Solutions for Homework 8, Math 291 Fall 2017

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1: Consider the function $f(x, y) = -xy^2 + 3y + x^2y$.

(a) Find all critical points, and compute the Hessian matrix at each critical point.

(b) Determine whether each critical point is a local minimum, a local maximum, a saddle, or if the Hessian is degenerate at the critical point.

(c) Sketch a contour plot of f in the vicinity of each critical point.

SOLUTION We first compute

$$\nabla f(x, y) = (y(2x - y), x^2 - 2xy + 3).$$

The critical points are the solutions of

$$\begin{aligned} y(2x - y) &= 0 \\ x^2 - 2xy + 3 &= 0\ . \end{aligned}$$

To satisfy the first equation we must have either $y = 0$ or $y = 2x$. If $y = 0$, the second equation says $x^2 + 3 = 0$, and this is impossible. If $y = 2x$, the second equation says that $x^2 - 4x^2 + 3 = 0$, or $x^2 = 1$. Hence we have two critical points $\mathbf{x}_1 = (1, 2)$ and $\mathbf{x}_2 = (-1, -2)$.

We next compute the Hessian:

$$[\mathrm{Hess}_f(x, y)] = \begin{bmatrix} 2y & 2x - 2y \\ 2x - 2y & -2x \end{bmatrix}\ .$$

Evaluating, $[\mathrm{Hess}_f(\mathbf{x}_1)] = \begin{bmatrix} 4 & -2 \\ -2 & -2 \end{bmatrix}$, and $[\mathrm{Hess}_f(\mathbf{x}_2)] = \begin{bmatrix} -4 & 2 \\ 2 & 2 \end{bmatrix}$.

For **(b)**, In both cases, the determinant is -12 so by Sylvester's Criterion, the critical points are both saddle points,

For **(c)**, we draw the sketches. First consider $\mathbf{x}_1$. The characteristic polynomial of $A = [\mathrm{Hess}_f(\mathbf{x}_1)]$ is $t^2 - 2t - 12$. Hence the eigenvalues are $\lambda_1 = 1 + \sqrt{13}$ and $\lambda_2 = 1 - \sqrt{13}$. From

$$A - \lambda_1 I = \begin{bmatrix} 3 - \sqrt{13} & -2 \\ -2 & -3 - \sqrt{13} \end{bmatrix}\ ,$$

we read off that $\mathbf{v} = (2, 3-\sqrt{13})$ is an eigenvector with eigenvalue λ_1. Normalizing we find that

$$\mathbf{u}_1 = \frac{1}{2(13-\sqrt{13})}(2, 3-\sqrt{13})$$

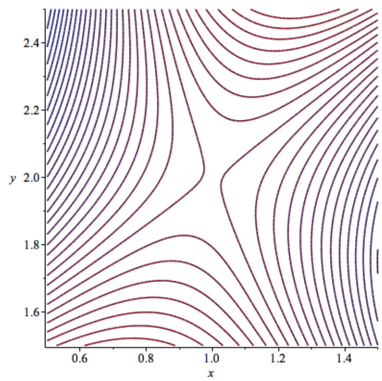
is a normalized eigenvector with eigenvalue λ_1. Since we know the eigenvectors with the other eigenvalue are orthogonal to $\mathbf{u}_1$, we can immediately write down

$$\mathbf{u}_2 = \frac{1}{2(13-\sqrt{13})}(3-\sqrt{13}, -2)\ .$$

Since $\lambda_1 \approx 4.60$ and $\lambda_2 \approx -2.60$, the contour curves near $\mathbf{x}_1$ will be hyperbolas, in shape more or less like those described by

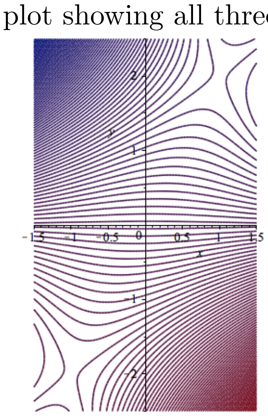
$$(4.60)u^2 - (2.60)v^2 = 1,$$

where the u axis runs along $\mathbf{u}_1$ and the v axis runs along $\mathbf{u}_2$. The asymptotes are the lines $(4.60)u^2 = (2.60)v^2$, and the hyperbolas are now easy to plot:



