Solutions for Homework 9, Math 291 Fall 2015

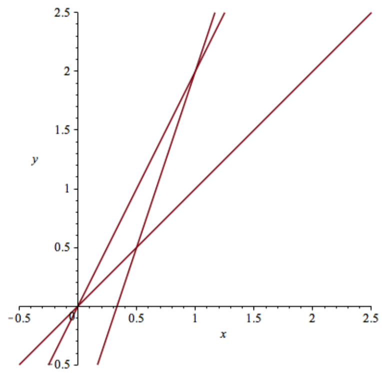
Eric A. Carlen[1]
Rutgers University

November 28, 2017

1. Let D be the triangle bounded by the three lines $y = x$, $y = 2x$ and $y = 3x - 1$. Let $f(x, y) = xy$. Compute

$$\int_D f(x,y)\mathrm{d}A \ .$$

SOLUTION The domain D is bounded by the three lines plotted below:



The vertices of the triangle are $(0, 0)$, $(1/2, 1/2)$ and $(1, 2)$, If we integrate in Cartesian coordinates, we will need to break D into two parts whether we integrate first in x or in y. We give the solution for integrating first in y. For $0 \le x \le 1/2$, the lower boundary is the line $y = x$, and the upper boundary is the line $y = 2x$ For $1/2 \le x \le 1$, the lower boundary is the line $y = 3x - 1$, and the upper boundary is the line $y = 2x$. That is, $D = D_1 \cup D_2$ where

$$D_1 = \{(x,y)\ \ 0 \le x \le 1/2 \qquad \text{and} \qquad x \le y \le 2x\ \}$$

and

$$D_2 = \{(x,y)\ \ 1/2 \le x \le 1 \qquad \text{and} \qquad 3x-1 \le y \le 2x\ \}$$

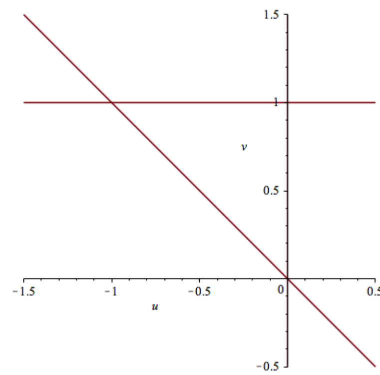
We have

$$\begin{aligned} \int_D f(x,y)\mathrm{d}A &= \int_0^{1/2}\left(\int_x^{2x} xy\mathrm{d}y\right)\mathrm{d}x + \int_{1/2}^1\left(\int_{3x-1}^{2x} xy\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^{1/2}\frac{3}{2}x^3\mathrm{d}x + \int_{1/2}^1 \frac{1}{2}(6x^2 - 5x^3 - x)\mathrm{d}x = \frac{13}{128} + \frac{3}{128} = \frac{1}{8}\ . \end{aligned}$$

Alternatively, we can change variables to avoid decomposing the domain D. Define $u = y - x$ and $v = 3x - y$. That is

$$(u, v) = \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix} (x, y) \quad \text{and hence} \quad (x, y) = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -3 & 1 \end{bmatrix} (u, v) \ .$$

It follows that the Jacobian determinant is given by $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{1}{2}$. Now note that $f(x, y) = xy = \frac{1}{4}(v - u)(v - 3u) = \frac{1}{4}(v^2 + 3u^2 - 4uv)$. $y = x$ translate into $u = 0$, $y = 3x - 1$ translates into $v = 1$, and $y = 2x$ translate into $u + v = 0$. This is much nicer triangle with vertices at $(1, 1)$, $(0, 0)$, and $(0, 1)$.



