Solutions for Homework 2, Math 291 Fall 2017

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1. Find a right handed orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that $\mathbf{u}_1$ is a positive multiple of $(4,4,7)$ and $\mathbf{u}_2$ is orthogonal to $(1,0,2)$. How many such bases are there?

SOLUTION: Since $\|(4,4,7)\| = 9$, we must take $\mathbf{u}_1 = \frac{1}{9}(4,4,7)$. We must choose $\mathbf{u}_2$ to be ortgogonal to both $(4,4,7)$ and $(1,0,2)$, so it must be a multiple of the cross product

$$(4,4,7) \times (1,0,2) = (8,-1,-4)$$

which has length 9. Hence we must take $\mathbf{u}_2 = \pm\frac{1}{9}(8,-1,-4)$. Finally we must take $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 = \pm\frac{1}{9}(-1,8,-4)$. Hence there are exactly two such bases. One is

$$\left\{\ \tfrac{1}{9}(4,4,7)\ ,\ \tfrac{1}{9}(8,-1,-4)\ ,\ \tfrac{1}{9}(-1,8,-4)\ \right\}$$

and the others has minus signs in from of the last two vectors.

2. Let ℓ_1 be the line parameterized by $\mathbf{x}(t) = (1,2,2) + t(0,3,3)$. Let ℓ_2 be the line parameterized by $\mathbf{y}(s) = s(2,1,2)$. Compute the distance between these two lines, and the points $\mathbf{x}_0$ on ℓ_1 and $\mathbf{y}_0$ on ℓ_2 such that $\|\mathbf{x}_0 - \mathbf{y}_0\| \le \|\mathbf{x}(t) - \mathbf{y}(s)\|$ for all s, t.

SOLUTION: Let $\mathbf{v} = (2,1,2)$ and $\mathbf{w} = (0,3,3)$. We compute an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that $\mathbf{u}_1$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, and such that $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$. This can be done by taking

$$\mathbf{u}_1 = \frac{\mathbf{v}\times\mathbf{w}}{\|\mathbf{v}\times\mathbf{w}\|}\ ,\quad \mathbf{u}_2 = \frac{\mathbf{v}\times\mathbf{u}_1}{\|\mathbf{v}\times\mathbf{u}_1\|}\quad \mathbf{u}_3 = \mathbf{u}_1\times\mathbf{u}_2\ .$$

Then

$$\begin{aligned}
\|\mathbf{x}(t) - \mathbf{y}(s)\|^2 &= \sum_{j=1}^{3}[((1,2,2) + t(0,3,3) - s(2,1,2))\cdot\mathbf{u}_j]^2\\
&= [(1,2,2)\cdot\mathbf{u}_1]^2 + [((1,2,2) + t(0,3,3))\cdot\mathbf{u}_2]^2\\
&+ [((1,2,2) + t(0,3,3) - s(2,1,2))\cdot\mathbf{u}_3]^2\ .
\end{aligned}$$

Computing we find and so

$$\|\mathbf{x}(t) - \mathbf{y}(s)\|^2 = \frac{1}{9} + \left(\frac{4}{3} + 3t\right)^2 + \left(\frac{8}{3} + 3t - 3s\right)^2\ .$$

To make this as small as possible, we choose

$$t = t_0 = -\frac{4}{9} \qquad \text{and then} \qquad s = s_0 = \frac{4}{9}\ .$$

Thus, the distance between the lines is $1/3$, and the closest points $\mathbf{x}(t_0) = (1,-1,-1)$ and $\mathbf{y}(s_0) = \frac{4}{9}(2,1,2)$.

3. Let ℓ_1 be the line passing through $(1,2,2)$ and $(1,5,5)$. Let ℓ_2 be the line passing through $\mathbf{x}_0$ and $\mathbf{x}_0 + (2,1,2)$. The set of all points $\mathbf{x}_0$ for which ℓ_2 meets ℓ_1 (i.e., such that $\ell_1 \cap \ell_2 \neq \emptyset$) is a plane. Find an equation for this plane written in the form $ax + by + cz = d$.

