Practice Final Exam, Math 291 Fall 2017

December 20, 2017

1: (a) Let $\mathbf{a} = (-1, 1, 2)$ and $\mathbf{b} = (2, -1, 1)$. Find all vectors $\mathbf{x}$, if any exist, such that

$$\mathbf{a} \times \mathbf{x} = (-2, 4, -3) \qquad \text{and} \qquad \mathbf{b} \cdot \mathbf{x} = 2 \ .$$

If none exist, explain why this is the case.

(b) Among all vectors $\mathbf{x}$ such that $(-1, 1, 2) \times \mathbf{x} = (-2, 4, -3)$, find the one that is closest to $(1, 1, 1)$.

SOLUTION (a) Since $\mathbf{a} = (-1, 1, 2)$ is orthogonal to $(-2, 4, -3)$, there is a solution, and one solution is given by

$$\mathbf{x}_0 := -\frac{1}{\|\mathbf{a}\|^2}\mathbf{a} \times (-2, 4, -3) = \frac{1}{6}(11, 7, 2) \ .$$

Then the set of all solutions of $\mathbf{a} \times \mathbf{x} = (-2, 4, -3)$ is

$$\mathbf{x}_0 + t\mathbf{a}$$

for all $t \in \mathbb{R}$. Taking the dot product with $\mathbf{b}$, we find $(\mathbf{x}_0 + t\mathbf{a}) \cdot \mathbf{b} = \dfrac{17}{6} - t$. Setting this equalt to 1, we find $t = \frac{5}{6}$, and so the unique solution is

$$\mathbf{x}_0 + \frac{5}{6}\mathbf{a} = (1, 2, 2) \ .$$

As you can check this does satsify both equations.

For **(b)** we choose t to minimize $\|(1, 1, 1) - (\mathbf{x}_0 + t\mathbf{a})\|^2$ and setting the derivative in t to be zero

$$0 = ((1, 1, 1) - \mathbf{x}_0 - t\mathbf{a}) \cdot \mathbf{a} = \frac{1}{6}(-5, -1, 4) \cdot (-1, 1, 2) - t6 = 2 - t6 \ .$$

Hence $t = 1/3$. The closest point is

$$\mathbf{x}_0 + \frac{1}{3}\mathbf{a} = \frac{1}{6}(11, 7, 2) + \frac{1}{6}(-2, 2, 4) = \frac{1}{2}(3, 3, 2) \ .$$

This is the closest point. Notice that $\dfrac{1}{2}(3, 3, 2) - (1, 1, 1) = \dfrac{1}{2}(1, 1, 0)$ is orthogonal to $\mathbf{a}$, as it must be.

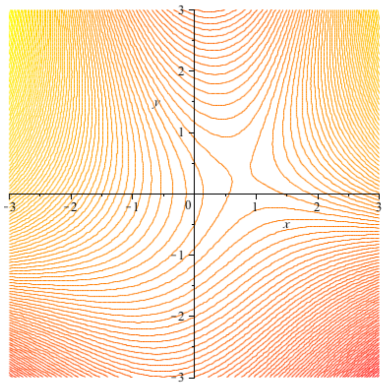
2: Let $f(x, y) = xy^2 - xy$.

(a) Find all of points at which the tangent plane to the graph of f is horizontal.

(b) Find the equation of the tangent plane to the graph of f at $(3/2, 1/3)$.

(c) Find the equation of the tangent line to the contour curve of f through the point $(3/2, 1/3)$.

(d) Could the following be a contour plot for f? Explain your answer to receive credit.



SOLUTION (a) We compute $\nabla f(x, y) = (y^2 - y, x(2y - 1))$. The tangent plane is horizontal at critical points. We get these by solving

$$\begin{aligned} y^2 &= y \\ x(2y-1) &= 0 \end{aligned}$$

From he first equation, we must have $y = 0$ or $y = 1$. If $y = 0$, the second gives $x = 0$. If $y = 1$, the second asl gives $x = 0$. So the two critical points are $\mathbf{x}_1 = (0, 0)$ and $\mathbf{x}_2 = (0, 1)$.

