Solutions for Homework 10, Math 291 Fall 2017

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1. Let $\mathcal{D} \subset \mathbb{R}^2$ be the region that is to the left of the parabola $x = y(2-y)$ and below the line $x - 2y + 4 = 0$. Let $\mathcal{C}$ be its boundary given the outward normal orientation. Let $\mathbf{F}(x,y) = (-2xy, 4y + xy)$ Calculate the flux integral $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}s$ both directly, and by making use of the Divergence Theorem.

SOLUTION Where the line and parabola intersect,

$$2y - y^2 = x = 2y - 4$$

so $y = \pm 2$, and then the points of intersection are $(-2, -8)$ and $(0, 2)$. We parameterize $\mathcal{C}$ in two pieces. Let $\mathcal{C}_1$ be parameterized by

$$\mathbf{x}_1(t) = (t(2,t), t) \quad t \in (-2, 2) \ ,$$

and let $\mathcal{C}_2$ be parameterized by

$$\mathbf{x}_1(t) = (2t-4, t) \quad t \in (-2, 2) \ .$$

Then $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$, taking into account that the parameterization of $\mathcal{C}_2$ is inconsistent with that of $\mathcal{C}$.

We now calculate

$$\begin{aligned}\int_{\mathcal{C}_1} \mathbf{F}\cdot\mathbf{N}\mathrm{d}s &= \int_{-2}^{2} \mathbf{F}(\mathbf{X}_1)\cdot(-\mathbf{X}_1'(t))^{\perp}\mathrm{d}t \\ &= \int_{-2}^{2} (-2t^4 + 8t^3 - 8t)\mathrm{d}t = -\frac{128}{5} \ .\end{aligned}$$

$$\begin{aligned}\int_{\mathcal{C}_2} \mathbf{F}\cdot\mathbf{N}\mathrm{d}s &= \int_{-2}^{2} \mathbf{F}(\mathbf{X}_2)\cdot(-\mathbf{X}_2'(t))^{\perp}\mathrm{d}t \\ &= \int_{-2}^{2} (8t - 8t^2)\mathrm{d}t = -\frac{128}{3} \ .\end{aligned}$$

Then we have

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}s = \int_{\mathcal{C}_1} \mathbf{F}\cdot\mathbf{N}\mathrm{d}s - \int_{\mathcal{C}_2} \mathbf{F}\cdot\mathbf{N}\mathrm{d}s = \frac{256}{15} \ .$$

Next we compute $\text{div}\mathbf{F}(x,y) = x - 2y + 4$. Hence

$$\begin{aligned}\int_{\mathcal{D}} \text{div}\mathbf{F}\text{d}A &= \int_{-2}^{2}\left(\int_{2y-4}^{y(2-y)}(x-2y+4)\text{d}x\right)\text{d}y\\ &= \int_{-2}^{2}\left(\frac{1}{2}y^4 - 4y^2 + 8\right)\text{d}y = \frac{256}{15}\ .\end{aligned}$$

We get the same answer as before, as we must on account of the Divergence Theorem.

2. Let $\mathcal{C}$ be the oriented curve in the plane that starts at $(0,0)$, and moves along straight line segments form this point to $(1,2)$, then from this point to $(-1,4)$, then from this point to $(-3,2)$, and finally then from this point to $(-2,0)$. Let $\mathbf{F}(x,y) = (x^3y + y^2x^2, x + y + x^2y + y^2x)$. Compute the flux integral $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{N}\text{d}s$.

It is not a lot more work to parameterize the four legs of $\mathcal{C}$ and compute the flux directly, but it is a good idea to use the Divergence Theorem for practice with setting up the multiple integrals.