SOLUTION: Two lines that intersect lie in a plane. Suppose that ℓ_1 and ℓ_2 intersect in some point $\mathbf{z}_0$. Then there is a non zero vector $\mathbf{a}$ so that the plane given by $\mathbf{a}\cdot(\mathbf{x}-\mathbf{z}_0) = 0$ contains both ℓ_1 and ℓ_2, and therefore $\mathbf{x}_0$. So $\mathbf{x}_0$ must satisfy the equation for this plane. We now determine this equation.

Since we can use any point on a line as the base point in a parameterization of the line, our lines are also parameterized by $\mathbf{x}(t) := \mathbf{z}_0 + t\mathbf{w}$ and $\mathbf{y}(s) := \mathbf{z}_0 + s\mathbf{v}$ where $\mathbf{w} = (0,3,3)$ and $\mathbf{v} = (2,1,2)$. So $\mathbf{x}(t)$ and $\mathbf{y}(s)$ must satisfy the equation of the plane for all s and t, respectively. This means that $\mathbf{a}\cdot\mathbf{v} = \mathbf{a}\cdot\mathbf{w} = 0$, and so $\mathbf{a}$ must be a multiple of $\mathbf{v}\times\mathbf{w} = (-3,-6,6)$. We may as well take $a = (-1,-2,2)$. Then since we may take any point in the plane as the base point, we may as well take any point on either line, such as $(1,2,2)$. The equation for the plane then is

$$-x - 2y + 2z = -1\ .$$

Thus, the lines intersect if and only if $\mathbf{x}_0$ satsifies this equation.

4. Let $h_{\mathbf{u}}$ be the Householder reflection determined by a unit vector $\mathbf{u}$:

$$h_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}\cdot\mathbf{u})\mathbf{u}\ .$$

a. Find a choice for $\mathbf{u}$ such that $h_{\mathbf{u}}(\mathbf{e}_1) = \frac{1}{9}(4,4,7)$

b. For this choice of $\mathbf{u}$, also compute $h_{\mathbf{u}}(\mathbf{e}_2)$ and $h_{\mathbf{u}}(\mathbf{e}_3)$. Verify that $\{h_{\mathbf{u}}(\mathbf{e}_1), h_{\mathbf{u}}(\mathbf{e}_2), h_{\mathbf{u}}(\mathbf{e}_3)\}$ is a left handed orthonormal basis.

SOLUTION a: We take $\mathbf{u} = \|\mathbf{e}_1 - \frac{1}{9}(4,4,7)\|^{-1}(\mathbf{e}_1 - \frac{1}{9}(4,4,7)) = \frac{1}{3\sqrt{10}}(5,-4,-7)$. It is now easy to check that for this $\mathbf{u}$, $h_{\mathbf{u}}(\mathbf{e}_1) = \frac{1}{9}(4,4,7)$.

b. We next compute that

$$h_{\mathbf{u}}(\mathbf{e}_2) = \frac{1}{45}(20,29,-28) \quad \text{and} \quad h_{\mathbf{u}}(\mathbf{e}_3) = \frac{1}{45}(35,-28,-4)\ .$$

5. Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be any three vectors in $\mathbb{R}^3$ such that

$$\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3) \neq 0\ .$$

We know that $\{h_{\mathbf{u}}(\mathbf{e}_1), h_{\mathbf{u}}(\mathbf{e}_2), h_{\mathbf{u}}(\mathbf{e}_3)\}$ is orthonormal. To see whether it is right handed or left handed, we compute $h_{\mathbf{u}}(\mathbf{e}_1)\times h_{\mathbf{u}}(\mathbf{e}_2)$, and the reuslt is that this is $-h_{\mathbf{u}}(\mathbf{e}_3)$, so that the basis is left handed. In fact, this always happens: Reflection tuns a right handed basis into a left handed basis.