(b) $\nabla f(3/2, 1/3) = -(2/9, 1/2)$ and $f(3/2, 1/3) = -1/3$. Therefore the equation for the tangent plane is

$$z = -\frac{1}{3} - (2/9, 1/2) \cdot (x - 3/2, y - 1/3) = -\frac{2}{9}x - \frac{1}{2}y + \frac{1}{6} .$$

(c) The line is normal to $\nabla f(3/2, 1/3) = -(2/9, 1/2)$, so that equation is $(x - 3/2, y - 1/3) \cdot (2/9, 1/2) = 0$ which simplifies to

$$\frac{2}{9}x + \frac{1}{2}y = \frac{1}{2} .$$

(d) No. It shows a critical point (a saddle) above the x-axis, and not critical points on the x-axis.

3: Let $f(x, y)$ be a differentiable function on $\mathbb{R}^2$ such that $f(0, 0) = 0$. Define a function $g(x, y)$ by

$$g(x, y) = \begin{cases} \dfrac{f(x, y)}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) \neq (0, 0) . \end{cases}$$

Suppose that f is continuously differentiable. Is it then necessarily the case that g is continuous? Justify your answer to receive credit.

SOLUTION If f is continuously differentiable, then with $\mathbf{a} := \nabla f(0, 0)$,

$$f(x, y) \approx \mathbf{a} \cdot (x, y) ,$$

and for all (x, y),

$$\lim_{t \to 0} \frac{f(tx, ty)}{t} \mathbf{a} \cdot (x, y) .$$

Then, for $\mathbf{a} \neq \mathbf{0}$,

$$\lim_{t\to 0} g(t\mathbf{a}_\perp) = 0 \qquad \text{and} \qquad \lim_{t\to 0} g(t\mathbf{a}) = 1 \ .$$

Hence, when $\mathbf{a} \neq \mathbf{0}$, g need not be continuous. For a specific counterexample, take $f(x, y) = x$. (However, it is true that under the additional assumption that $\nabla f(0, 0) = \mathbf{0}$, g is continuous.)

4: Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = (t, t^2/2, t^3/3)$.

(a) Compute the curvature $\kappa(t)$ as a function of t, and show that $\lim_{t\to\infty} \kappa(t) = 0$.

(b) Compute the angle $\theta(t)$ between $\mathbf{T}(t)$ and $\mathbf{a}(t)$ as a function of t, and show that $\lim_{t\to\infty} \theta(t) = 0$. Comment on the relation between this limit, and the limit in part **(a)**.

(c) Find the equation of the osculating plane at $t = 1$.

(d) Compute the distance from the origin to the osculating plane at $t = 1$.

SOLUTION (a) We compute $\mathbf{v}(t) = (1, t, t^2)$ and $\mathbf{a}(t) = (0, 1, 2t)$. Then $v(t) = \sqrt{1 + t^2 + t^4}$ and

$$\mathbf{T}(t) := \frac{1}{\sqrt{1 + t^2 + t^4}}(1, t, t^2) \ .$$

It follows that

$$\mathbf{a}_\perp(t) = \frac{1}{1 + t^2 + t^4}(-2t^3 - 1, 1 - t^4, t^3 + 2t) \quad \text{and} \quad \|\mathbf{a}_\perp(t)\| = \frac{\sqrt{1 + 4t^2 + t^4}}{\sqrt{1 + t^2 + t^4}} \ .$$

Then since $v^2(t)\kappa(t)\|\mathbf{a}_\perp(t)\|$,

$$\kappa(t) = \frac{\sqrt{1 + 4t^2 + t^4}}{(1 + t^2 + t^4)^{3/2}} \ .$$

Since $\lim_{t\to\infty} \|\mathbf{a}_\perp(t)\| = 1$ and since $\lim_{t\to\infty} v^2(t) = \infty$, it is evident that $\lim_{t\to\infty} \kappa(t) = 0$.

(b) We compute

$$\frac{\mathbf{a}(t) \cdot \mathbf{v}(t)}{\|\mathbf{a}(t)\|\|\mathbf{v}(t)\|} = \frac{2t^3 + t}{\sqrt{4t^2 + 1}\sqrt{t^4 + t^2 + 1}} \ .$$

Therefore,

$$\lim_{t\to\infty} \frac{\mathbf{a}(t) \cdot \mathbf{v}(t)}{\|\mathbf{a}(t)\|\|\mathbf{v}(t)\|} = 1 \ ,$$

and $\frac{\mathbf{a}(t)\cdot\mathbf{v}(t)}{\|\mathbf{a}(t)\|\|\mathbf{v}(t)\|} = \cos(\theta(t))$. Therefore, $\lim_{t\to\infty} \theta(t) = 0$. For large t, the acceleration lines up with the direction of motion, so the curvature goes to zero.