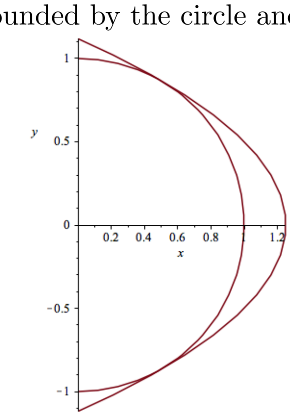
Integrating in v first we have $-u \le v \le 1$ for $-1 \le u \le 0$ and so

$$\begin{aligned} \int_D f(x, y) \mathrm{d}A &= \int_{-1}^{0} \left(\int_{-u}^{1} \frac{v^2 + 3u^2 - 4uv}{4} \frac{1}{2} \mathrm{d}v \right) \mathrm{d}u \\ &= \frac{1}{8} \int_{-1}^{0} \left(\frac{16}{3} u^3 + 3u^2 - 2u + \frac{1}{3} \right) \mathrm{d}u = \frac{1}{8} \ . \end{aligned}$$

2. Let D be the region in the plane that is to the right of the unit circle, and to the left of the parabola $x = 5/4 - y^2$. Let $f(x, y) = x^2 + y^2$.

$$\int_D f(x, y) \mathrm{d}A \ .$$

SOLUTION The domain D is bounded by the circle and parabola plotted below:



To find the points where the parabola and circle meet, we note that at these points, $y^2 = 1 - x^2$ and $y^2 = 5/4 - x$. Therefore, $x^2 - x = -1/4$ so that $x = 1/2$ and then $y = \pm\sqrt{3}/2$. Hence the ponts of intersection are $(1/2, \sqrt{3}/2)$ and $(1/2, -\sqrt{3}/2)$.

For this problem, it is simplest to ingtegrate first in x since the region D can be described as

$$D = \{(x,y)\ \ \sqrt{1-y^2} \le x \le 5/4 - y^2 \quad \text{and} \quad -\sqrt{3}/2 \le y \le \sqrt{3}/2\ \}\ .$$

Hence

$$\begin{aligned}\int_D f(x,y)\mathrm{d}A &= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \left(\int_{\sqrt{1-y^2}}^{5/4-y^2} (x^2+y^2)\mathrm{d}x \right) \mathrm{d}y \\ &= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \left(\frac{1}{3}\left(\frac{5}{4} - y^2\right)^3 - \frac{1}{3}(1-y^2)^{3/2} + y^2\left(\frac{5}{4} - y^2 - \sqrt{1-y^2}\right)\right) \mathrm{d}y \\ &= \frac{267}{560}\sqrt{3} - \frac{1}{6}\pi\ .\end{aligned}$$

3. Let D be the set in $\mathbb{R}^2$ that is given by

$$x^2 \le y \le 2x^2 \qquad \text{and} \qquad x^3 \le y \le 2x^3\ .$$

Let $f(x,y) = \dfrac{x}{y}$. Compute $\int_D f(x,y)\mathrm{d}A$.

SOLUTION We introduce the variables $u = y/x^2$ and $v = y/x^3$ This defines the transformation

$$\mathbf{u}(x,y) = (u(x,y), v(x,y)) = (y/x^2, y/x^3)\ .$$

To find the inverse transformation $\mathbf{x}(u,v)$, we solve for x and y in terms of u and v: $x = u/v$, and then $y = x^2u = u^3/v^2$. Thus,

$$\mathbf{x}(u,v) = (u/v, u^3/v^2)\ .$$

For the function $f(x,y) = xy$, $f(\mathbf{x}(u,v)) = u^4/v^2$. The Jacobain matrix of the transformation $\mathbf{x}(u,v)$ is

$$D_{\mathbf{x}}(u,v) = \begin{bmatrix} 1/v & -u/v^2 \\ 3u^2/v^2 & -2u^3/v^3 \end{bmatrix}\ .$$

Taking the absolute value of the determinant, we find the Jacobian determinant is

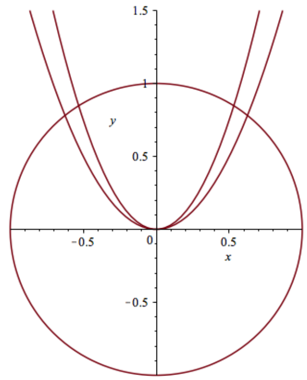
$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{u^3}{v^4}\ .$$

We apply the change of variables formula to conclude that

$$\begin{aligned}\int_D f(x,y)\mathrm{d}A &= \int_1^2 \left(\int_1^2 u^4/v^2 \frac{u^3}{v^4}\mathrm{d}v \right) \mathrm{d}u \\ &= \left(\int_1^2 u^7 \mathrm{d}u \right) \left(\int_1^2 v^{-2}\mathrm{d}v \right) = \frac{255}{16}\ .\end{aligned}$$

4. Let D be the region in upper right quadrant of $\mathbb{R}^2$ that is inside the circle $x^2 + y^2 = 1$, and between the parabolas $y = 2x^2$ and $y = 3x^2$. Compute $\int_D xy\mathrm{d}A$.

