Solutions for Homework 3, Math 291 Fall 2017

Eric A. Carlen[1]
Rutgers University

October 12, 2017

1. Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (1 + 2\cos t + 2\sin t\ ,\ 1 - 2\cos t + \sin t\ ,\ \cos t - 2\sin t)\ .$$

Compute $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$ and the speed, curvature and torsion $v(t)$, $\kappa(t)$ and $\tau(t)$.

SOLUTION: Differentiating, we compute

$$\mathbf{v}(t) = (\,-2\sin t + 2\cos t\ ,\ 2\sin t + \cos t\ ,\ -\sin t - 2\cos t)$$

and

$$\mathbf{a}(t) = (\,-2\cos t - 2\sin t\ ,\ 2\cos t - \sin t\ ,\ -\cos t + 2\sin t)\ .$$

We then compute $\mathbf{v}(t)\cdot\mathbf{v}(t) = 9$, and so $v(t) = 3$ and

$$\mathbf{T}(t) = \frac{1}{3}(\,-2\sin t + 2\cos t\ ,\ 2\sin t + \cos t\ ,\ -\sin t + 2\cos t)\ .$$

Next,

$$\mathbf{v}(t)\times\mathbf{a}(t) = (3,6,6)\ ,$$

and so

$$\mathbf{B}(t) = \frac{1}{\|\mathbf{v}(t)\times\mathbf{a}(t)\|}\mathbf{v}(t)\times\mathbf{a}(t) = \frac{1}{3}(1,2,2)\ .$$

Since $\mathbf{B}(t)$ is constant, the torsion $\tau = 0$. Finally

$$\mathbf{N}(t) = \mathbf{B}(t)\times\mathbf{T}(t) = \frac{1}{3}(\,-2\sin t - 2\cos t\ ,\ -\sin t + 2\cos t\ ,\ 2\sin t - \cos t)\ ,$$

and

$$\kappa(t) = \frac{\|\mathbf{v}\times\mathbf{a}\|}{v^3(t)} = \frac{9}{27} = \frac{1}{3}\ .$$

2. Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (2t + t^2\ ,\ 2t\ ,\ t + t^2 + t^3/3\ )\ .$$

Compute $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$ and the speed, curvature and torsion $v(t)$, $\kappa(t)$ and $\tau(t)$.

SOLUTION: Differentiating, we compute

$$\mathbf{v}(t) = \big(2(1+t)\ ,\ 2\ ,\ (1+t)^2\big)$$

and

$$\mathbf{a}(t) = \big(2\ ,\ 0\ ,\ 2(1+t)\big)\ .$$

We then compute $\mathbf{v}(t)\cdot\mathbf{v}(t) = 4(1+t)^2 + 4 + (1+t)^4 = (2+(1+t)^2)^2$, and so $v(t) = 2+(1+t)^2$ and

$$\mathbf{T}(t) = \frac{1}{2+(1+t)^2}\big(2(1+t)\ ,\ 2\ ,\ (1+t)^2\big)\ .$$

Differentiating, we find

$$\mathbf{T}'(t) = \frac{2}{(2+(1+t)^2)^2}\big(2-(1+t)^2\ ,\ -2(1+t)\ ,\ 2(1+t)\ \big)\ .$$

Then we compute $\|\mathbf{T}'(t)\| = 2/(2+(1+t)^2) = v(t)\kappa(r)$. Therefore, $\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t)$ and hence $\kappa(t) = \|\mathbf{T}'(t)\|/v(t)$,

$$\kappa(t) = \frac{2}{(2+(1+t)^2)^2}\ ,$$

and

$$\mathbf{N}(t) = \frac{1}{2+(1+t)^2}\big(2-(1+t)^2\ ,\ -2(1+t)\ ,\ 2(1+t)\ \big)\ .$$

Notice that $\lim_{t\to\infty}\kappa(t) = 0$, and this makes sense since

$$\lim_{t\to\infty} t^{-3}\mathbf{x}(t) = \big(0,0,1/3\big)\ ,$$

so that for large t, the curve looks like a straight line, and a straight line has zero curvature.