Next, since $[\mathrm{Hess}_f(\mathbf{x}_2)] = -[\mathrm{Hess}_f(\mathbf{x}_1)]$, the eigenvalues of $[\mathrm{Hess}_f(\mathbf{x}_2)]$ are obtained by changing the signs of the eigenvalues of $[\mathrm{Hess}_f(\mathbf{x}_1)]$, and the eigenvectors are exactly the same. Hence the contour plot will have exactly the same contours, except that the curves will represent different values. But these are not indicated our plot anyhow, so the plots for $\mathbf{x}_2$ is the same as the one given above for $\mathbf{x}_1$.

Finally, the following is a contour plot showing all three critical points:



Notice that the symmetry you see in this plots corresponds to the fact that $f(-x,-y) = f(x,y)$ for all (x,y). This was the reason we did not have to compute eigenvalues and eigenvectors twice.

2: Consider the function $f(x,y) = \dfrac{x^2y}{1+2x^4+y^4}$.

(a) Find all critical points in the open upper right quadrant; i.e., the set of (x, y) with $x, y > 0$. Compute the Hessian matrix at each such critical point.

(b) Determine whether each such critical point is a local minimum, a local maximum, a saddle, or if the Hessian is degenerate at the critical point.

(c) Sketch a contour plot of f in the vicinity of each such critical point.

SOLUTION We first compute

$$\nabla f(x, y) = (1 + 2x^4 + y^4)^{-2}(2xy(2x^4 - y^4 - 1), x^2(1 + 2x^4 - 3y^4)).$$

The critical points in the quadrant with $x, y > 0$ are the solutions of

$$\begin{aligned} 2x^4 - y^4 &= 1 \\ 2x^4 - 3y^4 &= -1 \ . \end{aligned}$$

Subtracting the second equation from the first, we obtain $2y^4 = 2$, so that $y^4 = 1$, and then the first equation gives us $x^4 = 1$. Since $x, y > 0$, this means $\mathbf{x}_1 = (1, 1)$ is the only critical point in the quadrant.

We next compute the Hessian and evaluate it at $\mathbf{x}_1 = (1, 1)$, finding

$$[\mathrm{Hess}_f(\mathbf{x}_1)] = \frac{1}{4}\begin{bmatrix} -4 & 2 \\ 2 & -3 \end{bmatrix} .$$

For **(b)**, note that the determinant is positive, and both diagonal entries are negative. Hence, by Sylvester's Criterion, both eigenvalues are negative, and $\mathbf{x}_1$ is a local maximum.

For **(c)**, we need to compute the eigenvalues and eigenvectors. The characteristic polynomial of A is $t^2 + 7t/4 + 1/2$, and its roots are the eigenvalues $\lambda_1 = (-7 + \sqrt{17})/8$ and $\lambda_2 = (-7 - \sqrt{17})/8$. To find the eigenvectors, we compute

$$A - \lambda_1 I = \frac{1}{8}\begin{bmatrix} -1 - \sqrt{17} & 4 \\ 4 & 1 - \sqrt{17} \end{bmatrix} .$$

From this we read off that $\mathbf{v} = (4, 1 + \sqrt{17})$ is an eigenvector with eigenvalue λ_1. Normalizing we find that

$$\mathbf{u}_1 = \frac{1}{2(17 + \sqrt{17})}(4, 1 + \sqrt{17})$$

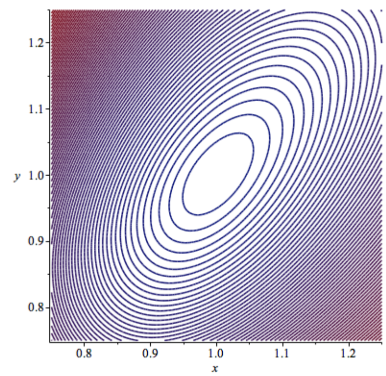
is a normalized eigenvector with eigenvalue λ_1. Since we know the eigenvectors with the other eigenvalue are orthogonal to $\mathbf{u}_1$, we can immediately write down

