Homework 5, Math 291 Fall 2017

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1: Let $f(x,y)$ be given by

$$f(x,y) = -xy^2 + 3y + x^2y \ .$$

(a) Find all critical points of f.

(b) Find the equation of the tangent plane to the graph of $z = f(x,y)$ at $(x,y) = (2,1)$.

(c) Suppose the positive y-axis runs due North, and the positive x-axis runs due East. If $z = f(x,y)$ represents the altitude at the point (x,y), and you pour out a glass of water at $(2,1)$, in what compass direction does the water run?

(d) Find all points (x,y) such that the normal vector to the tangent plane at (x,y) is a multiple of $(1,1,1)$.

SOLUTION We compute

$$\nabla f(x,y) = (\,-y^2 + 2xy\ ,\ -2xy + 3 + x^2)\ .$$

The critical points are the solutions of

$$\begin{aligned} -y^2 + 2xy &= 0 \\ -2xy + 3 + x^2 &= 0 \end{aligned}$$

The first equation is satisfied if and only if either $y = 0$ or $y = 2x$. If $y = 0$, the second equation becomes $x^2 + 3 = 0$, which has no solutions. If $y = 2x$, the second equation becomes $x^2 = 1$, and so we get two critical points $\mathbf{x}_1 = (1,2)$ and $\mathbf{x}_2 = (\,-1,-2)$.

For **(b)**, since $\nabla f(2,1) = (3,3)$, the normal vector is $(3,3,-1)$. Then with $\mathbf{X} = (x,y,z)$ and $\mathbf{x}_0 = (2,1,f(2,1)) = (2,1,5)$, the equation of the tangent plane is $\mathbf{A}\cdot(\mathbf{X} - \mathbf{X}_0)$ which simplifies to

$$3x + 3y - z = 4\ .$$

For **(c)** Since the gradient points Northeast, the water runs Southwest.

For **(c)** The normal to the tangent plane at (x,y) is is is $(\,-y^2+2xy\ ,\ -2xy+3+x^2,-1)$. This vector is a multiple of $(1,1,1)$ exactly when the cross product of the two vectors is zero. Computing the cross product we find

$$(\,-2xy + 4 + x^2\ ,y^2 - 2xy - 1\ ,\ 4xy - y^2 - x^2 - 3\ )\ .$$

Setting this equal to **0** gives us the system

$$\begin{aligned} -2xy + 4 + x^2 &= 0 \\ y^2 - 2xy - 1 &= 0 \\ 4xy - y^2 - x^2 - 3 &= 0 \end{aligned}$$

Subtracting the first equation from the second we get $y^2 = x^2 + 5$. Solving the second equation for x we get

$$x = \frac{y^2 - 1}{2y} \quad \text{so that} \quad x^2 = \frac{y^4 - 2y^2 + 1}{4y^2} \ .$$

Then $y^2 = x^2 + 5$ becomes

$$y^2 = \frac{y^4 - 2y^2 + 1}{4y^2} + 5 \quad \text{so that} \quad 3y^4 - 18y^2 - 1 = 0 \ .$$

Introducing $t = y^2$, we have quadratic equation in t. The two solutions are

$$3 \pm \frac{2}{3}\sqrt{21} \ .$$

Only the larger one is positive, and hence an admissible value for y^2. Hence

$$y^2 = 3 + \frac{2}{3}\sqrt{21} \quad \text{and} \quad x^2 = -2 + \frac{2}{3}\sqrt{21} \ .$$

Since $y^2 - 1 > 0$, we see from the second equation, $2xy = y^2 - 1$, the x and y have the same sign. Hence there are exactly two such points

$$\left(\sqrt{-2 + \frac{2}{3}\sqrt{21}}\ ,\ \sqrt{3 + \frac{2}{3}\sqrt{21}}\right) \quad \text{and} \quad \left(-\sqrt{-2 + \frac{2}{3}\sqrt{21}}\ ,\ -\sqrt{3 + \frac{2}{3}\sqrt{21}}\right) \ .$$

2: Let $f(x, y)$ be given by

$$f(x, y) = xy^2 - x^4 - y^4 \ .$$

(a) Find all critical points of f.