SOLUTION We compute

$$\text{div}\mathbf{F}(x,y) = 3x^2y + 2y^2x + 1 + x^2 + 2xy\ .$$

Let $\mathcal{C}_2$ be the straight line segment running from $(-2,0)$ to $(0,0)$, which we parameterize by

$$\mathbf{x}_2(t) = (t-2, 0)\ ,\qquad t \in (0,2)\ .$$

Along $\mathcal{C}_2$, $\mathbf{N} = -\mathbf{e}_2$, and so

$$\mathbf{F}\cdot\mathbf{N} = x$$

all along $\mathcal{C}_2$. Thus

$$\int_{\mathcal{C}_2} \mathbf{F}\cdot\mathbf{N}\text{d}s = \int_0^2 (2-t)\text{d}t = 2\ .$$

Now let $\mathcal{D}$ be the polygonal region enclosed by $\mathcal{C} + \mathcal{C}_2$. We write this as a union of to pieces:

$$\mathcal{D}_1 = \{-2 - y/2 \le x \le y/2\ , y \in (0,2)\ \}$$

and

$$\mathcal{D}_2 = \{y - 5 \le x \le 3 - y\ , y \in (2,4)\ \}$$

We compute

$$\begin{aligned}\int_{\mathcal{D}_1} \text{div}\text{d}A &= \int_0^2\left(\int_{-2-y/2}^{y/2}(3x^2y + 2y^2x + 2xy + x^2 + 1)\text{d}x\right)\text{d}y\\ &= \int_0^2\left(\frac{1}{4}y^4 - \frac{5}{12}y^3 + \frac{1}{2}y^2 + 7y + \frac{14}{3}\right)\text{d}y = \frac{123}{5}\ .\end{aligned}$$

We then compute

$$\begin{aligned}\int_{\mathcal{D}_2} \text{div}\text{d}A &= \int_2^4\left(\int_{y-5}^{3-y}(3x^2y + 2y^2x + 2xy + x^2 + 1)\text{d}x\right)\text{d}y\\ &= \int_2^4\left(2y^4 - \frac{82}{3}y^3 + 106y^2 + 100y + \frac{176}{3}\right)\text{d}y = -\frac{272}{15}\ .\end{aligned}$$

Hence

$$\int_{\mathcal{D}} \mathrm{div}\mathbf{F}\mathrm{d}A = \frac{97}{15}$$

and then

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}s = \frac{97}{15} - 2 = \frac{67}{15} \ .$$

3: Let $\mathcal{S}$ be the part of the surface in $\mathbb{R}^3$ given by $\sqrt{x^2+y^2} = 8 - z$ that lies inside the cylinder $x^2 + y^2 = 4$. With $\mathbf{F} = (2yz - y^2, x^2z - 2x, x^2y)$, compute the flux

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \ ,$$

where $\mathbf{N}$ is taken to point outward from the z-axis.

SOLUTION We first solve this by direct calculation of the given flux integral. For this, we parameterize the surface in cylindrical coordinates using

$$\mathbf{X}(r,\theta) = \big(r\cos\theta, r\sin\theta, 8 - r\big)$$

We compute

$$\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta) = r\big(\cos\theta, \sin\theta, 1\big) \ .$$

This points outward from the z-axis, so it is the orientation we want. We now compute

$$\mathbf{F}(\mathbf{X}(r,\theta))\cdot\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta) = r^2\cos\theta\sin\theta(14 - 2r - r\sin\theta + 8r\cos\theta) \ .$$

Integrating over $\theta \in (0, 2\pi)$ gives zero by symmetry, and so

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = 0 \ .$$

It is easier to use Stokes' Theorem to replace $\mathcal{S}$ by $\mathcal{S}_2$, the disk of radius 2 at height $z = 6$ that has the same boundary as $\mathcal{S}$, and the same orientation if we give it the upward normal.

On $\mathcal{S}_2$, $\mathbf{N} = \mathbf{e}_3$ everywhere, and so $\mathbf{F}\cdot\mathbf{N} = x^2y$. It is cleat that when we integrate this over the disk $\{(x,y) \ : \ x^2+y^2 \le 4\}$ we get zero, by symmetry.