Next,

$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{2+(1+t)^2}\big(2(1+t)\ ,\ -(1+t)^2\ ,\ -2\big)\ .$$

Differentiating, we find

$$\mathbf{B}'(t) = \frac{2}{(2+(1+t)^2)^2}\big(2-(1+t)^2\ ,\ -2(1+t)\ ,\ 2(1+t)\ \big)\ .$$

Since $\mathbf{B}'(t) = -v(t)\tau(t)\mathbf{N}(t)$,

$$\tau(t) = \frac{\mathbf{B}'(t)\cdot\mathbf{N}(t)}{v(t)} = \frac{2}{(2+(1+t^2))^2}$$

3. One of the two curves given in Exercises 1 and 2 is a planar curve. Which one is it? Justify your answer, and find an equation for the plane containing the curve.

SOLUTION: A planar curve is a curve with zero torsion. The curve from Exercise 1 had zero torsion, but the curve from Exercise 2 did not. Hence the curve from Exercise 1 is the planar curve, The normal to the plane is given by the constant $\mathbf{B}$ vector, namely

$$\mathbf{B} = \frac{1}{3}\big(1,2,2\big)\ .$$

For the base point of the plane, we may take any point on the plane. We choose $\mathbf{x}(0) = (3, -1, 1)$. Hence the equation of the plane is

$$(1, 2, 2) \cdot (x - 3, y + 1, z - 1) = 0 \ ,$$

which works out to

$$x + 2y + 2z = 3 \ .$$

4. Let $\mathbf{x}(t)$ be the curve given in Exercise 2. Compute the function $s(t)$ giving the arc length traveled along the curve for $t > 0$ starting at $t = 0$. You will find that $s(t)$ can be written as a so-called *depressed cubic* in the variable $1 + t$. That is, for certain numbers A, B and C, $s(t) = A + B(t + 1) + C(t + 1)^3$. This cubic is depressed because the second power is missing. **EXTRA CREDIT** Scipone del Ferro (1465-1526), a student and later lecturer at the University of Bologna, one of the first universities in Europe, figured out how to find the roots of depressed cubics. Look up his story on Wikipedia, and use his formula to solve for $t(s)$, and find the arc length parameterization of this curve.

SOLUTION: We have seen that the speed of the curve in Exercise 2 is given by

$$v(t) = 2 + (1 + t)^2 \ .$$

Integrating we find

$$s(t) = \int_0^t v(r)\mathrm{d}r = 2t + \frac{1}{3}(1 + t)^3 - \frac{1}{3} \ .$$

EXTRA CREDIT SOLUTION: We have found

$$s(t) = 2t + \frac{1}{3}(1 + t)^3 - \frac{1}{3} = -\frac{7}{3} + 2(1 + t) + \frac{1}{3}(1 + t)^3 \ .$$

To find the inverse function $t(s)$, we must solve

$$s = -\frac{7}{3} + 2(1 + t) + \frac{1}{3}(1 + t)^3$$

for t. Introducing the variable $u = 1 + t$, this amounts to solving

$$(s + 7/3) = 2u + \frac{1}{3}u^3$$

for u. The unique real root of this equation is

$$u = \frac{1}{2}\left(12(s + 7/3) + 4\sqrt{32 + 9(s + 7/3)^2}\right)^{1/3} - 4\left(12(s + 7/3) + 4\sqrt{32 + 9(s + 7/3)^2}\right)^{-1/3} \ .$$

Thus,

$$t(s) = \frac{1}{2}\left(12(s + 7/3) + 4\sqrt{32 + 9(s + 7/3)^2}\right)^{1/3} - 4\left(12(s + 7/3) + 4\sqrt{32 + 9(s + 7/3)^2}\right)^{-1/3} - 1 \ .$$

Substituting $t(s)$ in for t in $\mathbf{x}(s)$ we obtain the arc length parameterization.

5. Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (1 + t)^{-2}(1 + 4t + t^2 \ , \ 2^{3/2}t^{1/2} \ , \ 2t + t^2 - 1)$$

for $t > 0$ Compute $v(t)$, $\kappa(t)$ and $\tau(t)$ for all $t > 0$. Also compute $\mathbf{T}(1)$, $\mathbf{N}(1)$ and $\mathbf{B}(1)$.

SOLUTION: The answers require careful computation, but are not so complicated once you get there: We shall see that

$$v(t) = \sqrt{\frac{2}{t}}\frac{\sqrt{1+2t}}{(1+t)^2} \ , \quad \kappa(t) = \frac{1}{t}\left(\frac{9t+5}{(1+t)^5(2t+1)}\right) \ , \quad \tau(t) = -3\frac{\sqrt{2t}(1+t)}{9t+5} \ .$$

We shall also see that

$$\mathbf{T}(1) = \frac{1}{\sqrt{3}}(0,-1,\sqrt{2}) \ , \quad \mathbf{N}(t) = \frac{1}{\sqrt{21}}(-3,-2\sqrt{2},-2) \ , \quad \mathbf{B}(t) = \frac{1}{\sqrt{7}}(2,-\sqrt{2},-1) \ .$$

Differentiating, we compute

$$\mathbf{v}(t) = \frac{1}{(1+t)^3\sqrt{t}}\left(2(1-t)\sqrt{t} \ , \ \sqrt{2}(1-3t) \ , \ 4\sqrt{t}\right)$$

and

$$\mathbf{a}(t) = \frac{1}{(1+t)^4 t^{3/2}}\left(4(t-2)t^{3/2} \ , \ 2^{-1/2}(15t^2-10t-1) \ , \ -12t^{3/2}\sqrt{t}\right) \ .$$

Since

$$\begin{aligned}\|(2(1-t)\sqrt{t} \ , \ \sqrt{2}(1-3t) \ , \ 4\sqrt{t})\|^2 &= 4t(1-t)^2 + 2(1-3t)^2 + 16t \\ &= 4t^3 + 10t^2 + 8t + 2 = (2+4t)(1+t)^2 \ ,\end{aligned}$$

we have

$$v(t) = \sqrt{\frac{2}{t}}\frac{\sqrt{1+2t}}{(1+t)^2} \qquad \text{and} \qquad \mathbf{T}(t) = \frac{1}{(1+t)\sqrt{2+4t}}\left(2(1-t)\sqrt{t} \ , \ \sqrt{2}(1-3t) \ , \ 4\sqrt{t}\right) \ .$$

Next we compute

$$\mathbf{a}_\perp(t) = \mathbf{a}(t) - (\mathbf{a}(t)\cdot\mathbf{T}(t))\mathbf{T}(t) \ .$$

First, we work out the dot product:

$$(\mathbf{a}(t)\cdot\mathbf{T}(t))^2 = \frac{1}{2}\left(\frac{(8t^2+5t+1)^2}{(2t+1)t^3(1+t)^6}\right) \ .$$

Then since

$$\|\mathbf{a}_\perp\|^2 = \|\mathbf{a}\|^2 - (\mathbf{a}\cdot\mathbf{T})^2 \ ,$$

computing

$$\|\mathbf{a}\|_2^2 = \frac{1}{2}\left(\frac{32t^3+33t^2+18t+1}{t^3(1+t)^6}\right) \ ,$$

and then subtracting we find

$$\|\mathbf{a}_\perp(t)\|^2 = \frac{9t+5}{t^2(1+t)^5(2t+1)} \ .$$

Since $\|\mathbf{a}_\perp(t)\|^2 = v^2(t)\kappa(t)$, we readily deduce

$$\kappa(t) = \frac{1}{t}\left(\frac{9t+5}{(1+t)^5(2t+1)}\right) \ .$$

The normal vector $\mathbf{N}(t)$ is given by $\|\mathbf{a}_\perp(t)\|^{-1}\mathbf{a}_\perp(t)$, and our computations so far readily yield

$$\mathbf{a}_\perp(t) = \frac{1}{t(2t+1)(1+t)^4}\big(\, -9t^2 - 4t + 1\ ,\ \sqrt{2t}(3t^2-6t-5)\ ,\ -2(4t^2+t-1)\ \big)\ .$$

and then

$$\mathbf{N}(t) = \frac{1}{\kappa(t)t(2t+1)(1+t)^4}\big(9t^2 + 4t - 1\ ,\ \sqrt{2t}(3t^2-6t-5)\ ,\ 2(4t^2+t-1)\ \big)\ .$$

where we have expressed the normalizing factor in terms of $\kappa(t)$ which is given above.