$$\mathbf{u}_2 = \frac{1}{2(17 + \sqrt{17})}(1 + \sqrt{17}, -4) \ .$$

Since $\lambda_1 \approx -0.36$ and $\lambda_2 \approx -1.39$, the contour curves near $\mathbf{x}_1$ will be ellipses, in shape more or less like those described by

$$(0.36)u^2 + (1.39)v^2 = 1,$$

where the u axis runs along $\mathbf{u}_1$ and the v axis runs along $\mathbf{u}_2$. Note the the u-axis is the major axis. It is now east to do the plot:



3: For any real number a, let $A(a)$ be the matrix

$$A(a) := \begin{bmatrix} 3 & 1 & a \\ 1 & 2 & a \\ a & a & 1 \end{bmatrix} .$$

For which values of a are all of the eigenvalues of $A(a)$ strictly positive? Justify your answer to receive credit.

SOLUTION By Sylvester's Theorem, $A(a)$ is positive definite if and only if

$$3 > 0 \ , \quad \det\left(\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\right) = 5 > 0 \quad \text{and} \quad \det(A(a)) = 5 - 3a^2 > 0 \ .$$

Hence $A(a)$ is strictly positive if and only if $a \in (-\sqrt{5/3}, \sqrt{5/3})$.

4: Let $a > b > 0$ be given. Consider the circle in the x, y plane in $\mathbb{R}^3$ parameterized by $a(\cos u, \sin u, 0)$, $0 \le u < 2\pi$. The set $\mathcal{S}$ of all points in $\mathbb{R}^3$ whose distance from this circle is exactly b is torus. It is parameterized by

$$\mathbf{X}(u,v) = a(\cos u, \sin u, 0) + b(\cos u \sin v, \sin u \sin v, \cos v) = (\cos u(a + b\sin v) \ , \ \sin u(a + b\sin v) \ , \ b\cos v) \ ,$$

where $0 \le u, v < 2\pi$.

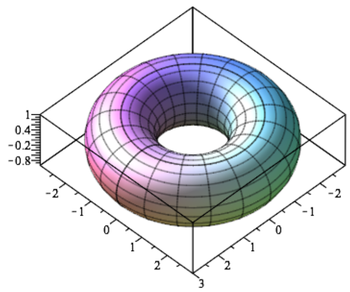
(a) Check that $\mathbf{X}_u(u,v)$ and $\mathbf{X}_v(u,v)$ are linearly independent for all u, v.

(b) Compute the equation for the tangent plane to the torus at $\mathbf{X}(\pi/4, \pi/3)$.

(c) Compute the principle curvatures at $\mathbf{X}(\pi/4, \pi/3)$, and find an orthonormal basis of eigenvectors of the tangent plane at $\mathbf{X}(\pi/4, \pi/3)$ given by eigenfunctions of the shape operator.

(d) Compute the Gaussian curvature of the torus at each point $\mathbf{X}(u,v)$.

SOLUTION Here is a plot of the torus for $a = 2$ and $b = 1$:



For **(a)**, we compute

$$\mathbf{X}_u(u,v) = \big(-a\sin(u) - b\sin(u)\sin(v)\ ,\ a\cos(u) + b\cos(u)\sin(v)\ ,\ 0\big)$$

and

$$\mathbf{X}_u(u,v) = \big(b\cos(u)\cos(v)\ ,\ b\sin(u)\cos(v)\ ,\ -b\sin(v)\big)\ .$$

We then compute

$$\mathbf{X}_u \times \mathbf{X}_v = -b(a + b\sin(v))\big(\cos(u)\sin(v)\ ,\ \sin(u)\sin(v), \cos(v)\big)\ .$$

