Solutions for Homework 7, Math 291 Fall 2017

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1: Let $f(x,y) = xy + 2x - 2y$. Find the minimum and maximum values of f on the region of $\mathbb{R}^2$ that lies below the parabola $y = 1 - x^2$ and above the x-axis, $y = 0$.

SOLUTION We compute $\nabla f(x,y) = (y+2, x-2)$. The unique critical point is $(2,-2)$ which lies outside the region.

One part of the boundary is given by $g_1(x,y) = 0$ where $g_1(x,y) = x^2+y-1$. Then $\nabla g_1(x,y) = (2x,-1)$. One this part of the boundary, Lagrange's equations are

$$y+2 = 2x(x-2) \qquad \text{and} \qquad y = 1-x^2 \ .$$

Eliminating y using the second equation, $3-x^2 = 2x^2-4x$, or $3x^3-4x-3 = 0$. The two roots of this quadratic equation are $(2\pm\sqrt{13})/3$, whach are close to 1.87 and -0.54. Only

$$\mathbf{x}_1 := ((2-\sqrt{13})/3, (4\sqrt{13}-8)/9)$$

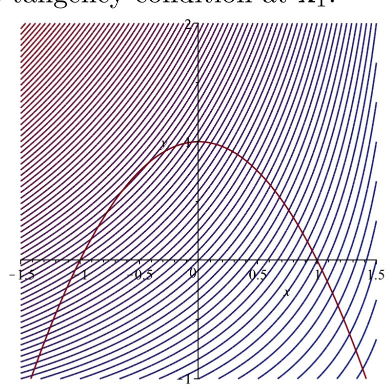
lies on this part of the boundary.

the other boundary is given by $g_2(x,y) = 0$ where $g_2(x,y) = y$. Then $\nabla g(x,y) = (0,1)$. Evidently $\nabla f = \lambda\nabla g$ is possible only if $y = -2$, but this is not in the region.

Finally, we have the points where the tw parts of the boundaty intersect, which are $\mathbf{x}_2 = (-1,0)$ and $\mathbf{x}_3 = (1,0)$. We compute

$$f(\mathbf{x}_1) = (16-26\sqrt{13})/27 \approx -2.88 \ , \quad f(\mathbf{x}_2) = -2 \ , \quad f(\mathbf{x}_3) = 2 \ .$$

Evidently, $\mathbf{x}_1$ is the minimizer, and the minimum value is $(16-26\sqrt{13})/27$ while $\mathbf{x}_3$ is the maximizer, and the maximum value is 2. Here is a contour plot of f together with the contraint curves, one being the x-axis. You can see the tangency condition at $\mathbf{x}_1$.



2: The intersection of the plane $2z = x + 9$ and the cone $z^2 = x^2 + y^2$ is an ellipse. Find the points on the ellipse that are closest to and furthest from the origin.

SOLUTION We seek to minimize and maximize $f(x, y, z) = x^2 + y^2 + z^2$ subject to $g_j(x, y, z) = 0$ for $j = 1, 2$ where

$$g_1(x, y, x) = x - 2z + 9 \qquad \text{and} \qquad g_2(x, y, z) = z^2 - x^2 - y^2 \ .$$

The set of points satisfying the two constraints is closed, bounded and non-empty. Hence there will be both minimizers and maximizers. By Lagrange's Theorem, the minimizers and maximizers $\mathbf{x}$ must satisfy

$$\nabla f(\mathbf{x}) \cdot \nabla g_1(\mathbf{x}) \times \nabla g_2(\mathbf{x}) = 0 \ .$$

The gives us

$$(x, y, z) \cdot (1, 0, -2) \times (-x, -y, z) = -2yz = 0 \ ,$$

and hence either $y = 0$ or $z = 0$.

If $y = 0$, the system $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$ reduces to

$$\begin{aligned} x &= 2z - 9 \\ x^2 &= z^2 \end{aligned}$$

and hence $0 = (2z - 9)^2 - z^2$ which means that $z^2 - 12z + 27 = 0$, and this has the roots $z = 3$ and $z = 9$. Then since $x = 2z - 9$, the corresponding x values are $x = -3$ and $x = 9$, respectively. Hence we have the Lagrange points

$$\mathbf{x}_1 = (-3, 0, 3) \qquad \text{and} \qquad \mathbf{x}_2 = (9, 0, 9) \ .$$

If $z = 0$, the system $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$ reduces to

$$\begin{aligned} x &= 9 \\ x^2 + y^2 &= 0 \end{aligned}$$

and this clearly has no solutions. Hence the minimizers and maximizers of the distance from the origin on the constraint curve must lie in $\{\mathbf{x}_1, \mathbf{x}_2\}$. Evidently, $(-3, 0, 3)$ is the point closest to the origin on the constrain curve; its distance is $3\sqrt{2}$. Likewise, $(9, 0, 9)$ is the point furthest from the origin on the constrain curve; its distance is $9\sqrt{2}$.