(c) We compute $\mathbf{v}(t) \times \mathbf{a}(t) = (t^2, -2t, 1)$, and so $\mathbf{v}(1) \times \mathbf{a}(1) = (1, -2, 1)$ and $\mathbf{x}(1) = (1, 1/2, 1/3)$. The equation is $(x - 1, y - 1/2, z - 1/3) \cdot (1, -2, 1) = 0$, which simplifies to

$$x - 2y + z = \frac{1}{3} \ .$$

(d) The distance is

$$\frac{1}{\sqrt{6}}|(1, 1/2, 1/3) \cdot (1, -2, 1)| = \frac{1}{3\sqrt{6}} \ .$$

5: Let $f(x, y) = \dfrac{xy}{(1 + x^2 + y^2)^2}$.

(a) Find all of the critical points of f, and for each of them, determine whether it is a local minimum, a local maximum, a saddle point, or if it cannot be classified through a computation of the Hessian.

(b) There is one critical point of f in the interior of the upper right quadrant. Let $\mathbf{x}_0 = (x_0, y_0)$ denote this critical point. Let $\mathbf{u} = (u, v)$ be a unit vector, and consider the directional second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(x_0 + tu, y_0 + tv)\bigg|_{t=0} .$$

Which choices of the unit vector (u, v) makes this as large as possible? What is the largest possible value? Also, which choices of the unit vector (u, v) makes this as small as possible, and what is the smallest possible value?

(c) Sketch a contour plot of f near (x_0, y_0).

(d) Does f have minimum and maximum values? If so, say what they are, and what the maximizers and minimizes are.

SOLUTION (a) We compute

$$\nabla f(x,y) = \frac{1}{(1+x^2+y^2)^3}(y(1-3x^2+y^2), x(1-3y^2+x^2)) .$$

The critical points (x, y) solve

$$\begin{aligned} y(1-3x^2+y^2) &= 0 \\ x(1-3y^2+x^2) &= 0 \end{aligned}$$

If $y = 0$, the second equation reduces to $x(1 + x^2) = 0$, so $x = 0$. Likewise, if $x = 0$, we conclude $y = 0$. Hence $\mathbf{z}_1 = (0, 0)$ is a critical point, and any other critical points satisfy

$$\begin{aligned} 1-3x^2+y^2 &= 0 \\ 1-3y^2+x^2 &= 0 \end{aligned}$$

Adding 3 times the second equation to the first, $4 - 8y^2 = 0$, or $y^2 = 1/2$. Hence $y = \pm 2^{-1/2}$. and then we find from the first equation that $3x^2 = 3/2$, so $x = \pm 2^{-1/2}$. This gives us four more critical points

$$\mathbf{z}_2 = (2^{-1/2}, 2^{-1/2}) , \quad \mathbf{z}_3 = (-2^{-1/2}, -2^{-1/2}) , \quad \mathbf{z}_4 = (-2^{-1/2}, 2^{-1/2}) , \quad \mathbf{z}_5 = (2^{-1/2}, -2^{-1/2}) .$$

We next compute the Hesisan:

$$\mathrm{Hess}_f(x,y)] = \frac{1}{(x^2+y^2+1)^4}\begin{bmatrix} 12xy(x^2-y^2-1) & g(x,y) \\ g(x,y) & 12xy(y^2-x^2-1) \end{bmatrix} ,$$

where

$$g(x,y) := -3(x^4+y^4) + 18x^2y^2 - 2(x^2+y^2) + 1 .$$

Then

$$\mathrm{Hess}_f(\mathbf{z}_1)] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\text{Hess}_f(\mathbf{z}_2)] = \text{Hess}_f(\mathbf{z}_3)] = \frac{1}{8}\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad \text{Hess}_f(\mathbf{z}_4)] = \text{Hess}_f(\mathbf{z}_5)] = \frac{1}{8}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} .$$

By Sylvester's criterion, $\mathbf{z}_1$ is a saddle, $\mathbf{z}_2$ and $\mathbf{z}_3$ are local maxima, and $\mathbf{z}_4$ and $\mathbf{z}_5$ are local minima.

