Homework 3, Math 291 Fall 2017

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1. Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = \big(1 + 2\cos t + 2\sin t \ , \ 1 - 2\cos t + \sin t \ , \ \cos t - 2\sin t\big) \ .$$

Compute $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$ and the speed, curvature and torsion $v(t)$, $\kappa(t)$ and $\tau(t)$.

2. Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = \big(2t + t^2 \ , \ 2t \ , \ t + t^2 + t^3/3 \ \big) \ .$$

Compute $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$ and the speed, curvature and torsion $v(t)$, $\kappa(t)$ and $\tau(t)$.

3. One of the two curves given in Exercises 1 and 2 is a planar curve. Which one is it? Justify your answer, and find an equation for the plane containing the curve.

4. Let $\mathbf{x}(t)$ be the curve given in Exercise 2. Compute the function $s(t)$ giving the arc length traveled along the curve for $t > 0$ starting at $t = 0$. You will find that $s(t)$ can be written as a so-called *depressed cubic* in the variable $1 + t$. That is, for certain numbers A, B and C, $s(t) = A + B(t+1) + C(t+1)^3$. This cubic is depressed because the second power is missing. **EXTRA CREDIT** Scipone del Ferro (1465-1526), a student and later lecturer at the University of Bologna, one of the first universities in Europe, figured out how to find the roots of depressed cubics. Look up his story on Wikipedia, and use his formula to solve for $t(s)$, the function giving t in terms of s.

5. Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (1+t)^{-2}\big(1 + 4t + t^2 \ , \ 2^{3/2}t^{1/2} \ , \ 2t + t^2 - 1\big)$$

for $t > 0$ Compute $v(t)$, $\kappa(t)$ and $\tau(t)$ for all $t > 0$. Also compute $\mathbf{T}(1)$, $\mathbf{N}(1)$ and $\mathbf{B}(1)$.

6. A curve $\mathbf{x}(t)$ in $\mathbb{R}^3$ is called a *spherical curve* if for all t, $\mathbf{x}(t)$ lies on some given sphere. The general equation of a sphere in $\mathbb{R}^3$ is $\|\mathbf{x} - \mathbf{c}\|^2 = r^2$ for some given $r > 0$ and $\mathbf{c} \in \mathbb{R}^3$. (Then r is the radius of the sphere, and $\mathbf{c}$ is the center.) Thus, $\mathbf{x}(t)$ is a spherical curve if and only if there exist $r > 0$ and $\mathbf{c} \in \mathbb{R}^3$ such that

$$\|\mathbf{x}(t) - \mathbf{c}\|^2 = r^2 \qquad \text{for all} \quad t \ . \tag{1}$$

a. Suppose that for some given $r > 0$ and $\mathbf{c} \in \mathbb{R}^3$, the curve $\mathbf{x}(t)$ satisfies (1) for all t. Differentiating both sides of (1), show that for all t

$$(\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{T}(t) = 0 \ . \tag{2}$$

b. Differentiating both sides of (2), show that for all t

$$(\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{N}(t) = -\frac{1}{\kappa(t)} \ , \tag{3}$$

and then show

$$(\mathbf{x}(t) - \mathbf{c}) \cdot \mathbf{B}(t) = \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)} \ . \tag{4}$$

7. Continuing with the previous exercise, show that if a curve in $\mathbb{R}^3$ satisfies (1) for all t for all t, then its speed, curvature and torsion are related by

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 = r^2 - \frac{1}{\kappa^2(t)} \ . \tag{5}$$

This is a necessary condition on v, κ and τ for a curve to be a spherical curve. Note that as a consequence of (5), $1/\kappa(t) \le r$ for all t. Give a geometric explanation for why the radius of curvature of a spherical curve cannot be larger than the radius of the sphere.

8. Continuing with the previous two exercises, consider a curve in $\mathbb{R}^3$ that satisfies (1) for all t. Then using the results from Exercise 6, show that

$$\mathbf{c} = \mathbf{x}(t) - (\mathbf{x}(t) - \mathbf{c}) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) - \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t) \ .$$

That is, if $\mathbf{x}(t)$ is a spherical curve, the curve $\mathbf{y}(t)$ given by

$$\mathbf{y}(t) = \mathbf{x}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) - \frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\mathbf{B}(t)$$

must necessarily be constant. Differentiate to show that $\mathbf{y}'(t) = 0$ if and only if (5) is satisfied.

9. Show that a curve $\mathbf{x}(t)$ is a spherical curve if and only if its speed, curvature and torsion are such that

$$\left(\frac{\kappa'(t)}{v(t)\tau(t)\kappa^2(t)}\right)^2 + \frac{1}{\kappa^2(t)}$$

is constant.

10. Show that curve in exercise 5 is a spherical curve, and find the center and radius of the sphere.