Practice Final Exam, Math 291 Fall 2017

December 15, 2017

NAME: ____

1: (a) Let $\mathbf{a} = (-1, 1, 2)$ and $\mathbf{b} = (2, -1, 1)$. Find all vectors $\mathbf{x}$, if any exist, such that

$$\mathbf{a} \times \mathbf{x} = (-2, 4, -3) \qquad \text{and} \qquad \mathbf{b} \cdot \mathbf{x} = 2\ .$$

If none exist, explain why this is the case.

(b) Let $\mathbf{a} = (-1, 1, 2)$ and $\mathbf{b} = (2, -1, 1)$. Find all vectors $\mathbf{x}$, if any exist, such that

$$\mathbf{a} \times \mathbf{x} = (2, 4, 3) \qquad \text{and} \qquad \mathbf{b} \cdot \mathbf{x} = 2\ .$$

If none exist, explain why this is the case.

(c) Among all vectors $\mathbf{x}$ such that $(-1, 1, 2) \times \mathbf{x} = (-2, 4, -3)$, find the one that is closest to $(1, 1, 1)$.

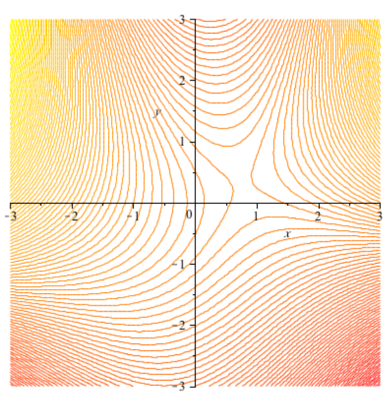
2: Let $f(x, y) = xy^2 - xy$.

(a) Find all of points at which the tangent plane to the graph of f is horizontal.

(b) Find the equation of the tangent plane to the graph of f at $(3/2, 1/3)$.

(c) Find the equation of the tangent line to the contour curve of f through the point $(3/2, 1/3)$.

(d) Could the following be a contour plot for f? Explain your answer to receive credit.



3: Let $f(x, y)$ be a differentiable function on $\mathbb{R}^2$ such that $f(0, 0) = 0$. Define a function $g(x, y)$ by

$$g(x,y) = \begin{cases} \dfrac{f(x,y)}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) \neq (0,0) \ . \end{cases}$$

Suppose that f is continuously differentiable. Is it then necessarily the case that g is continuous? Justify your answer to receive credit.

4: Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = (t, t^2/2, t^3/3)$.

(a) Compute the curvature $\kappa(t)$ as a function of t, and show that $\lim_{t\to\infty} \kappa(t) = 0$.

(b) Compute the angle $\theta(t)$ between $\mathbf{T}(t)$ and $\mathbf{a}(t)$ as a function of t, and show that $\lim_{t\to\infty} \theta(t) = 0$. Comment on the relation between this limit, and the limit in part **(a)**.

(c) Find the equation of the osculating plane at $t = 1$.

(d) Compute the distance from the origin to the osculating plane at $t = 1$.

5: Let $f(x,y) = \dfrac{xy}{(1+x^2+y^2)^2}$.

(a) Find all of the critical points of f, and for each of them, determine whether it is a local minimum, a local maximum, a saddle point, or if it cannot be classified through a computation of the Hessian.

(b) There is one critical point of f in the interior of the upper right quadrant. Let $\mathbf{x}_0 = (x_0, y_0)$ denote this critical point. Let $\mathbf{u} = (u, v)$ be a unit vector, and consider the directional second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(x_0 + tu, y_0 + tv)\bigg|_{t=0} .$$

Which choices of the unit vector (u, v) makes this as large as possible? What is the largest possible value? Also, which choices of the unit vector (u, v) makes this as small as possible, and what is the smallest possible value?

(c) Sketch a contour plot of f near (x_0, y_0).

(d) Does f have minimum and maximum values? If so, say what they are, and what the maximizers and minimizes are.