3: Let $f(x, y) = x + yz$. Consider the curve given by the pair of equations $3x^2 + 4y^2 + z^2 = 1$ and $z = x^2 + y^2$.

(a) Write down the system of three equations in three unknowns that must be satisfied by any maximizers of f on the curve, according to Lagrange's Theorem.

(b) As in Exercise 2, one of the equations may be used to eliminate z from this system of three equations, resulting in a system of two equations in two unknowns.

(c) Draw graphs, by any means, showing the solution sets of the two equations on the x, y plane, Count the points of intersection an determine the number of solutions. Then pick a starting point near each solution, and run one step of Newton's method to get an improved solution.

SOLUTION Let $g_1(x,y,z) = 3x^2+4y^2+z^2-1$ and let $g_2(x,y,z) = z-x^2-y^2$. Then Lagrange's Theorem tells us that any minimizers or maximizers of f on the constraint curve must satisfy $\nabla f(\mathbf{x})\cdot\nabla g_1(\mathbf{x})\times\nabla g_2(\mathbf{x}) = 0$. Computing the triple product, we find the equation

$$4x(y^2-z^2)+z(4y-6x)+8y = 0\ .$$

This equation, together with the two constraint equations is the system to be solved to find minimizers and maximizers:

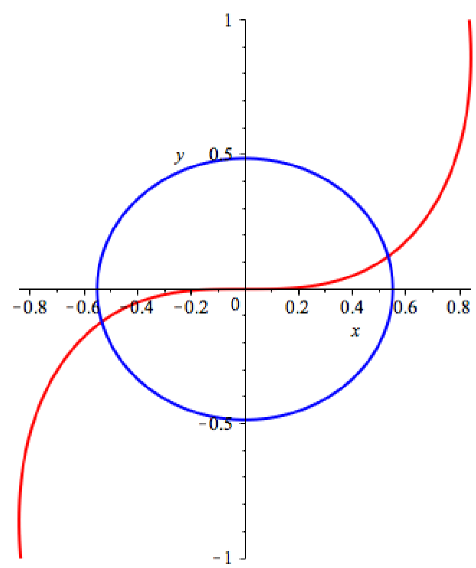
$$\begin{aligned} 4x(y^2-z^2)+z(4y-6x)+8y &= 0 \\ 3x^2+4y^2+z^2 &= 1 \\ z-x^2-y^2 &= 0 \end{aligned}$$

For **(b)**, we substitute $z = x^2+y^2$ from the third equation into the first two to obtain:

$$\begin{aligned} 4x^3(x^2-2y^2)+4xy^4+6x^3-4y^3+2xy^2-4x^2y-8y &= 0 \\ 3x^2+4y^2+(x^2+y^2)^2-1 &= 0\ . \end{aligned}$$

The second equation describes a curve that is very close to being a circle of radius $1/2$. If the coefficient of x^2 were 4 instead of 3, the equation would describe a circle of radius R with $R^2 = \sqrt{5}-2$. In this case, $R \approx 0.4858682712$. If the coefficient of y^2 were 3 instead of 4, the equation would describe a circle of radius R with $R^2 = (\sqrt{13}-3)/2$. In this case, $R \approx 0.5502505225$. The actual curve must lie between these two circles, so it is well approximated by a circle of radius $1/2$.

For points (x,y) on the circle of radius $1/2$, the third and fifth order terms are relatively small compared to the first order term. Neglecting higher order terms, the first equation is approximated by $8y = 0$, or more simple, $y = 0$. In particular, the contour curve described by the first equation passes through the origin and is tangent to the x-axis at the origin. Using methods from earlier chapters, we can easily sketch the curve inside the circle of radius $1/2$. Here is a plot showing both curves:



We see that there are exactly two solutions. One gives us the maximizer, and one the minimizer. We take as out starting points $(1/2,0)$ and $(-1/2,0)$. Since $z>0$ at each of the, f will evidently be maximized at the solution in the upper right quadrant, and minimized at the solution in the lower left quadrant.

