**5.2.** (a) $\nabla f = (2xy + y - y^2, x^2 + x - 2xy)$. Critical points are solutions of the pair of equations

$$2xy + y - y^2 = y(2x + 1 - y) = 0,\ x^2 + x - 2xy = x(x + 1 - 2y) = 0.$$

There are four critical points:

$y = 0, x = 0$ gives the point $(0, 0)$;

$y = 0, x + 1 - 2y = 0$ gives $(-1, 0)$;

$y = 2x + 1, x = 0$ gives $(0, 1)$;

$y = 2x + 1,\ x = 2y - 1$ gives $(1/3, -1/3)$.

(b) The level curve of $f$ through $(1, 1)$ is perpendicular to $\nabla f(1, 1) = (2, 0)$ and has equation

$$\nabla f(1, 1) \bullet (\mathbf{x} - (1, 1)) = 0, \quad \textit{or} \quad 2(x - 1) = 0, \quad \textit{or} \quad x = 1.$$

(c) The first could *not* be a contour plot of $f$ as it shows a crossing at $(1, 1)$. Gradients are $\mathbf{0}$ at points of crossing but $\nabla f(1, 1) \neq \mathbf{0}$.

The second at least has the right tangent line at $(1, 1)$.

**5.4.** (a) $\nabla f = (1 + x^2 + y^2)^{-3}(y(1 + y^2 - 3x^2), x(1 + x^2 - 3y^2))$. The five critical points are $(x, y) = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, and $(0, 0)$.

$f(0, 0) = 0$; $f(1/\sqrt{2}, 1/\sqrt{2}) = f(-1/\sqrt{2}, -1/\sqrt{2}) = 1/8$; and the value of $f$ at the other two critical points is $-1/8$.

(b) The center one. The simplest explanation is that $f(x, y)$ doesn't change when $x$ and $y$ are interchanged, so the contour plot doesn't change upon reflection in the line $y = x$. Another reason is the apparent location of critical points.

**5.6.** $g(x, y) = x^2 + y^2$. Lagrange's method gives the equations

$$\begin{aligned} \det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \det \begin{bmatrix} 4(x+y)^3 + 2(x-y) & 4(x+y)^3 - 2(x-y) \\ 2x & 2y \end{bmatrix} \\ = 8(x+y)^3(y-x) + 2(x-y)(y+x) = 0 \\ x^2 + y^2 = 1 \end{aligned}$$

If $y = x$ then $x = y = \pm 1/\sqrt{2}$, where $f(x, y) = (2/\sqrt{2})^4 = 4$.

If $y \neq x$ then we can divide by $y - x$ and 2 to get

$$4(x + y)^3 - (x + y) = 0$$

If $y = -x$ then $y = -x = \pm 1/\sqrt{2}$, where $f(x, y) = (2/\sqrt{2})^2 = 2$.

If $y \neq -x$ we can divide further by $x + y$, to get

$$4(x + y)^2 - 1 = 0.$$

Then $x + y = \pm 1/2$, $x^2 + ((\pm 1/2) - x)^2 = 1$, $2x^2 \pm x - (3/4) = 0$, $x = (\pm 1 \pm \sqrt{7})/4$, $y = (\pm 1 \mp \sqrt{7})/4$, $x - y = \pm\sqrt{7}/2$, $f(x, y) = (1/16) + (7/4) = 29/16$.

The maximizers are $\pm(1/\sqrt{2}, 1/\sqrt{2})$ (maximum value 4). The minimizers are $\pm((1 \pm \sqrt{7})/4, (1 \mp \sqrt{7})/4)$ (minimum value 29/16).

**5.8.** Since the constraint region is compact, maximizers and minimizers exist.

Four of the five critical points (see problem 5.4) are $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. They satisfy $|x|+|y| = 2/\sqrt{2} = \sqrt{2} > 1$ so they are not in the constraint region. The fifth critical point, $(0, 0)$, is in the running with $f(0, 0) = 0$.