SOLUTION The domain D is bounded by the parabolas and the circle plotted below, and lies in the upper right quadrant:



Parabolas of the form $y = ax^2$ actualy have a decent description in polar coordinates; they can be described by giving θ are a functions of r. To see this, we translate $y = ax^2$ into $r\sin\theta = ar^2\cos^2\theta = ar^2 = ar^2\sin^2\theta$. Hence,

$$\sin^2\theta + \frac{1}{ar}\sin\theta = 1\ .$$

Solving for $\sin\theta$, the relvant root is the one that goes to zero as r goes to zero, so

$$\sin\theta = \frac{\sqrt{(2ar)^2+1}-1}{2ar}\ .$$

Taking the inverse sine function, we obtain $\theta = \arcsin\left(\frac{\sqrt{(2ar)^2+1}-1}{2ar}\right)$. Therefore, if for $a > 0$ we defined the function $\theta_a(r) = \arcsin\left(\frac{\sqrt{(2ar)^2+1}-1}{2ar}\right)$, our region D is the set of points whose polar coordinates satisfy

$$\theta_2(r) \le \theta \le \theta_3(r) \qquad \text{and} \qquad 0 \le r \le 1\ .$$

Since $f(r\cos\theta, r\sin\theta) = r^2\cos\theta\sin\theta$ and $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$,

$$\begin{aligned}
\int_D f(x,y)\mathrm{d}A &= \int_0^1 r^3\left(\int_{\theta_2(r)}^{\theta_3(r)}\cos\theta\sin\theta\mathrm{d}\theta\right)\mathrm{d}r \\
&= \frac{1}{2}\int_0^1 r^3(\sin^2(\theta_3(r)) - \sin^2(\theta_3(r)))\mathrm{d}r \\
&= \frac{1}{72}\int_0^1 r\left(9\sqrt{16r^2+1} - 4\sqrt{36r^2+1} - 5\right)\mathrm{d}r \\
&= \frac{(17)^{3/2}}{768} - \frac{(37)^{3/2}}{3888} - \frac{1145}{62208} \approx 0.01497429640\ .
\end{aligned}$$

Another approach is to let D_3 be the region above $y = 3x^2$ and inside the circle, and let D_2 be the region above $y = 3x^2$ and inside the circle. Then $D_3 \subset D_2$, and D is the part of D_2 that is not contined in D_3, up to boundary points that do not affect the integral. Hence

$$\int_D f(x,y)\mathrm{d}A = \int_{D_2} f(x,y)\mathrm{d}A - \int_{D_3} f(x,y)\mathrm{d}A\ .$$

We can calculate both integrals on the right efficiently by denfining D_a to be the region above $y = ax^2$ and inside the circle for $a > 0$. The intersection of the parabola and circle is given by $x^2 + a^2x^4 = 1$. This is a quadratic equation in x^2. The unique positive root is $x^2 = \dfrac{\sqrt{1+4a^2}-1}{2a^2}$. Therefore, the value of x at the intrection of the circle and right branch of the parabola is x_a, where we define

$$x_a = \left(\frac{\sqrt{1+4a^2}-1}{2a^2}\right)^{1/2} .$$

Then

$$D_a = \left\{ \ (x,y) \ : \ ax^2 \le y \le \sqrt{1-x^2} \quad \text{and} \quad 0 \le x \le x_a \ \right\} .$$

Therefore,

$$\begin{aligned}
\int_{D_a} f(x,y)\mathrm{d}A &= \int_0^{x_a} x \left(\int_{ax^2}^{\sqrt{1-y^2}} y\mathrm{d}y \right) \mathrm{d}x \\
&= \frac{1}{2}\int_0^{x_a} \left(x - x^3 - a^2x^5 \right) \mathrm{d}x \\
&= \frac{x_a^2}{4} - \frac{x_a^4}{8} - a^2\frac{x_a^6}{12} .
\end{aligned}$$