a. Show that
$$|\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)| = |\mathbf{v}_2 \cdot (\mathbf{v}_3 \times \mathbf{v}_1)| = |\mathbf{v}_3 \cdot (\mathbf{v}_1 \times \mathbf{v}_2)| \ .$$

b. Let D be the common value considered in part **(a)**. Define vectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ by
$$\mathbf{w}_1 = \frac{1}{D}\mathbf{v}_2 \times \mathbf{v}_3 \ , \qquad \mathbf{w}_2 = \frac{1}{D}\mathbf{v}_3 \times \mathbf{v}_1 \ , \quad \text{and} \quad \mathbf{w}_3 = \frac{1}{D}\mathbf{v}_1 \times \mathbf{v}_2 \ .$$

Show that for all $1 \le i, j \le 3$,
$$\mathbf{v}_i \cdot \mathbf{w}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \ .$$

SOLUTION a: Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be any three vectors in $\mathbb{R}^3$. By the tripple product identity, and then the commutativity of the dot product, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$. That is,
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \ ,$$
which says that the triple product is unchanged under any cyclic permulation of the vectors. This implies an even stronger result than you were asked to prove, namely,
$$\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) = \mathbf{v}_2 \cdot (\mathbf{v}_3 \times \mathbf{v}_1) = \mathbf{v}_3 \cdot (\mathbf{v}_1 \times \mathbf{v}_2) \ .$$

Also for use in part **b**, note that by this invariance under cyclic permuations, the triple prodcut of three vectors is zero if any two of them are equal since by such a permuationswe can bring the two equal vectors into the cross product.

For **b**, we compute, using the remark made just above,
$$\mathbf{v}_1 \cdot \mathbf{w}_1 = \frac{1}{D}\mathbf{v}_1 \cdot \mathbf{v}_2 \times \mathbf{v}_3 = 1 \ , \quad \mathbf{v}_2 \cdot \mathbf{w}_1 = \frac{1}{D}\mathbf{v}_2 \cdot \mathbf{v}_2 \times \mathbf{v}_3 = 0 \quad \text{and} \quad \mathbf{v}_3 \cdot \mathbf{w}_1 = \frac{1}{D}\mathbf{v}_3 \cdot \mathbf{v}_2 \times \mathbf{v}_3 = 0 \ .$$

This proves the result for all i and $j = 1$. The proof for other vlaues of j goes the same way.

6. Let the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be as in Exercise **6**.

a. Show that $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \mathbb{R}^3$. **Hint:** We know that the span of any set of non-zero vectors in $\mathbb{R}^3$ is either a line through the origin, a plane through the origin, or is all of $\mathbb{R}^3$. So to show $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \mathbb{R}^3$ it suffices to show that whenever $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) \neq 0$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ cannot possibly lie on any one plane through the origin.

b. Show that for every $\mathbf{x} \in \mathbb{R}^3$, there are unique values of t_1, t_2 and t_3 such that
$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3 \ ,$$
and that for each $j = 1, 2, 3$,
$$t_j = \mathbf{w}_j \cdot \mathbf{x} \ .$$

SOLUTION a: $\mathrm{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ is a non-zero subspace of $\mathbb{R}^3$ and hence if it is not all of $\mathbb{R}^3$, it is either a line of a plane through the origin. In either case, if it is not all of $\mathbb{R}^3$, it lies in a plane through the origin. To show this is impossible, it suffices to show that there is no plane through the origin continaing $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The plane through the origin containing $\{\mathbf{v}_1, \mathbf{v}_2\}$ has the equation $(\mathbf{v}_1 \times \mathbf{v}_2)\cdot \mathbf{x} = 0$, but $\mathbf{v}_3$ does not satisfy this equation since $(\mathbf{v}_1 \times \mathbf{v}_2)\cdot \mathbf{v}_3 = D \neq 0$. Hence Span$(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$ is not contianed in any plane, and it must be all of $\mathbb{R}^3$.