4: Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies inside the sphere $x^2+y^2+z^2 = 4$, and above the graph of $z = 1/\sqrt{x^2+y^2}$. Let $\mathbf{F} = (y + z^2, x + z^2, 2z(x+y))$ and let $\mathbf{N}$ be the outward normal to $\mathcal{S}$, the boundary of $\mathcal{V}$. Compute the total flux

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \ .$$

SOLUTION It will be much easier if we use the Divergence Theorem rather than direct calculation. We compute $\mathrm{div}\mathbf{F}(x,y,z) = 2(x+y)$. Since the region $\mathcal{V}$ is symmetric in x and y,

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = \int_{\mathcal{V}} \mathrm{div}\mathbf{F}\mathrm{d}V = 0 \ .$$

5: Consider the two vector fields

$$\mathbf{F} = (yz^2 - 2xy, xz^2 - x^2, 2xyz) \qquad \text{and} \qquad \mathbf{G} = (z^2, y, x) \ .$$

(a) Compute the divergence of $\mathbf{F}$ and $\mathbf{G}$.

(b) Let $\mathcal{S}$ be the unit sphere, and $\mathbf{N}$ its outward normal. Compute

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \qquad \textit{and} \qquad \int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S \ .$$

Justify your answers to receive credit.

SOLUTION a We compute $\mathrm{div}\mathbf{F} = 2y(x-1)$ and $\mathrm{div}\mathbf{G} = 1$.

For **b** we use the Divergence Theorem to compute, first for $\mathbf{G}$.

$$\int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S = \int_{\mathcal{V}} \mathrm{div}\mathbf{G}\mathrm{d}V = \mathrm{vol}(\mathcal{V}) = \frac{4\pi}{3} \ .$$

6: Consider the two vector fields

$$\mathbf{F} = (y+z^2, x+z^2, 2zx+2zy) \qquad \text{and} \qquad \mathbf{G} = (y+z^2, x+z^2, 2x+2y) \ .$$

(a) Compute the divergence $\mathbf{F}$ and $\mathbf{G}$.

(b) Let $\mathcal{V}$ be the intersection of the ball of radius 1 centered at the origin, and the ball of radius 1 centered at $(1,0,0)$. Let $\mathcal{S}$ be the boundary of $\mathcal{V}$ oriented with the outward unit normal $\mathbf{N}$. Compute

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \qquad \textbf{and} \qquad \int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S \ .$$

SOLUTION a We compute $\mathrm{div}\mathbf{F} = 2(x+y)$ and $\mathrm{div}\mathbf{G} = 0$.

For **b** we use the Divergence Theorem to compute

$$\int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S = \int_{\mathcal{V}} \mathrm{div}\mathbf{G}\mathrm{d}V = \mathrm{vol}(\mathcal{V}) = 0 \ .$$

We have to work a little harder for $\mathbf{F}$. Again by the Divergence Theorem,

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = 2\int_{\mathcal{V}} (x+y)\mathrm{d}V \ .$$

The region is symmetric in y and so the integral over y is zero. We are left with computing x over the region

$$\mathcal{V} = \{(x,y,z) \ : \ 1-\sqrt{1-y^2+z^2} \le x \le \sqrt{1-y^2+z^2} \ , 0 \le y^2+z^2 \le 3/4\} \ .$$

Using cylindrical coordinates about the x-axis, We find

$$\begin{aligned} \int_{\mathcal{V}} x\mathrm{d}V &= 2\pi\int_0^{\sqrt{3/4}} \left(\int_{1-\sqrt{1-r^2}}^{\sqrt{1-r^2}} x\mathrm{d}x\right) r\mathrm{d}r \\ &= 2\pi\int_0^{\sqrt{3/4}} \left(\sqrt{1-r^2}-\frac{1}{2}\right) r\mathrm{d}r \\ &= \frac{5}{48} \ . \end{aligned}$$

Finally, we multiply by 2 to find

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = \frac{5}{24} \ .$$

7: As in Exercise 3, let $\mathcal{S}$ be the part of the surface in $\mathbb{R}^3$ given by $\sqrt{x^2+y^2} = 8 - z$ that lies inside the cylinder $x^2 + y^2 = 4$. With $\mathbf{F} = (2yz - y^2, x^2z - 2x, x^2y)$, Use Stokes' Theorem evaluate the flux

$$\int_{\mathcal{S}} \text{curl}\mathbf{F} \cdot \mathbf{N} \mathrm{d}S \ ,$$

where $\mathbf{N}$ is taken to point outward from the z-axis, by computing a line integral.

