Solutions for Homework 4, Math 291 Fall 2017

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1: Let $f(x, y)$ be given by

$$f(x,y) = \begin{cases} \dfrac{y\sin(xy)}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \ . \end{cases}$$

At which points $(x_0, y_0) \in \mathbb{R}^2$ is the function f continuous? Justify your answer.

SOLUTION: Since $\lim_{t\to 0} \frac{sint}{t} = 1$, when $x^2 + y^2$ is small, $\sin(xy) \approx xy$. Hence we expect that if we replace $\sin(x)$ by xy we do not change the outcome. Therefore, consider

$$g(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \ . \end{cases}$$

The identity $(a-b)^2 = a^2 + b^2 - 2ab$ is relevant. Take $a = x$ and $b = y^4$, so that $(x - y^2)^2 = x^2 + y^4 - 2xy^2$, Along the parabola $x = y^2$, we have $x^2 + y^4 - 2xy^2$ and hence $g(x, y) = 1/2$. On the other hand, along the x and y axes, $g(x, y) = 0$. Hence $g(x, y)$ is not continuous, since

$$\lim_{n\to\infty} g(1/n, 0) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} g(1/n^2, 1/n) = 1/2 \ .$$

Going back to f, we obviously have that $\lim_{n\to\infty} f(1/n, 0) = 0$ and

$$\lim_{n\to\infty} f(1/n^2, 1/n) = \lim_{n\to\infty} \frac{1}{2}\frac{\sin(1/n^3)}{1/n^3} = \frac{1}{2}\lim_{t\to 0}\frac{\sin(t)}{t} = \frac{1}{2} \ .$$

Hence, f is not continuous at $(0, 0)$.

2: Let $f(x, y)$ and $g(x, y)$ be given by

$$f(x,y) = \begin{cases} \dfrac{x^2y^3}{x^4+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \qquad \text{and} \qquad g(x,y) = \begin{cases} \dfrac{x^5}{x^4+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \ .$$

(a) Is the function f continuous at $(0, 0)$? Justify your answer.

(b) Is the function g continuous at $(0,0)$? Justify your answer.

SOLUTION: (a) The identity $(a-b)^2 = a^2+b^2-2ab$ is relevant as in the last problem. This time we take $a = x^2$ and $b = y^3$. We see that when $x^2 = y^6$. $f(x,y) = 1/2$. Hence we consider the sequence $\{\mathbf{x}_n\}$ with $\mathbf{x}_n = (1/n^3, 1/n^2)$. We find

$$\lim_{n\to\infty} f(1/n^3, 1/n^2) = 1/2\ ,$$

and since $f = 0$ everywhere along the x and y axes, f is discontinuous.

For **(b)** Since there is no non-zero power of y in the numerator, the identity $(a-b)^2 = a^2+b^2-2ab$ is not relevant. Instead, we introduce polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ and then we have that for $(x,y) \neq (0,0)$,

$$0 \leq |g(x,y)| = \frac{|x|^5}{x^4+y^6} = r\frac{|\cos^5\theta|}{\cos^4\theta + r^2\sin^6\theta} = r\frac{|\cos\theta|}{1+r^2\sin^2\theta\tan^4\theta} \leq r = \|\mathbf{x}\|\ .$$

Hence, by the squeeze principle, since $\lim_{\mathbf{x}\to 0}\|\mathbf{x}\| = 0$, $\lim_{\mathbf{x}\to 0} g(\mathbf{x}) = 0$, and hence g is continuous at $\mathbf{0}$, and then everywhere else as well, since only at $\mathbf{x} = (0,0)$ does the issue of dividing by zero arise.

3: Let $f(x,y)$ and $g(x,y)$ be given by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y) = (0,0) \end{cases} \qquad \text{and} \qquad g(x,y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y) = (0,0) \end{cases} .$$

(a) Is the function f continuous at $(0,0)$? Is is bounded on the closed unit disc $\{(x,y)\ :\ x^2+y^2 \leq 1\}$? Justify your answers.

(b) Is the function g continuous at $(0,0)$? Is is bounded on the closed unit disc $\{(x,y)\ :\ x^2+y^2 \leq 1\}$? Justify your answer.