Next we compute $\mathbf{a}(t)\times\mathbf{v}(t)$. Taking the cross product we have

$$\mathbf{v}(t)\times\mathbf{a}(t) = \frac{1}{t^{3/2}(1+t)^6}\big(2^{3/2}(3t+1)\ ,\ -8t^{3/2}\ ,\ -2^{1/2}(3t^2+1)\ \big)\ .$$

Then since

$$\|(\, -2^{3/2}(3t+1)\ ,\ 8t^{3/2}\ ,\ 2^{1/2}(3t^2+1)\ )\|^2 = 18t^4 + 64t^3 + 84t^2 + 48t + 10 = (18t+10)(1+t)^3\ ,$$

$$\mathbf{B}(t) = \frac{1}{\sqrt{9t+5}(1+t)^{3/2}}\big(2(3t+1)\ ,\ -4\sqrt{2}t^{3/2}\ ,\ -3t^2-1\ \big)$$

and

$$\|\mathbf{a}(t)\times\mathbf{v}(t)\| = \frac{\sqrt{18t+10}}{t^{3/2}(1+t)^{9/2}}\ .$$

Now to get the torsion, use the identities

$$v(t)\tau(t) = \mathbf{N}'(t)\cdot\mathbf{B}(t) = -\mathbf{B}'(t)\cdot\mathbf{N}(t).$$

The result is given above.

Now it is also a simple matter to evaluate $\mathbf{T}(1)$, $\mathbf{N}(1)$ and $\mathbf{B}(1)$ to obtain the result given above.

6. A curve $\mathbf{x}(t)$ in $\mathbb{R}^3$ is called a *spherical curve* if for all t, $\mathbf{x}(t)$ lies on some given sphere. The general equation of a sphere in $\mathbb{R}^3$ is $\|\mathbf{x}-\mathbf{c}\|^2 = r^2$ for some given $r > 0$ and $\mathbf{c}\in\mathbb{R}^3$. (Then r is the radius of the sphere, and $\mathbf{c}$ is the center.) Thus, $\mathbf{x}(t)$ is a spherical curve if and only if there exist $r > 0$ and $\mathbf{c}\in\mathbb{R}^3$ such that

$$\|\mathbf{x}(t)-\mathbf{c}\|^2 = r^2 \qquad \text{for all} \quad t\ . \tag{1}$$

a. Suppose that for some given $r > 0$ and $\mathbf{c}\in\mathbb{R}^3$, the curve $\mathbf{x}(t)$ satisfies (1) for all t. Differentiating both sides of (1), show that for all t

$$(\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{T}(t) = 0\ . \tag{2}$$

b. Differentiating both sides of (2), show that for all t

$$(\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{N}(t) = -\frac{1}{\kappa(t)}\ , \tag{3}$$

and then show

$$(\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{B}(t) = \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\ . \tag{4}$$

SOLUTION: Differentiating both sides of $\|\mathbf{x}(t)-\mathbf{c}\|^2 = r^2$ in t, we find $2(\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{v} = 0$. Then since $\mathbf{v}(t) = v(t)\mathbf{T}(t)$, we obtain (2). Now differentiating (2) we get two terms:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{T}(t) = \mathbf{v}\cdot\mathbf{T}(t) + (\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{T}'(t)\ .$$

Using $\mathbf{v}(t) = v(t)\mathbf{T}(t)$ and $\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t)$, This reduces to $(\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{N}(t) = -1/\kappa(t)$, which is (3). Finally, differentiating (3) in t, we get

$$\frac{\kappa'(t)}{\kappa^2(t)} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{N}(t) = \mathbf{v}(t)\cdot\mathbf{N}(t) + (\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{N}'(t)\ .$$