Let's find the maximizers in the first quadrant of the constraint set, which is the triangle bordered by the positive coordinate axes and the line $x + y = 1$. On the axes (including the triangle's vertices), $f(x, y) = 0$. On the line $x + y = 1$, Lagrange's method gives

$$\begin{aligned}\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ 1 & 1 \end{bmatrix} &= \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \\ &= (1 + x^2 + y^2)^{-3}(y(1 + y^2 - 3x^2) - x(1 + x^2 - 3y^2)) = 0.\end{aligned}$$

Thus $y(1 + y^2 - 3x^2) - x(1 + x^2 - 3y^2) = 0$, $y^3 + 3xy^2 - 3x^2y - x^3 + y - x = 0$,

$$(y - x)(x^2 + 4xy + y^2 + 1) = 0.$$

If $y - x = 0$ then $y = x$ and on the line $x + y = 1$, we get the point $(1/2, 1/2)$, a candidate. Similarly we get one point in each quadrant for a total of four: $(\pm 1/2, \pm 1/2)$.

Otherwise $x^2+4xy+y^2+1 = 0$ and substituting $y = 1-x$, $x^2+4x-4x^2+1-2x+x^2+1 = -2x^2+2x+2 = 0$, $x = (1 \pm \sqrt{5})/2$. These values of $x$ are not in the interval $0 \leq x \leq 1$, so there are no more candidates on the "northeastern" line segment. Similarly there are no more on any of the line segments.

Checking the five candidates, we see that the maximizers are $(1/2, 1/2)$ and $(-1/2, -1/2)$, where $f$ has the maximum value 1/9, and the minimizers are $(1/2, -1/2)$ and $(-1/2, 1/2)$, where $f$ has the minimum value $-1/9$.

**5.10.** The constraint set is a closed "inverted bowl". In particular it's compact so there exist maximizers and minimizers. The Lagrange method applied to the hemispherical $x^2 + y^2 + z^2 = 1$ will give all possible maximizers and minimizers except possibly on the circular rim $x^2 + y^2 = 1$, $z = 0$, where a separate analysis should be used. The separate analysis is easy, since $xyz = 0$ on the circular rim. However, at suitable points of the bowl, $xyz$ will positive, and at other points, $xyz$ will be negative, so we can ignore the circular rim as well as any point where $xyz = 0$.

On the hemisphere we use Lagrange multipliers, and solve

$$\nabla(xyz) = \lambda \nabla(x^2 + y^2 + z^2) \text{ and } x^2 + y^2 + z^2 = 1,\ z > 0.$$

This yields

$$\begin{aligned} yz &= 2\lambda x \\ xz &= 2\lambda y \\ xy &= 2\lambda z \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

Since we can ignore points where $x$, $y$, or $z$ is 0, we can divide by $x$, $y$, and $z$ if we wish. This gives $yz/x = 2\lambda = xz/y = xy/z$. Therefore $y/x = x/y$ and $z/y = y/z$. So $x^2 = y^2 = z^2$. Substituting in the constraint equation, $x^2 = y^2 = z^2 = 1/3$, $z = 1/\sqrt{3}$. There are four candidates: $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, 1/\sqrt{3})$. The maximizers are $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and $(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, where $xyz = 1/3\sqrt{3}$. The other two candidates are minimizers, with $xyz = -1/3\sqrt{3}$.

**5.12.** Both curves, $2x - x^2 = y$ and $(x-1)^2 + y^2 = 1$, pass through $(0,0)$, $(1,1)$, and $(2,0)$. The parabola $y = 2x - x^2$, opening downward, lies below the upper semicircle $y = \sqrt{1 - (x-1)^2} = \sqrt{2x - x^2}$ for $0 \le x \le 2$, since $0 \le 2x - x^2 \le 1$ on that interval.

Therefore the constraint region has upper boundary $y + x^2 - 2x = 0$ and lower boundary $y^2 + x^2 - 2x = 0$, $y \le 0$. The end points, where the two pieces of boundary meet, are $(0,0)$ and $(2,0)$.

A. Critical points of $f$ in the interior: $\nabla f = (2x, 2)$ is never $\mathbf{0}$.

B. Lagrange method on the upper boundary $y + x^2 - 2x = 0$:

$$\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \det \begin{bmatrix} 2x & 2 \\ 2x - 2 & 1 \end{bmatrix} = -2x + 4 = 0,$$

$x = 2$, $y = 0$.

