Solutions for Homework 6, Math 291 Fall 2017

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1: Let A be the matrix $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$.

(a) Compute A^{-1}m the inverse matrix of A.

(b) Find the solution of the equations

$$A\mathbf{x} = (3,2) \ , \qquad A\mathbf{x} = (2,2) \ , \qquad A\mathbf{x} = (\,-1,7) \ .$$

SOLUTION: **(a)** We compute $\det(A) = -1$, and then $A^{-1} = \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}$. The solutions in **(b)** are given by

$$\begin{aligned} \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}(3,2) &= (1,0) \\ \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}(3,2) &= (4,-2) \\ \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}(3,2) &= (38,-23) \end{aligned}$$

Note: To solve $A\mathbf{x} = (3,2)$, we do not need to use the inverse if we notice that $(3,2)$ is the first column of A, and recall that applying any matrix to $\mathbf{e}_j$ gives the jth column of that matrix. This is why the solution is $\mathbf{e}_1 = (1,0)$.

2: Let A be the matrix $A = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Compute A^{-1}m the inverse matrix of A.

(b) Find the solution of the equations

$$A\mathbf{x} = (3,2,1) \ , \qquad A\mathbf{x} = (2,2,1) \ , \qquad A\mathbf{x} = (\,-1,7,1) \ .$$

SOLUTION: Write $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. We then compute

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 &= (-1, 2, -1) \\ \mathbf{v}_1 \times \mathbf{v}_3 &= (3, -5, 1) \\ \mathbf{v}_2 \times \mathbf{v}_3 &= (5, -9, 2)\ . \end{aligned}$$

We then compute $\det(A) = \mathbf{v}_1 \times \mathbf{v}_2 \cdot \mathbf{v}_3 = -1$, and then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \mathbf{v}_2 \times \mathbf{v}_3 \\ \mathbf{v}_1 \times \mathbf{v}_3 \\ \mathbf{v}_1 \times \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -5 & 9 & -2 \\ -3 & 5 & -1 \\ 1 & -2 & 1 \end{bmatrix}\ .$$

For **(b)** since $(3, 2, 1)$ is the first column of A, the solution of $A\mathbf{x} = (3, 2, 1)$ is $(1, 0, 0)$. For the others, we multiply by the inverse to fond the solutions $(6, -3, -1)$ and $(66, -37, -14)$, respectively.

3: Let A be the 3×4 matrix $A = \begin{bmatrix} 6 & -6 & 4 & -7 \\ -2 & 2 & 1 & -7 \\ -3 & 3 & -9 & 7 \end{bmatrix}$.

(a) Apply the Gram-Schmidt Algorithm to the columns of A to produce an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$. Let Q be the matrix $Q = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. Let R be the 3×4 matrix whose i, j entry is $\mathbf{u}_i \cdot \mathbf{v}_j$ where $\mathbf{v}_j$ is the jth column of Q. Verify that $A = QR$.

(b) Find all solutions of $R\mathbf{x} = \mathbf{0}$, and then all solutions of $A\mathbf{x} = \mathbf{0}$.

(c) For $\mathbf{b} = (7, 7, 7)$, find all solutions of $R\mathbf{x} = \mathbf{b}$, and then find all solutions of $A\mathbf{x} = \mathbf{b}$.

SOLUTION: (a) Write $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$. Note that $\mathbf{v}_2 = -\mathbf{v}_1$, so these two vectors yield the same direction, and $\mathbf{v}_2$ will not contribute anything in the Gram-Schmidt Algorithm. We can skip it without any computation since we noticed this. We then proceed to construct $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ by doing the usual computations. Writing the result in the form $Q = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ we find

$$Q = \frac{1}{7} \begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{bmatrix}\ .$$

We then compute

$$R = Q^T A = \begin{bmatrix} 7 & -7 & 7 & -7 \\ 0 & 0 & 7 & -7 \\ 0 & 0 & 0 & 7 \end{bmatrix}\ .$$

Now that we have Q and R we can multiply Q and R to verify that $A = QR$. Even more simply, it suffices to verify that $Q^T Q = I$ since then Q^T is the inverse of Q, and so the definition of R becomes $R = Q^{-1}A$, which means that $A = QR$. But $Q^T Q = I$ is equivalent to the columns of Q being orthonormal, so we don't need to do any computation at all.

For **(b)**, Let $\mathbf{x} = (x, y, z, w)$, and suppose that $R\mathbf{x} = 0$. The corresponding system of equations (after eliminating a common factor of 7) is

$$\begin{aligned} x - y + z - w &= 0 \\ z - w &= 0 \\ w &= 0\ . \end{aligned}$$

Thus $R\mathbf{x} = 0$ if and only if $w = z = 0$ and $y = x$. Thus the general solution has the form

$$(x, -x, 0, 0) = x(1, -1, 0, 0) \ .$$

The space of solutions is one the one dimensional subspace of $\mathbb{R}^4$ spanned by the vector $(1, -1, 0, 0)$.