(d) Yes, since $\lim_{\|bx\|\to\infty} f(\mathbf{x}) = 0$, and the minima and maxima must be critical points. Hence $\mathbf{z}_2$ and $\mathbf{z}_3$ are maxima and $\mathbf{z}_4$ and $\mathbf{z}_5$ are minima. Hence

(b) The one critical point is $\mathbf{z}_2$, and so $\mathbf{x}_0 = \mathbf{z}_2$. $\text{Hess}_f(\mathbf{x}_0)] = \frac{1}{8}\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$. This matrix is double symmetric and so its eigenvalues are $\mu_1 = -2$ and $\mu_2 = -4$. The corresponding eigenvectors are

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}(1,1) \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}}(1,-1) .$$

The maximizers are $\pm\mathbf{u}_1$ and the minimizers are $\pm\mathbf{u}_2$.

(c) In the usual u, v coordinates, near $\mathbf{x}_0$ the contour curves are level curves are close to the level curves of

$$-2u^2 - 4v^2 .$$

these are ellipses with the major axis along the u-axis, and the major axis should be $\sqrt{2}$ times as long as the minor axis. The u-axis runs along the line through $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{u}_1$, which asses through $\mathbf{x}_0$ and makes an angle of $\pi/4$ with respect to the x-axis. You sketch must clearly show the aspect ratio.

6: Let $f(x,y) = xy$. Let D denote the region in the plane consisting of all of the points (x,y) such that

$$x^2 + 4y^2 \leq 6 .$$

Find the minimum and maximum values of f in D. Also, find all of the minimizers and maximizers in D.

SOLUTION We compute $\nabla f(x,y) = (y,x)$, and so the only critical point is $(0,0)$. Next,

$$\det\left(\begin{bmatrix} y & x \\ 2x & 2y \end{bmatrix}\right) = 2(y^2 - x^2)$$

so Lagrange's equations yield

$$\begin{aligned} y^2 &= x^2 \\ x^2 + 4y^2 &= 6 . \end{aligned}$$

Hence $5x^2 = 6$, so that $x \pm \sqrt{6/5}$ and then $y = \pm\sqrt{6/5}$. The maximizers are

$$(\sqrt{6/5}, \sqrt{6/5}) \quad \text{and} \quad -(\sqrt{6/5}, \sqrt{6/5})$$

and the minimizers are

$$(\sqrt{6/5}, -\sqrt{6/5}) \quad \text{and} \quad (-\sqrt{6/5}, \sqrt{6/5}) .$$

The maximum and minimum values are $6/5$ and $-6/5$ respectively.

7: (a) Let D be the set in the positive quadrant of $\mathbb{R}^2$ that bounded by

$$\begin{aligned} y &= x \\ y &= \sqrt{3}x \\ y &= x^2 + y^2 \end{aligned}$$

Let $f(x,y) = \sqrt{1 + x^2 + y^2}$. Compute $\int_D f(x,y)\mathrm{d}A$.
(b) Let D be the set in $\mathbb{R}^2$ that is given by

$$1 \le \frac{y}{x^2} \le 2 \qquad \text{and} \qquad 1 \le \frac{x}{y^2} \le 2\ .$$

Let $f(x,y) = \dfrac{1}{x^2y^2}$. Compute $\int_D f(x,y)\mathrm{d}A$.