C. Lagrange method on the lower boundary $y^2 + x^2 - 2x = 0$, $y \le 0$:

$$\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \det \begin{bmatrix} 2x & 2 \\ 2x - 2 & 2y \end{bmatrix} = 4xy - 4x + 4 = 0, \quad y = 1 - (1/x).$$

Since $y \le 0$, $x \le 1$. Substituting in the constraint equation, $(1 - (1/x))^2 + x^2 - 2x = 0$. Let $u = x + (1/x)$. Then $u^2 - 2u - 1 = 0$, $u = 1 \pm \sqrt{2}$. Since $x \ge 0$, also $u \ge 0$, so $u = 1 + \sqrt{2}$.

Then $x + (1/x) = u$, $x^2 - ux + 1 = 0$, $x = (u \pm \sqrt{u^2 - 4})/2 = 0.531\cdots$ or $1.883\cdots$. But we know that $x \le 1$ so only the first of these roots is relevant. Then $y = 1 - (1/x) = -0.883\cdots$ and $f(x,y) = -1.484\cdots$.

D. Endpoints $(0,0)$ ($f(0,0) = 0$) and $(2,0)$ ($f(2,0) = 4$).

E. Roundup. The maximizer is $(2,0)$, where $f(2,0) = 4$, and the minimizer is $(0.531\cdots, -0.883\cdots)$, where $f(x,y) = -1.484\cdots$.

**5.14.** See the text where this is now done.

**5.16.** The constraint curve is an ellipse. Let $f(x, y) =$ the distance from $(x, y)$ to the line $y = 3 - 2x$. *The level curves of $f$ are the lines parallel to $y = 3-2x$.* Reasoning geometrically, the maximizer(s) and minimizer(s) will be points on the given ellipse where the tangent line is parallel to $y = 3 - 2x$. That is, they will be points where the tangent line has slope $-2$.

So our equations are: $y' = -2$ and the ellipse equation. Differentiating the ellipse equation $x^2+y^2+xy = 1$ gives $2x + 2yy' + xy' + y = 0$. Substituting $y' = -2$, $2x - 2x - 4y + y = 0$, $y = 0$. Therefore $x = \pm 1$, and we have the two points $(\pm 1, 0)$

The constraint set is compact so maximizers and minimizers exist, and we therefore must have found them. From a sketch, $(1, 0)$ is closer to the line $y = 3 - 2x$ than $(-1, 0)$ is, so $(1, 0)$ is the closest and $(-1, 0)$ the furthest point.

(Note. The above solution depends on knowing that the line $y = 3 - 2x$ does not meet the ellipse. This is because substituting $y = 3 - 2x$ in the ellipse equation gives

$$x^2 + x(3 - 2x) + (3 - 2x)^2 = 0$$

or $3x^2 - 9x + 9 = 0$, an equation with no real solutions.)

**5.18.** This can be solved by elementary solid geometry, or by the Lagrange method. The geometric solution, briefly, is to draw the line connecting $(1, 2, 3)$ with the center $(0, 0, 0)$ of the sphere; the opposite ends of that diameter of the sphere give the closest and furthest points.

Solution by Lagrange method: We wish to find the maximizer and minimizer of the square distance

$$f(x, y, z) = (x - 1)^2 + (y - 2)^2 + (z - 3)^2$$

subject to the constraint

$$g(x, y, z) = x^2 + y^2 + z^2 = 1.$$

The constraint set is compact so maximizer and minimizer exist. Using Lagrange multipliers gives $\nabla f = \lambda \nabla g$, so we have to solve

$$\begin{aligned} 2(x - 1) &= \lambda \cdot 2x \\ 2(y - 2) &= \lambda \cdot 2y \\ 2(z - 3) &= \lambda \cdot 2z \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

If $x = 0$, then the first equation gives $2(x - 1) = 0$, $x = 1$, contradiction. Therefore $x \neq 0$. Similarly $y \neq 0$ and $z \neq 0$. The first three equations then give

$$\lambda = \frac{x - 1}{x} = \frac{y - 2}{y} = \frac{z - 3}{z}.$$

Therefore

$$1 - \frac{1}{x} = 1 - \frac{2}{y} = 1 - \frac{3}{z}, \text{ so } \frac{1}{x} = \frac{2}{y} = \frac{3}{z}.$$

So $z = 3x$, $y = 2x$, and the constraint equation gives $14x^2 = 1$, $x = \pm 1/\sqrt{14}$.

The minimizer is $-(1/\sqrt{14})(1, 2, 3)$, and the maximizer is $(1/\sqrt{14})(1, 2, 3)$.