Evaluating this for $a = 2$ and $a = 3$ we find

$$\int_{D_2} f(x,y)\mathrm{d}A = \frac{(17)^{3/2}-25}{768} \qquad \text{and} \qquad \int_{D_3} f(x,y)\mathrm{d}A = \frac{(37)^{3/2}-55}{3888} .$$

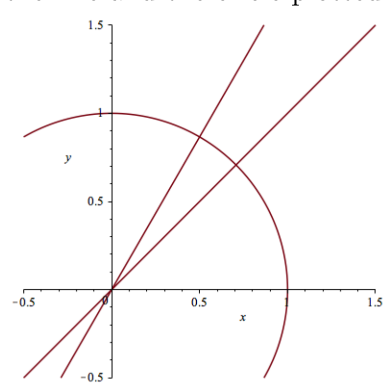
Subtracting, we find the same result we found before usng polar coordinates – as was necessarily the case.

5: (a) Let D be the set in the positive quadrant of $\mathbb{R}^2$ that bounded by

$$\begin{aligned}
y &= x \\
y &= \sqrt{3}x \\
y &= x^2 + y^2
\end{aligned}$$

Let $f(x,y) = \sqrt{1+x^2+y^2}$. Compute $\int_D f(x,y)\mathrm{d}A$.

The domain D is bounded by the line and the circle plotted below:



This is readily described in polar coodinate as the set with

$$\frac{\pi}{4} \le \theta \le \frac{\pi}{3} \qquad \text{and} \qquad 0 \le r \le 1\ .$$

Since $f(r\cos\theta, r\sin\theta) = \sqrt{1+r^2}$,

$$\int_D f(x,y)\mathrm{d}A = \left(\int_{\pi/4}^{\pi/3} \mathrm{d}\theta\right)\left(\int_0^1 \sqrt{1+r^2}r\mathrm{d}r\right) = \frac{\pi}{12}\frac{\sqrt{8}-1}{3}\ .$$

6. Let D be the region in the upper right quadrant between the curves

$$x = \frac{1}{y^2} \qquad \text{and} \qquad x = \frac{4}{y^2}$$

and between the curves

$$y = x^2 \qquad \text{and} \qquad y = 4x^2\ .$$

Compute $\int_D (x^2+y^2)\mathrm{d}A$.

7. Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that is inside the cylinder

$$x^2 + y^2 = y$$

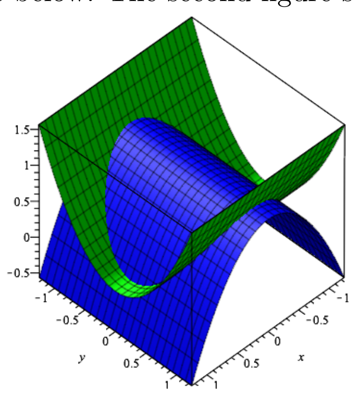
and bounded above and below by

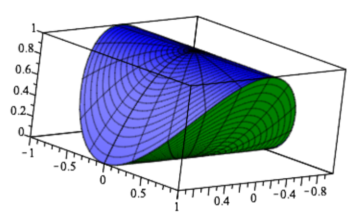
$$z = x^2 + y^2 \qquad \text{and} \qquad z = \sqrt{x^2+y^2}\ .$$

Compute the volume of this region.

8. (a) Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies below the graph of $z = 1 - x^2$, and above the graph of $z = y^2$. Compute the volume of $\mathcal{V}$.

SOLUTION The domain $\mathcal{V}$ is bounded by the "parabolic cyllinders", parts of which are shown in the first figure below. The second figure shows $\mathcal{V}$ itself:





We first find the intersection of the two bounding surfaces, at which we have $1 - x^2 = z = y^2$, so $x^2 + y^2 = 1$: The projection of the curve of intersection along the boundart onto the x, y plane is the unit circle. At any point x, y inside the unit circle the portion of the line through $(x, y, 0)$ parallel to the z-axis has

$$y^2 \le z \le 1 - x^2\ ,$$

We can describe $\mathcal{V}$ conveniently lin cylindrical coorinates $(x, y, z) = (r\cos\theta, r\sin\theta, z)$.

$$\mathcal{V} = \left\{(r\cos\theta, r\sin\theta, z)\ :\ 0 \le r \le 1\ ,\quad 0 \le \theta < 2\pi \quad \text{and} \qquad r^2\sin^2\theta \le x \le 1 - r^2\cos^2\theta\ \right\}$$