SOLUTION: (a) The identity $(a-b)^2 = a^2+b^2-2ab$ is relevant as in the last problem. This time we take $a = x^2$ and $b = |y|$. We see that when $x^2 = |y|$. $|f(x,y)| = 1/2$. Hence we consider the sequence $\{\mathbf{x}_n\}$ with $\mathbf{x}_n = (1/n^2, 1/n)$. We find

$$\lim_{n\to\infty} f(1/n^3, 1/n^2) = 1/2\ ,$$

and since $f = 0$ everywhere along the x and y axes, f is discontinuous. However, since for $(x,y)\neq(0,0)$,

$$|f(x,y)| = \frac{x^2|y|}{x^4+y^2} \leq \frac{1}{2}\ ,$$

the function f is bounded, not only on the unit circle, but on the whole x,y plane.

For **(b)**, we can again try to apply the identity $(a-b)^2 = a^2+b^2-2ab$ with $a = x^2$ and $b = |y|$. Since $(a-b)^2 > 0$ for all a and b, $a^2+b^2 \geq 2ab$, and so with $a = x^2$ and $b = |y|$,

$$x^4+y^2 \geq 2x^2|y|\ .$$

Hence, for $(x,y)\neq(0,0)$,

$$|g(x,y)| = \frac{x^2y^2}{x^4+y^2} = |y|\frac{x^2|y|}{x^4+y^2} \leq \frac{|y|}{2} \leq \frac{\|\mathbf{x}\|}{2}\ .$$

Since the right hand side converges to 0 as $\mathbf{x}$ converges to $\mathbf{0}$, the squeeze principle says that g is continuous at $(0,0)$. Since $g(x,y)$ is a rational function, it is continuous wherever the denominator is not zero. Hence g is continuous everywhere.

Since g is continuous, and since the closed unit disc is a closed and bounded set, f is bounded on this set by the theorem asserting that continuous functions have maximizers and minimizers in any closed, bounded set. However, we can be more explicit. Just above, we have derived the inequality

$$|g(x,y)| \leq \frac{\|\mathbf{x}\|}{2} \ .$$

This says that in fact $|g(x,y)| \leq 1/2$ everywhere on the unit disc. We now have an explicit value for the upper bound, namely $1/2$. The general theorem only says that there is *some* finite bound. Still, even this is useful, as we have seen!

4: Let $f(x,y)$ be a differentiable function on $\mathbb{R}^2$ such that $f(0,0) = 0$. Define a function $g(x,y)$ by

$$g(x,y) = \begin{cases} \dfrac{f(x,y)}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \ . \end{cases}$$

Suppose that f is continuously differentiable. Is it then necessarily the case that g is continuous? Justify your answer to receive credit.

SOLUTION: If $f(x,y)$ is differentiable at $(0,0)$, then all directional derivatives exist at the origin, and so for all unit vectors $\mathbf{u} = (\cos\theta, \sin\theta)$, we have that

$$\lim_{h\to 0} \frac{1}{h}[f(\mathbf{0} + h\mathbf{u}) - f(\mathbf{0})] = \lim_{h\to 0} \frac{1}{h} f(h\mathbf{u}) = \nabla f(\mathbf{0}) \cdot \mathbf{u} \ .$$

To relate this to the problem at hand, for any $(x,y) \neq (0,0)$, define $h = \sqrt{x^2+y^2}$, and then $\mathbf{u} = h^{-1}\mathbf{x}$, which is a unit vector. Then

$$g(h\mathbf{u}) = g(x,y) = \frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{f(h\mathbf{u}}{h} \ ,$$

and so

$$\lim_{h\to 0} g(h\mathbf{u}) = \nabla f(\mathbf{0}) \cdot \mathbf{u} = \|\nabla f(\mathbf{0})\| \cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\nabla f(\mathbf{0})$. We have shown that if $(\mathbf{0}) \neq \mathbf{0}$, the limit depends on θ, the angle of approach. Hence it is not generally that case that g is continuous. However, it may be seen that under the additional condition that $\nabla f(\mathbf{0}) = \mathbf{0}$, the function g is continuous.