Notice that $\big(\cos(u)\sin(v)\ ,\ \sin(u)\sin(v), \cos(v)\big)$ is a unit vector: It is noting but the usual "latitude and longitude" parameterization of the sphere, except here v ranges over $(0, 2\pi)$ and not only over $(0, \pi)$. Hence

$$\|\mathbf{X}_u \times \mathbf{X}_v\| = b|a + b\sin(v)|\ .$$

Here is where the condition $0 < b < a$ cones in: This guarantees that $a + b\sin(v) \neq 0$ for any v. Hence $\|\mathbf{X}_u \times \mathbf{X}_v\| > 0$ for all u, v, and this means that $\{\mathbf{X}_u(u,v), \mathbf{X}_v(u,v)\}$ is linearly independent for all u, v.

For **(b)**, by the calculations made above, the unit normal to the surface is given by

$$\mathbf{N}(u,v) = -\big(\cos(u)\sin(v)\ ,\ \sin(u)\sin(v), \cos(v)\big)\ .$$

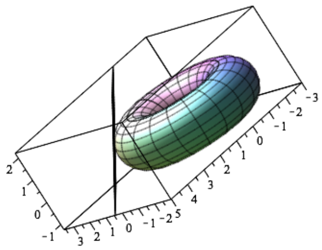
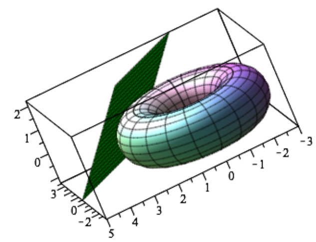
The equation for the tangent plane to the surface at $\mathbf{X}(u_0, v_0)$ is

$$(\big(x, y, z\big) - \mathbf{X}(u_0, v_0)) \cdot \mathbf{N}(u_0, v_0) = 0\ .$$

Evaluating, we find $\mathbf{X}(\pi/4, \pi/3) = \big(a/\sqrt{2} + b\sqrt{6}/4\ , a/\sqrt{2} + b\sqrt{6}/4\ , 1/2\big)$ and $\mathbf{N}(\pi/4, \pi/3) = \big(-\sqrt{6}/4\ ,\ -\sqrt{6}/4\ ,\ -1/2\big)$. The equation is

$$\frac{\sqrt{6}}{4}x + \frac{\sqrt{6}}{4}y + \frac{1}{2}z = \frac{\sqrt{3}a}{2} + b\ .$$

Here are two plots of the torus and its tangent plane at $\mathbf{X}(\pi/4, \pi/3)$ for two different perspectives:

Notice also that the entire torus lies to one side of the tangent plane (apart from the point of contact). Moreover, this is the positive side of the tangent plane with the orientation we are using since $\mathbf{N}(\pi/4, \pi/3)$ points toward the side of the tangent plane containing the origin.

For **(c)**, we compute the First and Second Fundamental Matrices: First, $E = \mathbf{X}_u \cdot \mathbf{X}_u = (a + b\sin(v))^2$, $F = \mathbf{X}_u \cdot \mathbf{X}_v = 0$ and $G = \mathbf{X}_v \cdot \mathbf{X}_v = b^2$. Hence the First Fundamental Matrix is

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} (a + b\sin(v))^2 & 0 \\ 0 & b^2 \end{bmatrix} .$$

Next we compute $L = \mathbf{N} \cdot \mathbf{X}_{uu} = \sin(v)(a + b\cos(v))$, $M = \mathbf{N} \cdot \mathbf{X}_{uv} = 0$ and $N = \mathbf{N} \cdot \mathbf{X}_{vv} = b$. Hence the Second Fundamental Matrix is

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \sin(v)(a + b\sin(v)) & 0 \\ 0 & b \end{bmatrix} .$$

The principle curvatures are the eigenvalues of the matrix

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \sin(v)/(a + b\sin(v)) & 0 \\ 0 & 1/b \end{bmatrix} .$$

Since the matrix is diagonal, the eigenvalues are the diagonal entries. Therefore, these are $\sin(v)/(a + b\sin(v))$ and $1/b$. These are independent of u, and evaluating at $v = \pi/3$ we find these are $(\sqrt{3}/2)(a + b\sqrt{3}/2)$ and $1/b$ respectively. Notice that these are both positive, which corresponds to the fact that the torus lies entirely on the positive side of the tangent plane at $\mathbf{X}(\pi/4, \pi/3)$.