(b) Find the maximum value of f over the whole plane, if there is one, and in that case, find all maximizers of f. Justify your answer.

(c) Find the minimum value of f over the whole plane, if there is one, and in that case, find all maximizers of f. Justify your answer.

(d) Consider the curve specified by

$$xy^2 - x^4 - y^4 = 0 \ .$$

Find all points on this curve where the tangent vector is vertical.

SOLUTION We compute

$$\nabla f(x, y) = (y^2 - 4x^3\ ,\ 2xy - 4y^3) \ .$$

The critical points are the solutions of

$$\begin{aligned} y^2 - 4x^3 &= 0 \\ 2xy - 4y^3 &= 0 \end{aligned}$$

The second equation is satisfied if and only if either $y = 0$ or $x = 2y^2$. If $y = 0$, the first equation requires $x = 0$, giving us the critical point $\mathbf{x}_1 = (0,0)$. If $x = 2y^2$, the first equation becomes$y^2 = 2^3y^6$. Hence $y = \pm 2^{-5/4}$, and then $x = 2^{-3/2}$. Hence we get two more critical points $\mathbf{x}_2 = (2^{-3/2}, 2^{-5/4})$ and $\mathbf{x}_3 = (2^{-3/2}, -2^{-5/4})$

For **(b)** and **(c)** Note that for large $x^2 + y^2$, the leading order approximation of f is $f(x,y) \approx -x^4 - y^4$. Hence f is not bounded below, so there is no minimizer. Moreover, outside and on the boundary of large closed disk, the values of f are strictly negative. We know f has a maximizer of this closed disk since f is continuous. The maximum cannot be on the boundary since $f(0,0) = 0$, and on the boundary all values of f are strictly negative. Hence at any maximizer $\nabla f = 0$, Evaluating f at out three critical points we find

$$f(\mathbf{x}_1) = 0 \quad \text{and} \quad f(\mathbf{x}_2) = f(\mathbf{x}_3) = 1/64 \ .$$

Hence the maximum value is $1/64$, and the maximizers are $\mathbf{x}_2$ and $\mathbf{x}_3$.

For **(d)** The tangent vector is vertical when the gradient is horizontal, and hence when the second component of the gradient is zero. Hence we have the system of equations

$$\begin{aligned} xy^2 - x^4 - y^4 &= 0 \\ 2xy - 4y^3 &= 0 \end{aligned}$$

The second equation is satisfied if and only if either $y = 0$ or $x = 2y^2$. If $y = 0$, the first equation requires $x = 0$, giving us the solution $\mathbf{z}_1 = (0,0)$. If $x = 2y^2$, the first equation becomes $y^4 = 2^4y^8$, so that $y = \pm 1/2$ and then $x = 1/2$. Hence there are two more such points, $\mathbf{z}_2 = (1/2, 1/2)$ and $\mathbf{z}_3 = (1/2, -1/2)$.

3: Let $f(x,y)$ be given by

$$f(x,y) = \frac{x^2y}{1 + x^4 + y^4} \ .$$

(a) Find all critical points of f.

(b) Find the maximum value of f over the whole plane, if there is one, and in that case, find all maximizers of f. Justify your answer.

(c) Find the minimum value of f over the whole plane, if there is one, and in that case, find all maximizers of f. Justify your answer.

(d) Consider the curve specified by

$$f(x,y) = \frac{1}{16} \ .$$

Find all points on this curve where the tangent vector is vertical.

SOLUTION We compute

$$\nabla f(x,y) = \frac{x}{(1 + x^4 + y^4)^2} (2y(x^4 - y^4 - 1) \ , \ x(1 + x^4 - 3y^4)) \ .$$

Notice that the pre-factor is zero exactly when $x = 0$. Hence every point on the y-axis is a critical point. Critical points (x, y) that are not on the y-axis must satisfy

$$\begin{aligned} x^4 - y^4 - 1 &= 0 \\ 1 + x^4 - 3y^4 &= 0 \end{aligned}$$

Subtracting the first equation from the second we see that $y^2 = 1$ and hence $y = \pm 1$. Next, $x^4 = 2$ so that $x = \pm 2^{1/4}$. The signs are uncoupled this time and so we have four critical points

$$\mathbf{x}_1 = (2^{1/4}\ ,\ 1)\ ,\quad \mathbf{x}_2 = (-2^{1/4}\ ,\ 1)\ ,\quad \mathbf{x}_3 = (2^{1/4}\ ,\ -1)\ ,\quad \mathbf{x}_4 = (-2^{1/4}\ ,\ -1)\ .$$