SOLUTION (a) In polar coordinates, $y = y^2 + x^2$ is $r \sin\theta = r^2$, or $r = \sin\theta$, the circle of radius $1/2$ centered at $(1/2, 0)$. In polar coordinates, $y = x$ is $\theta = \pi/4$, and in polar coordinates, $y = \sqrt{3}x$ is $\theta = \pi/3$. Hence the region is described by

$$0 \le r \le \sin\theta \quad \text{and} \quad \pi/4 \le \theta \le \pi/3\ .$$

Then

$$\int_D f(x,y)\mathrm{d}A = \int_{\pi/4}^{\pi/3} \left(\int_0^{\sin\theta} r\sqrt{1+r^2}\mathrm{d}r \right) \mathrm{d}\theta = \frac{2}{3} \int_{\pi/4}^{\pi/3} \left((1 + \sin^2\theta)^{3/2} - 1 \right) \mathrm{d}\theta$$

(b) Note that both x and y are positive, so the set lies in the upper-right quadrant. Define

$$u(x,y) = y/x^2 \quad \text{and} \quad v(x,y) = x/y^2\ .$$

Then $x = v^{1/3}u^{-2/3}$ and $y = u^{1/3}v^{-2/3}$. Hence

$$|\det([D_{\mathbf{x}}(\mathbf{u})])| = \frac{1}{3}u^{-4/3}v^{-4/3}\ .$$

Next

$$f(x,y) = \frac{1}{x^2y^2} = u^{-2/3}v^{-2/3}\ .$$

Hence

$$\int_D f(x,y)\mathrm{d}A = \left(\int_1^2 u^{-2}\mathrm{d}u \right) \left(\int_1^2 v^{-2}\mathrm{d}v \right) = \frac{1}{4}\ .$$

8: Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that is bounded by the surfaces

$$\begin{aligned} \sqrt{x^2+y^2} &= z^3 \\ \sqrt{x^2+y^2} &= 10 - z \end{aligned}$$

Compute the volume of $\mathcal{V}$ **and** the total surface area of its boundary. (There are two pieces to the boundary.
(a) Compute the $\int_{\mathcal{V}}(x^2+y^2)\mathrm{d}V$.

(b) Compute the total surface area of the boundary of $\mathcal{V}$.

SOLUTION (a) In cylindrical coordinates, we have $r = z^3$ and $r = 10 - z$. Where these surfaces meet, $z^3 + z = 10$. One obvious solution is $z = 2$ and $z^3 + z - 10 = (z-2)(z^2 + 2z + 5)$ so there are no other real roots. Hence the description of $\mathcal{V}$ in cylindrical coordinates is given by

$$r^{1/3} \leq z \leq 10 - r \ , \quad 0 \leq \theta \leq 2\pi \qquad \text{and} \qquad 0 \leq r \leq 8 \ .$$

(If you choose to integrate in r before z, you will need two regions, one for $0 \leq z \leq 2$ and one for $2 \leq z \leq 8$. This is obviously not the way to go.)

Therefore,

$$\int_{\mathcal{V}} (x^2 + y^2)\mathrm{d}V = 2\pi \int_0^8 r^3 \left(\int_{r^{1/3}}^{10-r} \mathrm{d}z \right) \mathrm{d}r = 2\pi \int_0^8 r^3 \left(10 - r - r^{1/3} \right) \mathrm{d}r \ .$$

(b) When computing the surface area, we will use cylindrical coordinates, and must break the integration up into the two surfaces anyhow. We may as well use r and θ as the parameters, eliminating z.

The lower surface is parameterized by

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, r^{1/3}) \ , \quad 0 \leq r \leq 8 \ , \quad 0 \leq \theta \leq 2\pi \ .$$

we compute

$$\mathbf{X}_r(r,\theta) = (\cos\theta, \sin\theta, \tfrac{1}{3}r^{-2/3}) \qquad \text{and} \qquad \mathbf{X}_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 0) \ .$$

Then

$$\|\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta)\| = \frac{r^{1/3}}{3}\sqrt{1 + 9r^{2/3}} \ .$$

The upper surface is parameterized by

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 10 - r) \ , \quad 0 \leq r \leq 8 \ , \quad 0 \leq \theta \leq 2\pi \ .$$

we compute

$$\mathbf{X}_r(r,\theta) = (\cos\theta, \sin\theta, -1) \qquad \text{and} \qquad \mathbf{X}_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 0) \ .$$

Then

$$\|\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta)\| = \sqrt{2}r \ .$$

Hence the total surface area is

$$2\pi \int_0^8 \left(\frac{r^{1/3}}{3}\sqrt{1 + 9r^{2/3}} + \sqrt{2}r \right) \mathrm{d}r \ .$$

9: Consider the two vector fields

$$\mathbf{F} = (yz, xz, xy) \qquad \text{and} \qquad \mathbf{G} = (z^2 + 2xy, x^2 - 2yz, 2xz - y^2) \ .$$

(a) Compute the divergence and curl of $\mathbf{F}$ and $\mathbf{G}$.

