Homework 10, Math 291, Fall 2017

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1. Let $\mathcal{D} \subset \mathbb{R}^2$ be the region that is to the left of the parabola $x = y(2-y)$ and below the line $x - 2y + 4 = 0$. Let $\mathcal{C}$ be its boundary given the outward normal orientation. Let $\mathbf{F}(x,y) = (-2xy, 4y + xy)$ Calculate the flux integral $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}s$ both directly, and by making use of the Divergence Theorem.

2. Let $\mathcal{C}$ be the oriented curve in the plane that starts at $(0,0)$, and moves along straight line segments form this point to $(1,2)$, then from this point to $(-1,4)$, then from this point to $(-3,2)$, and finally then from this point to $(-2,0)$. Let $\mathbf{F}(x,y) = (x^3y + y^2x^2, x + y + x^2y + y^2x)$. Compute the flux integral $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}s$.

It is not a lot more work to parameterize the four legs of $\mathcal{C}$ and compute the flux directly, but it is a good idea to use the Divergence Theorem for practice with setting up the multiple integrals.

3: Let $\mathcal{S}$ be the part of the surface in $\mathbb{R}^3$ given by $\sqrt{x^2+y^2} = 8 - z$ that lies inside the cylinder $x^2 + y^2 = 4$. With $\mathbf{F} = (2yz - y^2, x^2z - 2x, x^2y)$, compute the flux

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \ ,$$

where $\mathbf{N}$ is taken to point outward from the z-axis.

4: Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$, and above the graph of $z = 1/\sqrt{x^2+y^2}$. Let $\mathbf{F} = (y + z^2, x + z^2, 2z(x+y))$ and let $\mathbf{N}$ be the outward normal to $\mathcal{S}$, the boundary of $\mathcal{V}$. Compute the total flux

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \ .$$

5: Consider the two vector fields

$$\mathbf{F} = (yz^2 - 2xy, xz^2 - x^2, 2xyz) \qquad \text{and} \qquad \mathbf{G} = (z^2, y, x) \ .$$

(a) Compute the divergence of $\mathbf{F}$ and $\mathbf{G}$.

(b) Let $\mathcal{S}$ be the unit sphere, and $\mathbf{N}$ its outward normal. Compute

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \qquad \textit{and} \qquad \int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S\ .$$

Justify your answers to receive credit.

6: Consider the two vector fields

$$\mathbf{F} = (y+z^2, x+z^2, 2zx+2zy) \qquad \text{and} \qquad \mathbf{G} = (y+z^2, x+z^2, 2x+2y)\ .$$

(a) Compute the divergence $\mathbf{F}$ and $\mathbf{G}$.

(b) Let $\mathcal{V}$ be the intersection of the ball of radius 1 centered at the origin, and the ball of radius 1 centered at $(1,0,0)$. Let $\mathcal{S}$ be the boundary of $\mathcal{V}$ oriented with the outward unit normal $\mathbf{N}$. Compute

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \qquad \textbf{and} \qquad \int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S\ .$$

7: As in Exercise 3, let $\mathcal{S}$ be the part of the surface in $\mathbb{R}^3$ given by $\sqrt{x^2+y^2} = 8-z$ that lies inside the cylinder $x^2+y^2=4$. With $\mathbf{F} = (2yz-y^2, x^2z-2x, x^2y)$, Use Stokes' Theorem evaluate the flux

$$\int_{\mathcal{S}} \mathrm{curl}\mathbf{F}\cdot\mathbf{N}\mathrm{d}S\ ,$$

where $\mathbf{N}$ is taken to point outward from the z-axis, by computing a line integral.

8: Consider the two vector fields

$$\mathbf{F} = (yz^2-2xy, xz^2-x^2, 2xyz) \qquad \text{and} \qquad \mathbf{G} = (z^2, y, x)\ .$$

(a) Compute the curls of $\mathbf{F}$ and $\mathbf{G}$.

(b) Let $\mathcal{S}$ be the part of the centered sphere of radius 2 that lies above the plane $x+y+z=1$, oriented with its unit normal $\mathbf{N}$ pointing upwards. Let $\mathcal{C}$ be the bounding curve with the consistent orientation. Compute

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\mathrm{d}s \qquad \textit{and} \qquad \int_{\mathcal{C}} \mathbf{G}\cdot\mathbf{T}\mathrm{d}s\ .$$

Justify your answers to receive credit.

(c) One of these vector fields is conservative. Identify the conservative vector field, and a potential function for it.

9: Consider the two vector fields

$$\mathbf{F} = (y+z^2, x+z^2, 2zx+2zy) \qquad \text{and} \qquad \mathbf{G} = (y+z^2, x+z^2, 2x+2y)\ .$$

(a) Compute the curl of $\mathbf{F}$ and $\mathbf{G}$.

(b) Let $\mathcal{S}$ be the part of the ellipsoidal surface

$$x^2 + \frac{1}{2}y^2 + \frac{1}{4}z^2$$

above the plane $z = 1$, oriented so the unit normal $\mathbf{N}$ points upwards. Let $\mathcal{C}$ be the bounding curve with the consistent orientation. Compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} \mathrm{d}s \qquad \textbf{and} \qquad \int_{\mathcal{C}} \mathbf{G} \cdot \mathbf{T} \mathrm{d}s \ .$$

(c) One of these vector fields is conservative. Identify the conservative vector field, and a potential function for it.