SOLUTION The boundary $\mathcal{C}$ of $\mathcal{S}$ is consistently parameterized by

$$\mathbf{x}(t) = (2\cos t, 2\sin t, 6) \qquad t \in (0, 2\pi) \ .$$

We compute

$$\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = 8\sin t - 8\cos^2 t \sin t + 48\cos^3 t + 40\cos^2 t - 48 \ .$$

We now integrate from $t = 0$ to $t = 2\pi$ and by symmetry the first three terms make no contribution. We find

$$\int_{\mathcal{S}} \mathbf{F} \cdot \mathbf{N} \mathrm{d}S = \int_0^{2\pi} (40\cos^2 t - 48)\mathrm{d}t = 20\pi - 96\pi = -76\pi \ .$$

8: Consider the two vector fields

$$\mathbf{F} = (yz^2 - 2xy, xz^2 - x^2, 2xyz) \qquad \text{and} \qquad \mathbf{G} = (z^2, y, x) \ .$$

(a) Compute the curls of $\mathbf{F}$ and $\mathbf{G}$.

(b) Let $\mathcal{S}$ be the part of the centered sphere of radius 2 that lies above the plane $x + y + z = 1$, oriented with its unit normal $\mathbf{N}$ pointing upwards. Let $\mathcal{C}$ be the bounding curve with the consistent orientation. Compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} \mathrm{d}s \qquad \textit{and} \qquad \int_{\mathcal{C}} \mathbf{G} \cdot \mathbf{T} \mathrm{d}s \ .$$

Justify your answers to receive credit.

(c) One of these vector fields is conservative. Identify the conservative vector field, and a potential function for it.

SOLUTION a. We compute $\text{curl}\mathbf{F} = \mathbf{0}$ and $\text{curl}\mathbf{G} = (0, 2z - 1, 0)$.

b. Since $\mathcal{C}$ is a closed curve and $\mathbf{F}$ is conservative, $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} \mathrm{d}s = 0$. It is best to use an adapted system of coordinates, Let $\mathbf{u}_3 = 3^{-1/2}(1, 1, 1)$ be the unit normal to the plane. It is easy to see that $\mathbf{u}_1 = 2^{-1/2}(1, -1, 0)$ is a unit vector that is orthogonal to $\mathbf{u}_3$. Define $\mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1 = 6^{-1/2}(1, 1, -2)$. We can now parameterize the circle easily. The center is the point $(1/3, 1/3, 1/3)$. Then

$$(x - 1/3)^2 + (y - 1/3)^2 + (z - 1/3)^2 = x^2 + y^2 + z^2 - \frac{2}{3}(x + y + z) + \frac{1}{3} = 4 - \frac{2}{3} + \frac{1}{3} = \frac{11}{3} \ .$$

Therefore, the radius of the circle is $\sqrt{11/3}$. It can now be parameterized as

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{3}(1, 1, 1) + \sqrt{11/3}\cos t\mathbf{u}_1 + \sqrt{11/3}\sin t\mathbf{u}_2 \\ &= \frac{1}{6}(2 + \sqrt{77}\cos t + \sqrt{22}\sin t)\mathbf{e}_1 - \frac{1}{6}(2 + \sqrt{77}\cos t + \sqrt{22}\sin t)\mathbf{e}_2 + \frac{1}{3}(1 - \sqrt{22}\sin t)\mathbf{e}_3 \ . \end{aligned}$$

From here one works out that

$$\mathbf{G}(\mathbf{x}(t))\cdot\mathbf{x}'(t) = -\frac{88}{9\sqrt{3}}\left(\cos^2 t + \frac{\cos t\sin t}{\sqrt{3}} - \frac{7}{16}\right) - \frac{11}{4}(\sqrt{66}\sin t - \sqrt{22}\cos t)\left(\cos^2 t - \frac{10}{11}\right) .$$

Most terms drop out upon integration in t, leaving only

$$\int_{\mathcal{C}} \mathbf{G}\cdot\mathbf{T}\mathrm{d}s = \int_0^{2\pi} -\frac{88}{9\sqrt{3}}\left(\cos^2 t - \frac{7}{16}\right)\mathrm{d}t = -\pi\frac{11}{9\sqrt{3}}$$