(b) One of the vector fields $\mathbf{F}$ and $\mathbf{G}$ is equal to $\nabla\varphi$ for some potential function φ. Which one is it? Find such a potential function.

(c) One of the vector fields $\mathbf{F}$ and $\mathbf{G}$ is equal to curl$\mathbf{A}$ for some vector potential $\mathbf{A}$. Which one is it? Find such a vector potential.

(d) Let $\mathcal{S}$ be the part of the ellipsoid $x^2 + y^2 + \frac{1}{4}z^2 = \frac{5}{4}$ that lies above the plane $z = 1$ with $\mathbf{N}$ pointing upwards. Compute

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \quad \text{and} \quad \int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S \ .$$

(e) Let C be the curve that is the intersection of the graph of $z = 1 - x^2$ with the cylinder $x^2 + y^2 = 1$, oriented so that it runs clockwise when viewed from above. Compute

$$\int_{C} \mathbf{F}\cdot\mathbf{T}\mathrm{d}s \quad \text{and} \quad \int_{C} \mathbf{G}\cdot\mathbf{T}\mathrm{d}s \ .$$

SOLUTION (a) We compute

$$\mathrm{curl}\mathbf{F} = \mathbf{0} \qquad \text{and} \qquad \mathrm{curl}\mathbf{G} = 0 \ ,$$

and

$$\mathrm{div}\mathbf{F} = \mathbf{0} \qquad \text{and} \qquad \mathrm{div}\mathbf{G} = 2(x+y-z) \ .$$

(b) Since both curl$\mathbf{F} = \mathbf{0}$ and curl$\mathbf{G} = \mathbf{0}$, both vector fields are gradients. We find

$$\mathbf{F} = \nabla\varphi \quad \text{where } \varphi(x,y,z) := xyz \quad \text{and} \qquad \mathbf{G} = \nabla\varphi \quad \text{where } \varphi(x,y,z) := x^2y - y^2z + z^2x \ .$$

(c) Since div$\mathbf{F} = 0$, but div$\mathbf{G} \neq 0$, it is $\mathbf{F}$. We find $\mathbf{F} = \mathrm{curl}\mathbf{A}$ where

$$\mathbf{A} = \frac{1}{2}(z^2x, x^2y, y^2z) \ .$$

(d) Since div$\mathbf{F} = 0$, we may replace $\mathcal{S}$ by $\widetilde{\mathcal{S}}$ where $\widetilde{\mathcal{S}}$ is the circle in the plane $z = 1$ with $x^2+y^2 = 1$. Then since $\mathbf{N} = (0,0,1)$. $\mathbf{F}\cdot\mathbf{N} = xy$ and by symmetry,

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = \int_{\widetilde{\mathcal{S}}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = 0 \ .$$

Next, let $\mathcal{V}$ be the region that lies the ellipsoid $x^2 + y^2 + \frac{1}{4}z^2 = \frac{5}{4}$ and above the plane $z = 1$. By symmetry in x and y,

$$\int_{\mathcal{V}} \mathrm{div}\mathbf{G}\mathrm{d}V = \int_{\mathcal{V}} (-z)\mathrm{d}V \ .$$

The region $\mathcal{V}$ is specified in cylindrical coordinates by

$$0 \le r \le 1 \ , \quad 0 \le \theta \le 2\pi \ , \quad 1 \le z \le \sqrt{5} \ .$$

Hence

$$\int_{\mathcal{V}} (-z)\mathrm{d}V = -2\pi\left(\int_1^{\sqrt{5}} z\mathrm{d}z\right)\left(\int_0^1 r\mathrm{d}r\right) = -\frac{5\pi}{2} \ .$$