Define the function

$$\mathbf{h}(x,y) = (4x^3(x^2-2y^2)+4xy^4+6x^3-4y^3+2xy^2-4x^2y-8y\ ,\ 3x^2+4y^2+(x^2+y^2)^2-1)\ .$$

Taking $\mathbf{x}_0 = (1/2,0)$, Newton's method gives us

$$\mathbf{x}_1 = \mathbf{x}_0 - [D_{\mathbf{h}}(\mathbf{x}_0)]^{-1}\mathbf{h}(\mathbf{x}_0)\ .$$

Evaluating this, we find

$$\mathbf{x}_1 = (1/2,0) - \left(\frac{1}{4}\begin{bmatrix} -23 & 36 \\ 14 & 0 \end{bmatrix}\right)\frac{1}{16}(-14,-3) = \frac{1}{2016}(1116,256) \approx (0.55,0.13)\ .$$

Notice that the graph suggests that $\mathbf{x}_1$ is a much better approximate solution than $\mathbf{x}_0$, and indeed it is: Evaluating, we find

$$h(\mathbf{x}_1) \approx (-0.0472\ ,\ 0.0932)\ .$$

The iteration converges rapidly form here.

We now repeat for $\mathbf{x}_0 = (-1/2,0)$. Newton's method gives us

$$\mathbf{x}_1 = \mathbf{x}_0 - [D_{\mathbf{h}}(\mathbf{x}_0)]^{-1}\mathbf{h}(\mathbf{x}_0)\ .$$

Evaluating this, we find

$$\mathbf{x}_1 = (-1/2,0) - \left(\frac{1}{4}\begin{bmatrix} -23 & 36 \\ -14 & 0 \end{bmatrix}\right)\frac{1}{16}(14,-3) = \frac{1}{2016}(900,256) \approx (0.45,-0.13)\ .$$

Notice that the graph suggests that $\mathbf{x}_1$ is a much better approximate solution than $\mathbf{x}_0$, and indeed it is, as before.

4: Find an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors for the matrix $A = \begin{bmatrix} 5 & 9 \\ 9 & -4 \end{bmatrix}$.

SOLUTION We compute $\det(A-\lambda I) = \lambda^2-\lambda-101$. The roots are $\lambda_\pm = (1\pm 9\sqrt{5})/2$. Then

$$A - \lambda_+ I = \frac{9}{2}\begin{bmatrix} 1-\sqrt{5} & 2 \\ 2 & -1-\sqrt{5} \end{bmatrix}\ .$$

Therefore,

$$\mathbf{v}_1 = (2,\sqrt{5}-1) \quad \text{and} \quad \mathbf{v}_2 = (1-\sqrt{5},2)$$

are eigenvectors of A with eigenvalues $(1+9\sqrt{5})/2$ and $(1-9\sqrt{5})/2$ respectively. Normalizing them, we find $\|\mathbf{v}_1\|^2 = \|\mathbf{v}_2\|^2 = 10-2\sqrt{5}$. Hence, our orthonormal basis $\{\mathbf{u}_1,\mathbf{u}_2\}$ is given by

$$\mathbf{u}_1 = \frac{1}{\sqrt{10-2\sqrt{5}}}(2,\sqrt{5}-1) \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sqrt{10-2\sqrt{5}}}(1-\sqrt{5},2)\ .$$

5: Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors for the matrix $A = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$.

SOLUTION We compute $\det(A - \lambda I) = -\lambda^3 + 9\lambda^2 - 18\lambda = -\lambda(\lambda-3)(\lambda-6)$. The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = 6$. Any eigenvector with eigenvalue 0 must be orthogonal to all three rows of $A = A - 0I$. Hence the third row must lie in the plane spanned by the first two. We can get the orthogonal vector we need by taking the cross product of the first two rows. After normalizing we find $\mathbf{u}_1 = \frac{1}{3}(-1, -2, 2)$ is a unit eigenvector of A with eigenvalue 0. Doing the same for $A - 3I$ and $A - 6I$ we find that $\mathbf{u}_3 = \frac{1}{3}(2, -2, -1)$ is a unit eigenvector of A with eigenvalue 3 and $\mathbf{u}_2 = \frac{1}{3}(2, 1, 2)$ is a unit eigenvector of A with eigenvalue 6. Since eigenvectors of a symmetric matrix with different eigenvalues are automatically orthogonal, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthonormal.

6: Consider the function $f(x,y) = xy^2 - \frac{1}{4}x^4 - \frac{1}{2}y^4$.

(a) Find all critical points, and compute the Hessian matrix at each critical point.

(b) Determine whether each critical point is a local minimum, a local maximum, a saddle, or if the Hessian is degenerate at the critical point.

(c) Sketch a contour plot of f in the vicinity of each critical point.

