Homework 6, Math 291 Fall 2017

Eric A. Carlen[1]
Rutgers University

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1: Let A be the matrix $A = \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$.

(a) Compute A^{-1}m the inverse matrix of A.

(b) Find the solution of the equations

$$A\mathbf{x} = (3,2)\ , \qquad A\mathbf{x} = (2,2)\ , \qquad A\mathbf{x} = (\,-1,7)\ .$$

2: Let A be the matrix $A = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Compute A^{-1}m the inverse matrix of A.

(b) Find the solution of the equations

$$A\mathbf{x} = (3,2,1)\ , \qquad A\mathbf{x} = (2,2,1)\ , \qquad A\mathbf{x} = (\,-1,7,1)\ .$$

3: Let A be the 3×4 matrix $A = \begin{bmatrix} 6 & -6 & 4 & -7 \\ -2 & 2 & 1 & -7 \\ -3 & 3 & -9 & 7 \end{bmatrix}$.

(a) Apply the Gram-Schmidt Algorithm to the columns of A to produce an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$. Let Q be the matrix $Q = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. Let R be the 3×4 matrix whose i,j entry is $\mathbf{u}_i \cdot \mathbf{v}_j$ where $\mathbf{v}_j$ is the jth column of A. Verify that $A = QR$.

(b) Find all solutions of $R\mathbf{x} = \mathbf{0}$, and then all solutions of $A\mathbf{x} = \mathbf{0}$.

(c) For $\mathbf{b} = (7,7,7)$, find all solutions of $R\mathbf{x} = \mathbf{b}$, and then find all solutions of $A\mathbf{x} = \mathbf{b}$.

4: (a) Let $\mathbf{f}(x,y) = ((x^3 - x^2)y\ ,\ xy + x - y)$. Compute $D_{\mathbf{f}}(x,y)$, and evaluate this at $(x,y) = (-1,1)$ to obtain the matrix $A = D_{\mathbf{f}}(-1,1)$. Compute the inverse of this matrix.

(b) Let $\mathbf{g}(u,v) = (u^2 - v^2 + uv\ ,\ u^3/3 - v^2)$. Compute $D_{\mathbf{g}}(u,v)$, and evaluate this at $(x,y) = (\,-2,-3)$ to obtain the matrix $B = D_{\mathbf{g}}(-2,-3)$. Compute the inverse of this matrix.

5: Let the functions $\mathbf{f}$ and $\mathbf{g}$ be defined as in Exercise 4. Since the range of $\mathbf{f}$ is $\mathbb{R}^2$, which is the domain of $\mathbf{g}$, the composition $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ of these functions is well-defined: Define

$$\mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))\ .$$

(a) Compute an explicit formula for $\mathbf{h}(x, y)$, and then compute the matrix $C = D_{\mathbf{h}}(-1, 1)$.

(b) Use the calculation of part **(a)** and Exercise 4 to verify the chain rule:

$$D_{\mathbf{g}\circ\mathbf{f}}(\mathbf{x}_0) = [D_{\mathbf{g}}(\mathbf{f}(\mathbf{x}_0))][D_{\mathbf{f}}(\mathbf{x}_0)]$$

for these functions at $\mathbf{x}_0 = (1, 2)$.

5: Let $\mathbf{f}(x, y) = (\,-yx + 1 + x^2\ ,\ x^2 + (y/4)^2 - 1)$.

(a) How many solutions of the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ are the in the upper-right quadrant? (The upper right quadrant is the set $\{(x, y :\ x, y \geq 0\}$.

(b) Take $\mathbf{x}_0 = (2, 1)$, and compute $\mathbf{f}(\mathbf{x}_0)$ and $[D_{\mathbf{f}}(\mathbf{x}_0)]$ Then solve the equation

$$\mathbf{f}(\mathbf{x}_0) + [D_{\mathbf{f}}(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0) = \mathbf{0}$$

for $\mathbf{x}$. Call the solution $\mathbf{x}_1$. Compute $\mathbf{f}(\mathbf{x}_1)$, and compare $\|\mathbf{f}(\mathbf{x}_1)\|$ and $\|\mathbf{f}(\mathbf{x}_0)\|$.

This is essentially the Newton's method problem from the previous homework done using matrix methods.