For **b**, By definition, the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ consists of all vectors of the form $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3$ for arbitrary t_1, t_2 and t_3. By part **a**, Span$(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \mathbb{R}^3$, and so for any $\mathbf{x}\ in \mathbb{R}^3$, there exist numbers t_1, t_2 and t_3 such that

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3\ .$$

Taking the dot product of both sides with $\mathbf{w}_1$, we find

$$\mathbf{w}_1\cdot\mathbf{x} = t_1(\mathbf{w}_1\cdot\mathbf{v}_1) + t_2(\mathbf{w}_1\cdot\mathbf{v}_2) + t_3(\mathbf{w}_1\cdot\mathbf{v}_3) = t_1\ .$$

Thus, we must have $t_1 = \mathbf{w}_1\cdot\mathbf{x}$. Likewise, we must have $t_2 = \mathbf{w}_2\cdot\mathbf{x}$ and $t_3 = \mathbf{w}_3\cdot\mathbf{x}$.

7. Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be given by

$$\mathbf{v}_1 = (1,0,1)\ ,\quad \mathbf{v}_2 = (1,1,1)\ ,\quad \mathbf{v}_3 = (1,2,3)\ .$$

a. Find vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ such that for all $1 \le i,j \le 3$,

$$\mathbf{v}_i\cdot\mathbf{w}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}\ .$$

b. Compute numbers t_1, t_2 and t_3 such that

$$t_1(1,0,1) + t_2(1,1,1) + t_3(1,2,3) = (12,-7,19)\ .$$

SOLUTION a: We compute

$$\mathbf{v}_2\times\mathbf{v}_3 = (1,-2,1)\ ,\qquad \mathbf{v}_3\times\mathbf{v}_1 = (2,2,-2) \quad\text{and}\quad \mathbf{v}_1\times\mathbf{v}_2 = (-1,0,1)\ .$$

Therefore, using the formula frompart **a** the previous exercise, $D = \mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3) = 2$, and then

$$\mathbf{w}_1 = \tfrac{1}{2}(1,-2,1)\ ,\qquad \mathbf{w}_2 = (1,1,-1) \quad\text{and}\quad \mathbf{w}_3 = \tfrac{1}{2}(-1,0,1)\ .$$

Fpr **b**, using the formula frompart **b** the previous exercise,

$$\begin{aligned} t_1 &= \mathbf{w}_1\cdot(12,-7,19) = \tfrac{45}{2} \\ t_2 &= \mathbf{w}_2\cdot(12,-7,19) = -14 \\ t_3 &= \mathbf{w}_3\cdot(12,-7,19) = \tfrac{7}{2}\ . \end{aligned}$$

8. Show that for all $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$,

$$(\mathbf{b}\times\mathbf{c})\cdot[(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})] = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|^2\ .$$

SOLUTION: Several people asked for hints on this. The hint was to either use Lagrange's identity or to use the idea of the proof of Lagrange's identity. We first solve it using Lagrange's identity

$$\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) = (\mathbf{x}\cdot\mathbf{z})\mathbf{y} - (\mathbf{x}\cdot\mathbf{z})\mathbf{y}\ .$$

Taking $\mathbf{x} = (\mathbf{c} \times \mathbf{a})$, $\mathbf{y} = \mathbf{a}$ and $\mathbf{z} = \mathbf{b}$, we have

$$(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}) = ((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b})\mathbf{a} - ((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{a})\mathbf{b} = ((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b})\mathbf{a} \ .$$

(We have used the fact proved above that if any two vectors in a triple product are equal, the triple product is zero.) Now taking the dot product with $\mathbf{b} \times \mathbf{c}$ we obtain

$$(\mathbf{b} \times \mathbf{c}) \cdot [(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})] = (\mathbf{b} \times \mathbf{c}) \cdot [((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b})\mathbf{a}] = ((\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b})((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}) \ .$$

By the invariance of the triple product under cyclic invariance, this is the same as $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|^2$.