SOLUTION We compute $\nabla f(x,y) = (y^2 - x^3, 2y(x - y^2))$. Hence if (x,y) is a critical point, either $y = 0$ or $x = y^2$. In the first case $x = 0$, while in the second case $y = \pm 1$. Hence we have three critical points,

$$\mathbf{x}_1 = (0,0) \ , \qquad \mathbf{x}_2 = (1,1) \quad \text{and} \quad \mathbf{x}_3 = (1,-1) \ .$$

We next compute $\text{Hess}_f(x,y)] = \begin{bmatrix} -3x^2 & 2y \\ 2y & 2x - 6y^2 \end{bmatrix}$. Substituting in,

$$\text{Hess}_f(\mathbf{x}_1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \ , \quad \text{Hess}_f(\mathbf{x}_2) = \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix} \ , \quad \text{Hess}_f(\mathbf{x}_3) = \begin{bmatrix} -3 & -2 \\ -2 & -4 \end{bmatrix} \ .$$

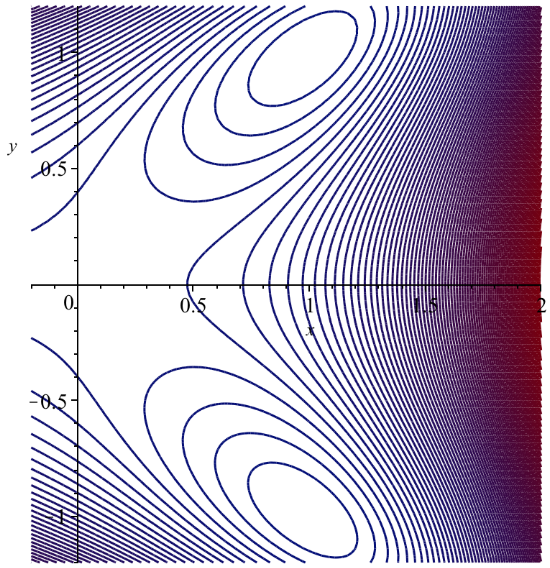
Evidently, the eigenvalues of $\text{Hess}_f(\mathbf{x}_1)$ are 0 and 0, so this critical point is degenerate. Next, $\det(\text{Hess}_f(\mathbf{x}_2)) = \det(\text{Hess}_f(\mathbf{x}_3)) = -1/2 < 0$, so $\mathbf{x}_2$ and $\mathbf{x}_3$ are saddle points.

Here is a contour plot near the three critical points. As our computations showed, the contours near $\mathbf{x}_2$ and $\mathbf{x}_3$ are ellipses.

To get the major and minor axes right, we compute the eigenvalues at $\mathbf{x}_2$, which are $\mu_\pm = (-7 \pm \sqrt{17})/2$. The eigenvector corresponding to μ_+ has the smallest absolute value, so the corresponding eigenvectors point along the majot axis. Such an eigenvector is given by

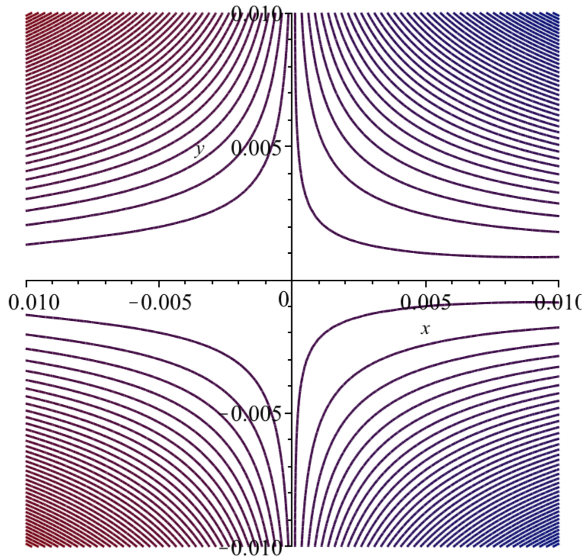
$$(2, (\sqrt{17} - 1)/2(\ .$$

so to get a sketch, draw an ellipse with the major axis parallel to this vector.



The contour plot near $\mathbf{x}_3$ is the mirror image of this point obtained by reflecting about the x axis.

Here is a detialed plot near $\mathbf{x}_1$, the degenerate critical point.



To generate this plot with pencil and paper, note that near $\mathbf{x}_1 = (0, 0)$, x^4 and y^4 are negligibly small compared with xy^2, so $f(x, y) \approx xy^2$. You get the local contour plot by graphing the curves $xy^2 = c$, or $x = c/y^2$, as you see here.