We must integrate in z first since the limits on z depend on both r and θ. After that we can do the other two integrations in any order since the limits are constant. It will be convenient to do θ secon and r third. Since the volume element for cylindrical coordinates is $r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$, the volume of $\mathcal{V}$ is

$$\begin{aligned} \int_{\mathcal{V}} 1\mathrm{d}V &= \int_0^1 \left(\int_0^{2\pi} \left(\int_{r^2\sin^2\theta}^{1-r^2\cos^2\theta} 1\mathrm{d}z \right) \mathrm{d}\theta \right) r\mathrm{d}r \\ &= \int_0^1 \left(\int_0^{2\pi} (1-r^2)\mathrm{d}\theta \right) r\mathrm{d}r \\ &= 2\pi \int_0^1 (1-r^2) r\mathrm{d}r = \frac{\pi}{2}\ . \end{aligned}$$

9. Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that is the intersection of the three cylinders of unit radius along the three coordinate axes. That is, $\mathcal{V}$ is the set of points (x, y, z) satisfying

$$y^2 + z^2 \le 1 \qquad x^2 + z^2 \le 1 \quad \text{and} \quad x^2 + y^2 \le 1\ .$$

Compute the volume of $\mathcal{V}$.

SOLUTION Done in class.

10. Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$, and above the graph of $z = 1/\sqrt{x^2 + y^2}$. Compute the volume of $\mathcal{V}$ **and** the total surface area of its boundary. (There are two pieces to the boundary.)

SOLUTION See the next to last example in Chapter 7, where this is solved in detail. (It pays to read...)

11. Let $\mathcal{S}$ be the surface that is the part of the graph of $z = 1 - \sqrt{x^2 + y^2}$ that lies inside the cylinder $(x-1)^2 + y^2 = 1$, which is a cylider of radius 1 running parallel to the z-axis. Let $f(x, y, z) = z$ Compute $\int_{\mathcal{S}} f\mathrm{d}S$. **Hint:** Write the equation $(x-1)^2 + y^2 = 1$ in polar coordinates.

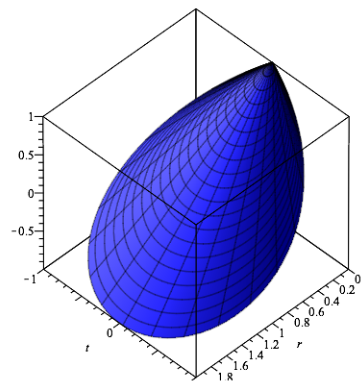
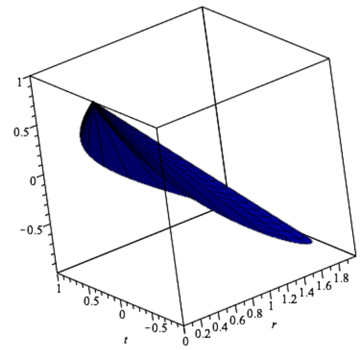
SOLUTION It is natrual to parameterize $\mathcal{S}$ using cylindrical coorinates $(x, y, z) = (r\cos\theta, r\sin\theta, z)$. Then we have $z = 1 - r$ on $\mathcal{S}$, while the equation for the bounding cylinder is $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. This is positive (and defines a radius) if and only if $\pi/2 \le \theta \le \pi/2$. Hence the surface is parameterized by

$$\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, 1 - r)$$

for

$$0 \le r \le 2\cos(t) \quad \text{and} \quad \pi/2 \le \theta \le \pi/2\ .$$

Here is a plot from two points of view:

We now compute

$$\mathbf{X}_r(r,\theta) = (\cos\theta, \sin\theta, -1) \qquad \text{and} \qquad \mathbf{X}_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 0)\ .$$

Then

$$\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta) = (r\cos\theta, r\sin\theta, r)\ .$$

Hence the surface area element is

$$\mathrm{d}S = \|\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta)\mathrm{d}r\|\mathrm{d}\theta = \sqrt{2}r\mathrm{d}r\|\mathrm{d}\theta\ .$$