Since $\mathbf{v}(t)$ is orthogonal to $\mathbf{N}(t)$, the first term on the right is zero. Since $\mathbf{N}'(t) = v(t)\tau(t)\mathbf{B}(t) - v(t)\kappa(t)\mathbf{T}(t)$, and since we have seen that $(\mathbf{x}(t)-\mathbf{c})$ is orthogonal to $\mathbf{T}(t)$, we get

$$\frac{\kappa'(t)}{\kappa^2(t)} = v(t)\tau(t)(\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{B}(t)\ ,$$

which is equivalent to (4).

7. Continuing with the previous exercise, show that if a curve in $\mathbb{R}^3$ satisfies (1) for all t for all t, then its speed, curvature and torsion are related by

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 = r^2 - \frac{1}{\kappa^2(t)}\ . \tag{5}$$

This is a necessary condition on v, κ and τ for a curve to be a spherical curve. Note that as a consequence of (5), $1/\kappa(t) \le r$ for all t. Give a geometric explanation for why the radius of curvature of a spherical curve cannot be larger than the radius of the sphere.

SOLUTION: For *any* orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$ we have

$$\mathbf{x}(t)-\mathbf{c} = ((\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{u}_1)\mathbf{u}_1 + ((\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{u}_2)\mathbf{u}_2 + ((\mathbf{x}(t)-\mathbf{c})\cdot\mathbf{u}_3)\mathbf{u}_3\ .$$

Here it is natural to use the curve's own particular orthonormal basis, namely

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}\ .$$

Using the dot products computed in the previous exercise,

$$\mathbf{x}(t)-\mathbf{c} = -\frac{1}{\kappa(t)}\mathbf{N}(t) + \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t)\ .$$

Therefore, by the Pythagorean Theorem,

$$r^2 = \|\mathbf{x}(t)-\mathbf{c}\|^2 = \frac{1}{\kappa^2(t)} + \left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2\ .$$

Regrouping terms, we have (5).

Since the left hand side of (5) is non-negative, we must have $1/\kappa(r) \le r$. The intersection of the osculating plane with the sphere on which the motion takes place is a circle of radius $\varrho(t)$ with $\varrho(t) \le r$. At time t, the instantaneous motion is along this circle, and hence the radius of curvature of the motion at time t is $\varrho(t) \le r$. Hence $1/\kappa(t) \le r$, as we found.

8. Continuing with the previous two exercises, consider a curve in $\mathbb{R}^3$ that satisfies (1) for all t. Then using the results from Exercise 6, show that

$$\mathbf{c} = \mathbf{x}(t) - (\mathbf{x}(t) - \mathbf{c}) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) - \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t) \ .$$

That is, if $\mathbf{x}(t)$ is a spherical curve, the curve $\mathbf{y}(t)$ given by

$$\mathbf{y}(t) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) - \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t)$$

must necessarily be constant. Differentiate to show that $\mathbf{y}'(t) = 0$ if and only if (5) is valid.

SOLUTION: Using the expansion of $\mathbf{x}(t) - \mathbf{c}$ from the previous exercise together with the identity $\mathbf{c} = \mathbf{x}(t) - (\mathbf{x}(t) - \mathbf{c})$, we immediately obtain

$$\mathbf{c} = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) - \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t) \ .$$

Differentiating, and using the Frenet-Seret formulae for $\mathbf{T}'(t)$, $\mathbf{N}'(t)$ and $\mathbf{B}'(t)$, we find

$$\begin{aligned}
\mathbf{0} &= v(t)\mathbf{T}(t) - \frac{\kappa'(t)}{\kappa^2(t)}\mathbf{N}(t) + \frac{1}{\kappa(t)}\mathbf{N}'(t) - \left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)'\mathbf{B}(t) - \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}'(t) \\
&= \left(v(t) - \frac{1}{\kappa(t)}v(t)\kappa(t)\right)\mathbf{T}(t) \\
&+ \left(-\frac{\kappa'(t)}{\kappa^2(t)} + \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}v(t)\tau(t)\right)\mathbf{N}(t) \\
&+ \left(\frac{1}{\kappa(t)}v(t)\tau(t) - \left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)'\right)\mathbf{B}(t) \ .
\end{aligned}$$

