Homework 2, Math 291 Fall 2017

Eric A. Carlen[1]
Rutgers University

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1. Find a right handed orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that $\mathbf{u}_1$ is a positive multiple of $(4, 4, 7)$ and $\mathbf{u}_2$ is orthogonal to $(1, 0, 2)$. How many such bases are there?

2. Let ℓ_1 be the line parameterized by $\mathbf{x}(t) = (1, 2, 2) + t(0, 3, 3)$. Let ℓ_2 be the line parameterized by $\mathbf{y}(s) = s(2, 1, 2)$. Compute the distance between these two lines, and the points $\mathbf{x}_0$ on ℓ_1 and $\mathbf{y}_0$ on ℓ_2 such that $\|\mathbf{x}_0 - \mathbf{y}_0\| \leq \|\mathbf{x}(t) - \mathbf{y}(s)\|$ for all s, t.

3. Let ℓ_1 be the line passing through $(1, 2, 2)$ and $(1, 5, 5)$. Let ℓ_2 be the line passing through $\mathbf{x}_0$ and $\mathbf{x}_0 + (2, 1, 2)$. The set of all points $\mathbf{x}_0$ for which ℓ_2 meets ℓ_1 (i.e., such that $\ell_1 \cap \ell_2 \neq \emptyset$) is a plane. Find an equation for this plane written in the form $ax + by + cz = d$.

4. Let $h_{\mathbf{u}}$ be the Householder reflection determined by a unit vector $\mathbf{u}$:

$$h_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{u})\mathbf{u} \ .$$

a. Find a choice for $\mathbf{u}$ such that $h_{\mathbf{u}}(\mathbf{e}_1) = \frac{1}{9}(4, 4, 7)$

b. For this choice of $\mathbf{u}$, also compute $h_{\mathbf{u}}(\mathbf{e}_2)$ and $h_{\mathbf{u}}(\mathbf{e}_3)$. Verify that $\{h_{\mathbf{u}}(\mathbf{e}_1), h_{\mathbf{u}}(\mathbf{e}_2), h_{\mathbf{u}}(\mathbf{e}_3)\}$ is a left handed orthonormal basis.

5. Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be any three vectors in $\mathbb{R}^3$ such that

$$\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) \neq 0 \ .$$

a. Show that

$$|\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)| = |\mathbf{v}_2 \cdot (\mathbf{v}_3 \times \mathbf{v}_1)| = |\mathbf{v}_3 \cdot (\mathbf{v}_1 \times \mathbf{v}_2)| \ .$$

b. Let D be the common value considered in part **(a)**. Define vectors $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ by

$$\mathbf{w}_1 = \frac{1}{D}\mathbf{v}_2 \times \mathbf{v}_3 \ , \qquad \mathbf{w}_2 = \frac{1}{D}\mathbf{v}_3 \times \mathbf{v}_1 \ , \quad \text{and} \quad \mathbf{w}_3 = \frac{1}{D}\mathbf{v}_1 \times \mathbf{v}_2 \ .$$

Show that for all $1 \leq i, j \leq 3$,

$$\mathbf{v}_i \cdot \mathbf{w}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \ .$$

6. Let the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be as in Exercise **5**.

a. Show that $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \mathbb{R}^3$. **Hint:** We know that the span of any set of non-zero vectors in $\mathbb{R}^3$ is either a line through the origin, a plane through the origin, or is all of $\mathbb{R}^3$. So to show $\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \mathbb{R}^3$ it suffices to show that whenever $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) \neq 0$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ cannot possibly lie on any one plane through the origin.

b. Show that for every $\mathbf{x} \in \mathbb{R}^3$, there are unique values of t_1, t_2 and t_3 such that

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3 \ ,$$

and that for each $j = 1, 2, 3$,

$$t_j = \mathbf{w}_j \cdot \mathbf{x} \ .$$

7. Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be given by

$$\mathbf{v}_1 = (1, 0, 1) \ , \quad \mathbf{v}_2 = (1, 1, 1) \ , \quad \mathbf{v}_3 = (1, 2, 3) \ .$$

a. Find vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ such that for all $1 \leq i, j \leq 3$,

$$\mathbf{v}_i \cdot \mathbf{w}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \ .$$

b. Compute numbers t_1, t_2 and t_3 such that

$$t_1(1, 0, 1) + t_2(1, 1, 1) + t_3(1, 2, 3) = (12, -7, 19) \ .$$

8 Show that for all $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$,

$$(\mathbf{b} \times \mathbf{c}) \cdot [(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})] = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|^2 \ .$$