For **(b)** and **(c)** Since it is clear that $\lim_{\mathbf{x}\to\infty} f(\mathbf{x}) = 0$, and since $f(1,1) = 1/3$ and $f(1,-1) = -1/3$, this function has both maximizers and minimizers, and they must be critical point. Evaluating, we find $f(\mathbf{x}_1) = f(\mathbf{x}_2) = 2^{-3/2}$ and $f(bx_3) = f(\mathbf{x}_4) = -2^{-3/2}$. Hence $2^{-3/2}$ is the maximum value and $\mathbf{x}_1$ and $\mathbf{x}_2$ are the maximizers, while $-2^{-3/2}$ is the minimum value and $\mathbf{x}_3$ and $\mathbf{x}_3$ are the minimizers.

(d) The tangent vector is vertical exactly where the gradient is horizontal, meaning that $\partial f/\partial y = 0$. Since

$$\frac{\partial}{\partial y} f(x,y) = \frac{x^2(1 + x^4 - 3y^4)}{1 + x^2 + y^2}\ ,$$

the equation $\dfrac{\partial}{\partial y} f(x,y) = 0$ is equivalent to

$$x^2(1 + x^4 - 3y^4) = 0\ .$$

Hence the points on the curve $f(x,y) = 1/16$ where the tangent is vertical satisfy the pair of equations

$$\begin{aligned} 16x^2y - 1 - x^4 - y^4 &= 0 \\ x^2(1 + x^4 - 3y^4) &= 0\ . \end{aligned}$$

Clearly, the second equation is satisfied if $x = 0$, but then the first equation becomes $-1 - y^4 = 0$, which has no solution. Likewise, if we set $y = 0$ in the first equation, we obtain $-1 - x^4 = 0$ which has no solutions. Hence $x \neq 0$ and $y \neq 0$. Since $x \neq 0$, we may divide the second equation by x^2, obtaining the system

$$\begin{aligned} 16x^2y - 1 - x^4 - y^4 &= 0 \\ 1 + x^4 - 3y^4 &= 0\ . \end{aligned}$$

Summing both sides the two equations, we obtain $4x^2y - y^4 = 0$. Since we found above that $y \neq 0$, we may divide by y to obtain $4x^2 = y^3$. Using this to eliminate x from the equation $16x^2y - 1 - x^4 - y^4 = 0$, we obtain $3y^4 - 1 - \dfrac{1}{16}y^6 = 0$. This is equivalent to

$$y^6 - 48y^2 + 16 = 0\ .$$

Introducing the variable $t = y^3$, we seek the roots of the cubic

$$t^3 - 48t^2 + 16 = 0\ .$$

Before actually solving this equation, let us look at its solution set. Define $p(t) = t^3 - 48t^2 + 16$. Then $p'(t) = 3t^2 - 96t = 3t(t-32)$ and $p''(t) = 6(t-16)$. From this we see that p has a local maximum at $t = 0$, and a local minimum at $t = 32$. Since $p(0) = 16$ and $p(32) = -16368$, the graph of $y = p(t)$ crosses the t axis 3 times; so there are there roots r_1, r_2 and r_3, one of which is negative and two of which are positive. We may as well suppose that $r_1 < r_2 < r_3$ so that r_2 and r_3 are the positive roots. We are only concerned with the positive roots since we got the cubic equation by setting $t = y^2$m and y^2 is never negative. Moreover, at the points we seek, we know that $4x^2 = y^3$. This means the y itself is positive, and $x = \frac{1}{2}y^{3/2}$ Hence there are two points at which the normal to the tangent plane is a multiple of $(1,1,1)$: These are

$$\left(\tfrac{1}{2}r_2^{3/3}, r_2^{1/2}\right) \qquad \text{and} \qquad \left(\tfrac{1}{2}r_3^{3/3}, r_3^{1/2}\right) .$$