We can also use Stokes' Theorem to do this. This will lead to simple calculustions because the curl of $\mathbf{G}$ is quite simple, and because $\mathcal{C}$ not only bounds $\mathcal{S}$, but it also bounds $\widetilde{\mathcal{S}}$, a disk in the plane $x+y+z=1$. The normal to this plane, with the consistent orientationm is $\mathbf{N} = 3^{-1/2}(1,1,1)$ everywhere on $\widetilde{\mathcal{S}}$. Hence by Stokes'Theorem,

$$\int_{\mathcal{C}} \mathbf{G}\cdot\mathbf{T}\mathrm{d}s = \int_{\widetilde{\mathcal{S}}}(0,2z-1,0)\cdot\mathbf{N}\mathrm{d}S = \frac{1}{\sqrt{3}}\int_{\widetilde{\mathcal{S}}}(2z-1)\mathrm{d}S .$$

Let $\langle z\rangle$ denote the aveage of z over $\widetilde{\mathcal{S}}$, and let A denote the area of $\widetilde{\mathcal{S}}$. Then

$$\frac{1}{\sqrt{3}}\int_{\widetilde{\mathcal{S}}}(2z-1)\mathrm{d}S = \frac{1}{\sqrt{3}}(2\langle z\rangle - 1)A .$$

Evidently $\langle z\rangle$ is the z-coordinate of the center of $\widetilde{\mathcal{S}}$; that is, $\langle z\rangle = 1/3$, and $A = \pi\frac{11}{3}$. Altogether,

$$\int_{\mathcal{C}} \mathbf{G}\cdot\mathbf{T}\mathrm{d}s = -\pi\frac{11}{9\sqrt{3}} .$$

which agrees with what we found before by direct calculuation. The calculuation uisng Stokes' Theorem is considerably shorter.

For **c**, we know that since curl$\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is a gradient. Integrating we find that

$$\varphi(x,y,z) = xyz^2 - x^2y$$

is a potential function for $\mathbf{F}$.

9: Consider the two vector fields

$$\mathbf{F} = (y+z^2, x+z^2, 2zx+2zy) \qquad \text{and} \qquad \mathbf{G} = (y+z^2, x+z^2, 2x+2y) .$$

(a) Compute the curl of $\mathbf{F}$ and $\mathbf{G}$.

(b) Let $\mathcal{S}$ be the part of the ellipsoidal surface

$$x^2 + \frac{1}{2}y^2 + \frac{1}{4}z^2 = 1$$

above the plane $z=1$, oriented so the unit normal $\mathbf{N}$ points upwards. Let $\mathcal{C}$ be the bounding curve with the consistent orientation. Compute

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\mathrm{d}s \qquad \textbf{and} \qquad \int_{\mathcal{C}} \mathbf{G}\cdot\mathbf{T}\mathrm{d}s .$$

(c) One of these vector fields is conservative. Identify the conservative vector field, and a potential function for it.

SOLUTION We compute curl$\mathbf{F} = \mathbf{0}$ and curl$\mathbf{G} = 2(1 - z, z - 1, 0)$.

For **b**, since $\mathbf{F}$ is conservative and $\mathcal{C}$ is closed, $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} \mathrm{d}s = 0$. Note that $\mathcal{C}$ is also the boundary of the ellipse in the plane $z = 1$ with $x^2 + y^2/2 \leq 3/4$; call this $\widetilde{\mathcal{S}}$, and the consistent unit normal is $\mathbf{N} = (0, 0, 1)$ everywhere on $\widetilde{\mathcal{S}}$. Thus curl$\mathbf{G} \cdot \mathbf{N} = 0$ everywhere on $\widetilde{\mathcal{S}}$, and the by Stokes' Theorem,

$$\int_{\mathcal{C}} \mathbf{G} \cdot \mathbf{T} \mathrm{d}s = 0 \ .$$

For **c**, we know that since curl$\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is a gradient. Integrating we find that

$$\varphi(x, y, z) = z^2(x + y) + xy$$

is a potential function for $\mathbf{F}$.