The coefficient of $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$ must all be zero since $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ is orthonormal. The coefficient of $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are clearly equal to zero, so the only new information is that the coefficient of $\mathbf{B}(t)$ is zero, namely

$$\frac{\tau(t)}{\kappa(t)} = \frac{1}{v(t)}\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)' \ .$$

In any case, we must find the conditions under which this identity is satisfied, and we do not want to directly differentiate, since this will bring $\kappa''(t)$ into the story, and we do not have any formulas for that. So left us go back to our necessary condition for a curve to remain on a sphere of radius r, namely (5) and take square roots to write

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right) = \pm\sqrt{r^2 - \frac{1}{\kappa^2(t)}} \ .$$

Differentiating both sides we find

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)' = \pm\frac{1}{\sqrt{r^2 - \frac{1}{\kappa^2(t)}}}\frac{\kappa'(t)}{\kappa^3(t)} \ .$$

Eliminating the square root using the identity just above, we see that for *either* choice of the sign in this identity,

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)' = \left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^{-1}\frac{\kappa'(t)}{\kappa^3(t)} = \frac{v(t)\tau(t)}{\kappa(t)} .$$

Thus, thus whenever (5) is satisfied for a curve $\mathbf{x}(t)$, and $\mathbf{y}(t)$ is defined as above, $\mathbf{y}'(t) = 0$ for all t.

9. Show that a curve $\mathbf{x}(t)$ is a spherical curve if and only if its speed, curvature and torsion are such that

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)}$$

is constant.

SOLUTION: We have seen that for a curve $\mathbf{x}(t)$ to be a spherical curve, it is necessary that its speed, curvature and torsion are such that (5) is satisfied, where r is the radius of the sphere. Hence the condition that $\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)}$ be constant is necessary.

On the other had, suppose that $\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)}$ is constant. Define r^2 to be its constant value. In the previous exercise, we have see that the curve $\mathbf{y}(t)$ defined by

$$\mathbf{y}(t) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) - \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t)$$

satisfies $\mathbf{y}'(t) = \mathbf{0}$, and hence $\mathbf{y}(t)$ is contant. Define $\mathbf{c} = \mathbf{y}(0)$, Then

$$\mathbf{x}(t) - \mathbf{c} = -\frac{1}{\kappa(t)}\mathbf{N}(t) + \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t) ,$$

and by the Pythagorean Theorem,

$$\|\mathbf{x}(t) - \mathbf{c}\|^2 = \left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)} = r^2 .$$

Hence the curve $\mathbf{x}(t)$ is spherical. Thus, (5) is a necessary and sufficient condition.

10. Show that curve in exercise 5 is a spherical curve, and find the center and radius of the sphere.

SOLUTION: We first compute

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)} .$$

By the previous exercise, if this is constant, then the curve is spherical, and this constant is the squared radius of the sphere. Using the results of Exercise 5, we compute

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 = \frac{(3t^2+1)^2}{(1+t)^3(9t+5)} ,$$

and

$$\frac{1}{\kappa^2(t)} = \frac{4(2t+1)^3}{(1+t)^3(9t+5)} .$$

Summing and simplifying, we find

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)} = 1 \ ,$$

so the curve is spherical, and the radius of the sphere is 1.

It remains to find the center $\mathbf{c}$ of the sphere. By the results above, this is given by

$$\mathbf{c} = \mathbf{x}(1) + \frac{1}{\kappa(1)}\mathbf{N}(1) - \frac{\kappa'(1)}{v(1)\tau(1)\kappa^2(1)}\mathbf{B}(1) \ .$$

Doing the computations, we find $\mathbf{c} = \big(1,0,0\big)$.

Now you can readily verify that indeed, as claimed

$$\|\mathbf{x}(t) - \big(1,0,0\big)\|^2 = 1$$

for all t.