By the Divergence Theorem,

$$\int_{\mathcal{V}} \mathrm{div}\mathbf{G}\mathrm{d}V = \int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S - \int_{\widetilde{\mathcal{S}}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S$$

with $\widetilde{\mathcal{S}}$ given as above. On $\widetilde{\mathcal{S}}$, $\mathbf{G}\cdot\mathbf{N} = 2x - y^2$. Hence

$$\int_{\widetilde{\mathcal{S}}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S = -\left(\int_0^1 r^3\mathrm{d}r\right)\left(\int_0^{2\pi} \sin^2\theta\mathrm{d}\theta\right) = -\frac{\pi}{4}\ .$$

Finally,

$$\int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S = -\frac{11\pi}{4}\ .$$

10: Define the points

$$\mathbf{p}_1 = (0,0,0)\ ,\quad \mathbf{p}_2 = (1,2,-3)\ ,\quad \mathbf{p}_3 = (4,1,-5)\ ,\quad \mathbf{p}_4 = (5,-1,-4)\ ,\quad \mathbf{p}_5 = (2,-1,-1)\ .$$

Notice that all 5 points lie in the plane $x+y+z=0$. Let $\mathcal{C}$ be the curve that runs in straight line segments from $\mathbf{p}_1$ to $\mathbf{p}_2$, then from $\mathbf{p}_2$ to $\mathbf{p}_3$, then from $\mathbf{p}_3$ to $\mathbf{p}_4$ and finally from $\mathbf{p}_4$ to $\mathbf{p}_5$. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = (z^2+2xy+z, x^2-2yz+x, 2xz-y^2+y)\ .$$

Compute $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s$ by making use of Stokes' Theorem.

SOLUTION (a) As we have seen in the previous problem,

$$(z^2+2xy, x^2-2yz, 2xz-y^2) = \nabla\varphi(x,y,z) \quad \text{where} \quad \varphi(x,y,z) = x^2y - y^2z + z^2x\ .$$

Hence

$$\mathrm{curl}\mathbf{F} = \mathrm{curl}(z,x,y) = (1,1,1)\ .$$

(You could also compute this directly without eliminating the gradient part.) Define $\mathbf{G}(x,y,z) = (z,x,y)$. Then $\mathbf{F} = \nabla\varphi + \mathbf{G}$.

We can close the curve by adding in $\widetilde{\mathcal{C}}$ which runs on the straight line segment from $\mathbf{p}_5$ to $\mathbf{p}_1$. This is parameterized by

$$\mathbf{x}(t) = (1-t)(2,-1,-1)\ ,\quad t\in(0,1)\ .$$

Then $\mathbf{x}'(t) = (-2,1,1)$, and

$$\begin{aligned}\int_{\widetilde{\mathcal{C}}} \mathbf{F}\cdot\mathbf{T}\mathrm{d}s &= \int_{\widetilde{\mathcal{C}}} \nabla\varphi\cdot\mathbf{T}\mathrm{d}s + \int_{\widetilde{\mathcal{C}}} \mathbf{G}\cdot\mathbf{T}\mathrm{d}s \\ &= \varphi(\mathbf{p}_1) - \varphi(\mathbf{p}_5) - 6\int_0^1 (1-t)\mathrm{d}t \\ &= -1-3 = -4\ .\end{aligned}$$

Since all of the points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ and $\mathbf{p}_5$ lie in the plane $x+y+z=0$, we define $\mathcal{S}$ to be the polygon in this plane with vertices $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ and $\mathbf{p}_5$. Then $\mathcal{S}$ is bounded by $\widetilde{\mathcal{C}}\cup\mathcal{C}$, and since the curve is traversed clockwise when viewed from above, the consistent unit normal is

$$\mathbf{N} = -\frac{1}{\sqrt{3}}(1,1,1)\ .$$

Hence $\mathbf{F} \cdot \mathbf{N} = -\sqrt{3}$ everywhere on $\mathcal{S}$, and so by Stokes' Theorem,

$$\int_{\tilde{\mathcal{C}} \cup \mathcal{C}} \mathbf{F} \cdot \mathbf{T} \mathrm{d}s = -\sqrt{3} \mathrm{area}(\mathcal{S}) \ .$$

With a simple sketch, one breaks $\mathcal{S}$ up into several triangles and rectangles, and computes that $\mathrm{area}(\mathcal{S}) = 19/2$. Hence

$$\int_C \mathbf{F} \cdot \mathbf{T} \mathrm{d}s = \frac{19\sqrt{3}}{2} + 4 \ .$$