mathbf{u}_1 = (x+y)/\sqrt{2}$ and $v = (\mathbf{x}-(0,0))\bullet\mathbf{u}_2 = (x-y)/\sqrt{2}$,

$$f(x,y) \approx_2 0 + \frac{1}{2}(4u^2 - 4v^2) \text{ as } (x,y) \to (0,0).$$

The level curves of $f$ near the origin are approximately the hyperbolas $u^2 - v^2 = K$ (and the lines $u = v$, $u = -v$, corresponding to $K = 0$). Here the $u$-axis is defined by $v = 0$, i.e., $y = x$, and the $v$-axis is defined by $u = 0$, i.e., $y = -x$. (Remark: It is not surprising, since $x^4$ and $y^4$ are much smaller than $xy$ near $(0,0)$, that the hyperbola $u^2 - v^2 = K$ is a hyperbola of the form $xy = K^*$, i.e., the $4xy$ term dominates $f$ near $(0,0)$.)

For $H = [Hess_f(1,1)]$, the same $\mathbf{u}_1$ as above is a unit eigenvector for $H$ corresponding to the eigenvalue $-8$, and $\mathbf{u}_2$ is a unit eigenvector for $H$ corresponding to the eigenvalue $-16$. Then letting $u = (\mathbf{x}-(1,1))\bullet\mathbf{u}_1 = (x+y-2)/\sqrt{2}$ and $v = (\mathbf{x}-(1,1))\bullet\mathbf{u}_2 = (x-y)/\sqrt{2}$,

$$f(x,y) \approx_2 2 + \frac{1}{2}(-16u^2 - 8v^2) \text{ as } (x,y) \to (1,1).$$

The level curves of $f$ near the origin are approximately the ellipses $16u^2+8v^2 = K$, whose semimajor axes are on the $v$-axis and semiminor axes are on the $u$-axis. Here the $u$-axis is defined by $v = 0$, i.e., $y = x$, and the $v$-axis is defined by $u = 0$, i.e., $y = 2 - x$.

Since $f$ is unchanged by the transformation $x \to -x$, $y \to -y$, the contour map of $f$ is symmetric with respect to the origin. So the approximate picture at $(-1,-1)$ is a mirror image (with respect to the origin) of the approximate picture at $(1,1)$.

**6.16.** (a) Same as in 6.14, except now $(1,1)$ and $(-1,-1)$ are local minima, and the Hessian is the negative of what it was in 6.14. (So the eigenvectors of the Hessian are the same $\mathbf{u}_1$ and $\mathbf{u}_2$ as there, but now the eigenvalues are 16 and 8.)

(b) The largest and smallest possible values of the second directional derivative are the eigenvalues of $[Hess_f(1,1)]$ and they arise from the directions $\mathbf{u}$ which are the corresponding unit eigenvectors of the Hessian. Let $\mathbf{u}_1 = (1/\sqrt{2})(1,1)$, an eigenvector for the eigenvalue 8. This choice of the unit vector $\mathbf{u}$ makes the second derivative as small as possible, namely equal to 8. Similarly the direction $\mathbf{u}_2 = (1/\sqrt{2})(1,-1)$ makes the second derivative as large as possible, namely equal to 16.

(c) The contours look exactly the same as in 6.14, but the *values* of $f$ associated with each contour are the negatives of the corresponding values in 6.14.

**6.18.** Neither.

$$[Hess_f] = \begin{bmatrix} 6xyz^2 & 3x^2z^2+4 & 6x^2yz \\ 3x^2z^2+4 & 0 & 2x^3z-3 \\ 6x^2yz & 2x^3z-3 & 2x^3y \end{bmatrix}, \text{ so } [Hess_f(1,1,1) = \begin{bmatrix} 6 & 7 & 6 \\ 7 & 0 & -1 \\ 6 & -1 & 2 \end{bmatrix}.$$

The determinants of the upper left $1\times 1$, $2\times 2$, and $3\times 3$ matrices are $6, -49, -176$. The sequence of signs is thus $+--$. Since the sequence is neither $+++$ nor $-+-$, the answer is "neither."