Then for $f(x,y,z) = x$, $f(\mathbf{X}(r,\theta)) = 1 - r$, and so

$$\begin{aligned}\int_{\mathcal{S}} f\mathrm{d}S &= \sqrt{2}\int_{-\pi/2}^{\pi/2}\left(\int_0^{2\cos(\theta)}(1-r)r\mathrm{d}r\right)\mathrm{d}\theta \\ &= \sqrt{2}\int_{-\pi/2}^{\pi/2}\left(2\cos^2(\theta) - \frac{8}{3}\cos^3(\theta)\right)\mathrm{d}\theta = \sqrt{2}\left(\pi - \frac{32}{9}\right)\ .\end{aligned}$$

12. Let $\mathcal{S}$ be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane $x + z = 1$. Compute $\int_{\mathcal{S}} f(x,y,z)\mathrm{d}S$ where $f(x,y,z) = y/\sqrt{x^2+y^2}$. To get full credit, carry the computations through to the point that only an integral over a single variable remains to be evaluated.

SOLUTION The intersection of the two surfaces is given by

$$1 - x^2 - y^2 = z = 1 - x$$

so that on the intersection, $x^2 + y^2 = x$. Note that the equations specifying the surface $\mathcal{S}$ are even in y. That is $(x,y,z) \in \mathcal{S} \iff (x,-y,z) \in \mathcal{S}$. On the other hand, the integrand is odd in y: $f(x,-y,z) = -f(x,y,z)$. Hence without any computation at all, we see that

$$\int_{\mathcal{S}} f(x,y,z)\mathrm{d}S = 0\ .$$

This is all that needs to be said for full credit. However, it is instructive to set up the integral, and to see how thiss result comes out of the computations.

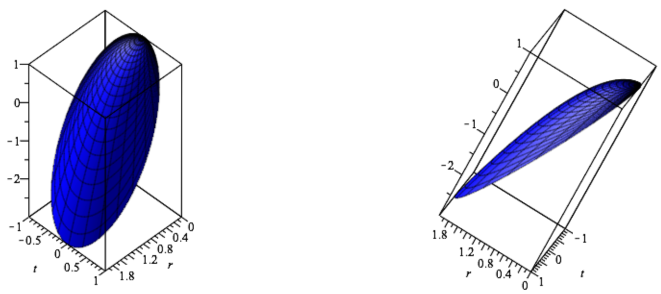
It is natrual to parameterize $\mathcal{S}$ using cylindrical coorinates $(x,y,z) = (r\cos\theta, r\sin\theta, z)$. Then we have $z = 1 - r^2$ on $\mathcal{S}$, while the equation $x^2 + y^2 = x$ describing the prjoection of the intersection of the surfaces ontl the x, y plane is $r = \cos\theta$. This is positive (and defines a radius) if and only if $\pi/2 \leq \theta \leq \pi/2$. Hence the surface is parameterized by

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 1 - r^2)$$

for

$$0 \leq r \leq \cos(t) \quad \text{and} \quad \pi/2 \leq \theta \leq \pi/2 \ .$$

Here is a plot from two points of view:



We now compute

$$\mathbf{X}_r(r,\theta) = (\cos\theta, \sin\theta, -2r) \qquad \text{and} \qquad \mathbf{X}_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 0) \ .$$

Then

$$\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta) = (2r^2\cos\theta, 2r^2\sin\theta, r) \ .$$

Hence the surface area element is

$$\mathrm{d}S = \|\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta)\|\mathrm{d}r\mathrm{d}\theta = \sqrt{1+4r^2}\,r\mathrm{d}r\mathrm{d}\theta \ .$$

Then for $f(x,y,z) = y/\sqrt{x^2+y^2}$, $f(\mathbf{X}(r,\theta)) = \sin\theta$, and so

$$\begin{aligned}\int_{\mathcal{S}} f\mathrm{d}S &= \int_{-\pi/2}^{\pi/2}\left(\int_0^{\cos(\theta)} \sin\theta\sqrt{1+4r^2}\,r\mathrm{d}r\right)\mathrm{d}\theta \\ &= \frac{1}{12}\int_{-\pi/2}^{\pi/2}\left((1+4\cos^2 t)^{3/2} - 1\right)\sin\theta\mathrm{d}\theta = 0 \ .\end{aligned}$$