You get full credit for getting to here. The rest is not really multivariable calculus *per se*, but it is very interesting for several reasons. The method for solving such equations tells us the we should substitute

$$t = 16 + 2w + \frac{128}{w} \tag{0.1}$$

into our equation. Let's take this for granted. The interesting thing is what comes next. In the new variable w, $p(t) = 0$ becomes

$$\frac{8}{w^3}(w^6 - 1022w^3 + 262144) = 0 ,$$

which is satisfied if and only if $w^6 - 1022w^3 + 262144 = 0$. This is a quadratic equation in w^2. (Note that $262144 = 2^{18}$, a first hint something reasonable is coming out.) Solving the quadratic, we find

$$w^3 = 511 \pm i\sqrt{1023} .$$

This is complex, but our equation has three real roots. Hence the set of roots do not change under complex conjugation, and the three real roots are what one gets plugging the three complex cube roots of $511 + i\sqrt{1023}$ into (0.1).

To find these, write $w^3 = 511 + i\sqrt{1023}$ in polar form: Note that $\sqrt{(511)^2 + 1023} = 512 = 2^9$. Define $\theta_0 = \arccos(511/512)$, we have $w^3 = 2^9 e^{i\theta_0}$. Then the three values of w are

$$w_1 = 8e^{i\theta_0/3} , \quad w_2 = 8e^{i\theta_0/3 + 2\pi/3} , \quad w_2 = 8e^{i\theta_0/3 + 4\pi/3} .$$

Next note that when $|w|^2 = 64$, as in our case.

$$16 + 2w + \frac{128}{w} = 16 + 2w + \frac{128}{|w|^2}w^* = 16 + 2(w + w^*) ,$$

which is real, as it must be. Using the three choices of w, we get the three real roots of the cubic:

$$r_3 = 16 + 32\cos((\arccos(511/512))/3) , \quad r_2 = 16 + 32\cos((\arccos(511/512))/3 + 4\pi/3)$$

These have the approximate values $r_2 \approx 0.58087572$ and $r_2 \approx 47.99305355$.

4: Let $f(x,y)$ be given by

$$f(x,y) = \frac{x^2y + y^2}{1 + x^4 + y^2} .$$

(a) Compute equations for the tangent planes to the graph of $z = f(x, y)$ at $(1, 1)$ and at $(-1, -1)$.

(b) These planes intersect in a line. Find a parameterization of the line.

SOLUTION We compute

$$\nabla f(x, y) = \frac{1}{(1 + x^4 + y^2)^2}(2xy(x^4 + 2x^2y - y^2 - 1)\ ,\ x^2 + x^6 + 2y + 2yx^4 - x^2y^2)\ .$$

Then we find $\nabla f(1, 1) = \frac{1}{9}(-2, 5)$ and $\nabla f(1, 1) = \frac{1}{3}(2, -1)$, and $f(1, 1) = \frac{2}{3}$ and $f(-1, -1) = 0$.

Therefore, the equation for the tangent plane at $(1, 1)$ is

$$(-2, 5, -9) \cdot (x - 1, y - 1, z - 2/3) = 0 \qquad \text{or equivalently} \quad -2x + 5y - 9z = -3$$

and the equation for the tangent plane at $(1, 1)$ is

$$(2, -1, -3) \cdot (x + 1, y + 1, z) = 0 \qquad \text{or equivalently} \quad 2x - y - 3z = -1\ .$$

For **(b)** we need one point on the intersection of the planes, which means one solutions of

$$\begin{aligned} -2x + 5y - 9z &= -3 \\ 2x - y - 3z &= -1\ . \end{aligned}$$

The point where the line crossed the x, y plane is given by these equations together with $z = 0$. Adding the two equations and using $z = 0$, we get $4y = -4$, so $y = -1$ and then $-2x = 2$ so that $x = -1$. The direction vector of the line i given by taking the cross product of their normals so we compute

$$(-2, 5, -9) \times (2, -1, -3) = (2, -1, -3) = -8(3, 3, 1)\ .$$

Hence the line is parameterized by

$$\mathbf{x}(t) = (-1, -1, 0) + t(3, 3, 1) = (3t - 1, 3t - 1, t)\ .$$