6: Let $f(x,y) = xy$. Let D denote the region in the plane consisting of all of the points (x,y) such that

$$x^2 + 4y^2 \leq 6 \ .$$

Find the minimum and maximum values of f in D. Also, find all of the minimizers and maximizers in D.

7: **(a)** Let D be the set in the positive quadrant of $\mathbb{R}^2$ that bounded by

$$\begin{aligned} y &= x \\ y &= \sqrt{3}x \\ y &= x^2 + y^2 \end{aligned}$$

Let $f(x,y) = \sqrt{1 + x^2 + y^2}$. Compute $\int_D f(x,y)\mathrm{d}A$.

(b) Let D be the set in $\mathbb{R}^2$ that is given by

$$1 \leq \frac{y}{x^2} \leq 2 \qquad \text{and} \qquad 1 \leq \frac{x}{y^2} \leq 2 \ .$$

Let $f(x,y) = \dfrac{1}{x^2y^2}$. Compute $\int_D f(x,y)\mathrm{d}A$.

8: Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that is bounded by the surfaces

$$\begin{aligned}\sqrt{x^2+y^2} &= z^3 \\ \sqrt{x^2+y^2} &= 10-z\end{aligned}$$

Compute the volume of $\mathcal{V}$ **and** the total surface area of its boundary. (There are two pieces to the boundary.

(a) Compute the $\int_{\mathcal{V}}(x^2+y^2)\mathrm{d}V$.

(b) Compute the total surface area of the boundary of $\mathcal{V}$.

9: Consider the two vector fields

$$\mathbf{F} = (yz, xz, xy) \qquad \text{and} \qquad \mathbf{G} = (z^2 + 2xy, x^2 - 2yz, 2xz - y^2) \ .$$

(a) Compute the divergence and curl of $\mathbf{F}$ and $\mathbf{G}$.

(b) One of the vector fields $\mathbf{F}$ and $\mathbf{G}$ is equal to $\nabla\varphi$ for some potential function φ. Which one is it? Find such a potential function.

(c) One of the vector fields $\mathbf{F}$ and $\mathbf{G}$ is equal to curl$\mathbf{A}$ for some vector potential $\mathbf{A}$. Which one is it? Find such a vector potential.

(d) Let $\mathcal{S}$ be the part of the ellipsoid $x^2 + y^2 + \frac{1}{4}z^2 = \frac{5}{4}$ that lies above the plane $z = 1$. Compute

$$\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{N} \mathrm{d}S \quad \text{and} \quad \int_{\mathcal{S}} \mathbf{G} \cdot \mathbf{N} \mathrm{d}S \ .$$

(e) Let C be the curve that is the intersection of the graph of $z = 1 - x^2$ with the cylinder $x^2 + y^2 = 1$, oriented so that it runs clockwise when viewed from above. Compute Compute

$$\int_{C} \mathbf{F} \cdot \mathbf{T} \mathrm{d}s \quad \text{and} \quad \int_{C} \mathbf{G} \cdot \mathbf{T} \mathrm{d}s \ .$$

10: Define the points

$$\mathbf{p}_1 = (0,0,0)\ ,\quad \mathbf{p}_2 = (1,2,-3)\ ,\quad \mathbf{p}_3 = (4,1,-5)\ ,\quad \mathbf{p}_4 = (5,-1,-4)\ ,\quad \mathbf{p}_5 = (2,-1,-1)\ .$$

Notice that all 5 points lie in the plane $x+y+z=0$. Let $\mathcal{C}$ be the curve that runs in straight line segments from $\mathbf{p}_1$ to $\mathbf{p}_2$, then from $\mathbf{p}_2$ to $\mathbf{p}_3$, then from $\mathbf{p}_3$ to $\mathbf{p}_4$ and finally from $\mathbf{p}_4$ to $\mathbf{p}_5$. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = (z^2 + 2xy + z, x^2 - 2yz + x, 2xz - y^2 + y)\ .$$

Compute $\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s$ by making use of Stokes' Theorem.