ALTERNATE SOLUTION: We define an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ built out of the given vectors, as in the proof of Lagrange's identity. Define

$$\begin{aligned} \mathbf{u}_1 &= \frac{1}{\|\mathbf{b} \times \mathbf{c}\|}\mathbf{b} \times \mathbf{c} \\ \mathbf{u}_2 &= \frac{1}{\|\mathbf{b}\|}\mathbf{b} \\ \mathbf{u}_3 &= \mathbf{u}_1 \times \mathbf{u}_2 \ . \end{aligned}$$

In this basis, it is clear that the right hand side equals $\|\mathbf{b} \times \mathbf{c}\|^2(\mathbf{a} \cdot \mathbf{u}_1)^2$. We now show that the left hand side is equal to this also.

Note that $\mathbf{b}$ is by definition a multiple of $\mathbf{u}_2$ and $\mathbf{c}$ is orthogonal to $\mathbf{u}_1$, so we have

$$\mathbf{b} \cdot \mathbf{u}_1 = \mathbf{b} \cdot \mathbf{u}_3 = \mathbf{c} \cdot \mathbf{u}_1 = 0 \ .$$

Since we can compute cross products in the usual way using coordinates with respect to any orthonormal basis, and since by defnition $\mathbf{b} \times \mathbf{c} = \|\mathbf{b} \times \mathbf{c}\|\mathbf{b} \times \mathbf{c}$, we compute

$$\begin{aligned} \mathbf{b} \times \mathbf{c} &= \|\mathbf{b} \times \mathbf{c}\|\mathbf{u}_1 \\ \mathbf{c} \times \mathbf{a} &= [(\mathbf{c} \cdot \mathbf{u}_2)(\mathbf{a} \cdot \mathbf{u}_3) - (\mathbf{c} \cdot \mathbf{u}_3)(\mathbf{a} \cdot \mathbf{u}_2)]\mathbf{u}_1 - (\mathbf{c} \cdot \mathbf{u}_3)(\mathbf{a} \cdot \mathbf{u}_1)\mathbf{u}_2 + (\mathbf{c} \cdot \mathbf{u}_2)(\mathbf{a} \cdot \mathbf{u}_3)\mathbf{u}_3 \\ \mathbf{a} \times \mathbf{b} &= \|\mathbf{b}\|(\mathbf{a} \cdot \mathbf{u}_1)\mathbf{u}_3 - \|\mathbf{b}\|(\mathbf{a} \cdot \mathbf{u}_3)\mathbf{u}_1 \ . \end{aligned}$$

By the invariance of triple product identity under cyclic permutation,

$$(\mathbf{b} \times \mathbf{c}) \cdot [(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})] = (\mathbf{c} \times \mathbf{a}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c})]$$

and from the above

$$[(\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c})] = -(\mathbf{a} \cdot \mathbf{u}_1)\|\mathbf{b}\|\|\mathbf{b} \times \mathbf{c}\|\mathbf{u}_2$$

Then

$$(\mathbf{c} \times \mathbf{a}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c})] = \|\mathbf{b}\|\|\mathbf{b} \times \mathbf{c}\|(\mathbf{c} \cdot \mathbf{u}_3)(\mathbf{a} \cdot \mathbf{u}_1)^2 \ .$$

Let $\mathbf{c} = \mathbf{c}_\parallel + \mathbf{c}_\perp$ be the decomposition of $\mathbf{c}$ into components parallel and perpendicular to $\mathbf{b}$. Then since $\mathbf{c}$ is orthogonal to $\mathbf{u}_1$, $\|\mathbf{c} \cdot \mathbf{u}_3\| = \|\mathbf{c}_\perp\|$ and of course $\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{c}_\perp$, so that $\|\mathbf{b}\|(\mathbf{c} \cdot \mathbf{u}_3) = \|\mathbf{b} \times \mathbf{c}\|$. Thus the left had side is equal to $\|\mathbf{b} \times \mathbf{c}\|^2(\mathbf{a} \cdot \mathbf{u}_1)^2$, as was to be shown.