For **(c)**, again let $\mathbf{x} = (x, y, z, w)$, and suppose that $R\mathbf{x} = (7, 7, 7)$. The corresponding system of equations (after eliminating a common factor of 7) is

$$\begin{aligned} x - y + z - w &= 1 \\ z - w &= 1 \\ w &= 1 \ . \end{aligned}$$

The last equation requires $w = 1$. Then the second equation requires $z = 2$. The first equation then requires $y = x$. Thus $R\mathbf{x} = (7, 7, 7)$ if and only if $w = 1$, $z = 2$ and $y = x$. Thus the general solution has the form

$$(x, -x, 2, 1) = x(1, -1, 0, 0) + (0, 0, 2, 1) \ .$$

To solve $A\mathbf{x} = (7, 7, 7)$, we do something similar: We multiply on the left by Q^T to obtain $Q^T A\mathbf{x} = Q^T(7, 7, 7) = (1, -5, -11)$. The corresponding system of equations (after eliminating a common factor of 7) is

$$\begin{aligned} x - y + z - w &= 1/7 \\ z - w &= -5/7 \\ w &= -11/7 \ . \end{aligned}$$

The last equation requires $w = -11/7$. Then the second equation requires $z = -16/7$. The last equation becomes $x - y = 6/7$ We get a particular solution by choosing $y = 0$, and then $x = 6/7$. Hence

$$\mathbf{x}_0 = \frac{1}{7}(6, 0, -16, -11) \ .$$

To get the general solution, we add to this the general solution of $R\mathbf{x} = \mathbf{0}$, which is also the general solution of $A\mathbf{x} = \mathbf{0}$ that we have found above. Hence the general solution is given by

$$\frac{1}{7}(6, 0, -16, -11) + x(1, -1, 0, 0) \ ,$$

as x ranges over the real line.

4: (a) Let $\mathbf{f}(x, y) = ((x^3 - x^2)y \ , \ xy + x - y)$. Compute $D_{\mathbf{f}}(x, y)$, and evaluate this at $(x, y) = (-1, 1)$ to obtain the matrix $A = D_{\mathbf{f}}(-1, 1)$. Compute the inverse of this matrix.

(b) Let $\mathbf{g}(u, v) = (u^2 - v^2 + uv \ , \ u^3/3 - v^2)$. Compute $D_{\mathbf{g}}(u, v)$, and evaluate this at $(x, y) = (-2, -3)$ to obtain the matrix $B = D_{\mathbf{g}}(-2, -3)$. Compute the inverse of this matrix.

SOLUTION: (a) We compute $D_{\mathbf{f}}(x, y) = \begin{bmatrix} (3x^2 - 2x)y & x^3 - x^2 \\ y + 1 & x - 1 \end{bmatrix}$. Evaluating we find $D_{\mathbf{f}}(-1, 1) = \begin{bmatrix} 5 & -2 \\ 2 & -2 \end{bmatrix}$. The inverse of this matrix is

$$\frac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & -5 \end{bmatrix} \ .$$

For **(b)** we compute $D_{\mathbf{g}}(u,v) = \begin{bmatrix} 2u+v & u-2v \\ u^2 & -2v \end{bmatrix}$. Evaluating we find $D_{\mathbf{g}}(-2,-3) = \begin{bmatrix} -7 & 7 \\ 4 & 6 \end{bmatrix}$. The inverse of this matrix is

$$\frac{1}{58}\begin{bmatrix} -6 & 4 \\ 4 & 7 \end{bmatrix} .$$

5: Let the functions $\mathbf{f}$ and $\mathbf{g}$ be defined as in Exercise 4. Since the range of $\mathbf{f}$ is $\mathbb{R}^2$, which is the domain of $\mathbf{g}$, the composition $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ of these functions is well-defined: Define

$$\mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x})) .$$

(a) Compute an explicit formula for $\mathbf{h}(x,y)$, and then compute the matrix $C = D_{\mathbf{h}}(-1,1)$.
(b) Use the calculation of part **(a)** and Exercise 4 to verify the Chain Rule:

$$D_{\mathbf{g}\circ\mathbf{f}}(\mathbf{x}_0) = [D_{\mathbf{g}}(\mathbf{f}(\mathbf{x}_0))][D_{\mathbf{f}}(\mathbf{x}_0)]$$

for these functions at $\mathbf{x}_0 = (1,2)$.

SOLUTION: (a) Making the substitution and simplifying, we find $\mathbf{h}(x,y) = (h_1(x,y), h_2(x,y))$ where

$$\begin{aligned} h_1(x,y) &= (x^6 - 2x^5 + 2x^4 - 2x^3 + 2x - 1)y^2 + (x^4 - x^3 - 2x^2 + 2x)y - x^2 \\ h_2(x,y) &= \frac{1}{3}(x^9 - 3x^8 + 3x^7 - x^6)y^3 - (x^2 - 2x + 1)y^2 + 2(x^2 - x)y - x^2 . \end{aligned}$$

Computing the Jacobian matrix and evaluating, we find

$$C = D_{\mathbf{h}}(-1,1) = \begin{bmatrix} -27 & 6 \\ 32 & -20 \end{bmatrix} .$$

Then since $\mathbf{f}(-1,1) = (-2,-3)$,

$$[D_{\mathbf{g}}(\mathbf{f}(\mathbf{x}_0))][D_{\mathbf{f}}(\mathbf{x}_0)] = [D_{\mathbf{g}}(-2,-3)][D_{\mathbf{f}}(-1,1)] ,$$

which is the product of the matrices found in part **(a)**. Using the results of part **(a)**,

$$[D_{\mathbf{g}}(-2,-3)][D_{\mathbf{f}}(-1,1)] = \begin{bmatrix} -8 & 1 \\ 9 & 4 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -27 & 6 \\ 32 & -20 \end{bmatrix} ,$$

in accordance with the Chain Rule.