Since the shape operator matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}$ is diagonal, $(1, 0)$ and $(0, 1)$ are a basis of eigenvectors of the shape operator. The corresponding tangent vectors, $[D_{\mathbf{X}}(\pi/4, \pi/3)](1, 0) = \mathbf{X}_u(1, 0)$ and $[D_{\mathbf{X}}(\pi/4, \pi/3)](0, 1) = \mathbf{X}_v(1, 0)$ are the corresponding orthogonal pairs of tangent vectors. (In fact, we have already seen that they are orthogonal.) Normalizing them we obtain our orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$:

$$\mathbf{u}_1 = \frac{1}{\sqrt{4a^2 + 4ab\sqrt{3} + 2b^2}}\left(-\sqrt{2}a - \sqrt{6}b/2 \ , \ -\sqrt{2}a - \sqrt{6}b/2 \ , \ 0\right)$$

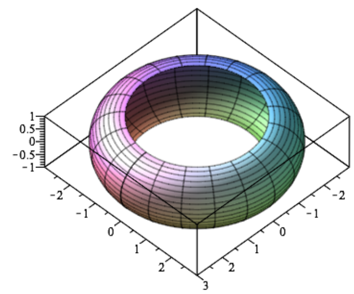
and

$$\mathbf{u}_2 = \frac{1}{4}(\sqrt{2} \ , \ \sqrt{2} \ , \ 2\sqrt{3}) \ .$$

For **(d)**, we have done all of the work already since we have computed the principle curvatures at each u and v. The Gaussian curvature is the product of these, which is

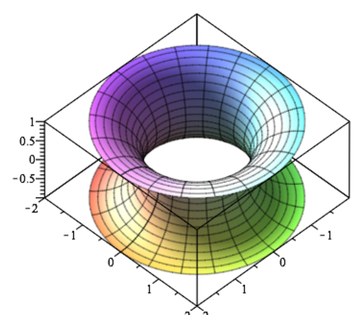
$$\frac{\sin(v)}{b(a+b\cos(v))}\ .$$

Thus the Gaussian curvature is positive when $0 < v < \pi$, and negative when $\pi < v < 2\pi$. These are the "outside" and "inside" of the tours respectively. The "outside" is the part parameterized by $0 \le v \le pi$, and here is a plot of only this part of the torus:

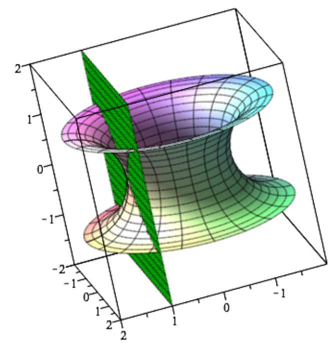


It is clear from the plot that at every point on this part of the torus, the entire torus will lie to one side (the side containing the origin) of the tangent plan at that point.

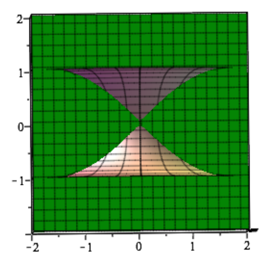
The "inside" is the part parameterized by $\pi \le v \le 2\pi$, and here is a plot of only this part of the torus, again for $a = 2$ and $b = 1$.



It is clear from the plot that at every point on this part of the torus, the parts of the torus will on both sides of the tangent plan at that point. Here is a plot showing the inside of the torus, and the tangent plane again for $a = 2$ and $b = 1$, at the point $\mathbf{X}(0, 3\pi/2) = (1, 0, 0)$:



If we view this from a point of the form $(x, 0, 0)$, with x large, so that we are looking "straight in" at the tangent plane, we see:



The part of the torus we still see is the part on the negative side of the tangent plane.