Solutions for the Final Exam, Math 291 Fall 2017

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Material from Part 1 of the Course:

1: (a) Let $\mathbf{a} = (2, -2, 3)$ and $\mathbf{b} = (2, -1, 2)$. Find a right-handed orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$ such that $\mathbf{u}_1$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{u}_2$ is orthogonal to $\mathbf{b}$. How many such bases are there?

(b) Find an equation for the plane through the origin that is spanned by $\mathbf{a}$ and $\mathbf{b}$; i.e., the plane parameteried by $\mathbf{x}(s,t) = s\mathbf{a} + t\mathbf{b}$, $(s,t) \in \mathbb{R}^2$. Also, compute the distance between $(1, 1, 1)$ and this plane.

(c) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be any orthonormal set of vectors. Compute the distance between the lines parameterized by $\mathbf{x}(s) = \mathbf{v}_1 + s\mathbf{v}_2$ and $\mathbf{y}(t) = \mathbf{v}_2 + t\mathbf{v}_3$. This does not require any long computations!

SOLUTION (a) We conpute $\mathbf{a} \times \mathbf{b} = (-1, 2, 2)$ hence we must take

$$\mathbf{u}_1 = \pm\frac{1}{3}(-1, 2, 2) \ .$$

Then since $\mathbf{u}_2$ is required to be orthogonal to $\mathbf{u}_1$ and $\mathbf{b}$, it must be a multiple of $(-1, 2, 2) \times (2, -1, 2) = (6, 6, -3)$, and hence we must take

$$\mathbf{u}_2 := \pm\frac{1}{3}(2, 2, -1) \ .$$

Hence there are 4 choices for the pair $\{\mathbf{u}_1, \mathbf{u}_2\}$. Once there are made, the requirement that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be right handed forces $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$. Hence there are exactly 4 such bases. Choosing the + signs above, we get

$$\mathbf{u}_1 = \frac{1}{3}(-1, 2, 2) \ , \quad \mathbf{u}_2 := \frac{1}{3}(2, 2, -1) \ , \qquad \mathbf{u}_3 := \frac{1}{3}(-2, 1, -2) \ .$$

(b) The equation can be written in the form $\mathbf{u} \cdot \mathbf{x} = 0$, where $\mathbf{u}$ is a unit vector that is orthogonal to $\mathbf{a}$ and $\mathbf{b}$. We have seen that $\mathbf{u}_1$ formn part **(a)** is such a vector. Hence the equation is $\mathbf{u}_1 \cdot \mathbf{x} = 0$ which simplified to

$$-x + 2y + 2z = 0 \ .$$

The distance form $(1, 1, 1)$ to this plane is $|\mathbf{u}_1 \cdot (1, 1, 1)| = 1$.
(c) We compute

$$\|\mathbf{x}(s) - \mathbf{y}(t)\|^2 = \|(\mathbf{v}_1 + s\mathbf{v}_2) - (\mathbf{v}_2 + t\mathbf{v}_3)\|^2 = \|\mathbf{v}_1 + (s-1)\mathbf{v}_2 + t\mathbf{v}_3\|^2 = 1 + (s-1)^2 + t^2 \ .$$

This is evidently minimized by choosing $s = 1$ and $t = 0$, and then the distance is 1.

2: Let $\mathbf{x}(t)$ be the parameterized curve given by

$$\mathbf{x}(t) = (2\cos t - 2\sin t, 4\cos t + 2\sin t, -4\cos t + \sin t)\ .$$

(a) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(b) Find the minimum and maximum values of the curvature and the times t at which they occur.

(c) Compute $\mathbf{T}(0)\cdot\mathbf{x}(t)$, $\mathbf{N}(0)\cdot\mathbf{x}(t)$, and $\mathbf{B}(0)\cdot\mathbf{x}(t)$. Find a system of equations that specifries the curve. What kind of curve is it?

SOLUTION (a) We compute

$$\mathbf{x}'(t) = (-2\sin t - 2\cos t, -4\sin t + 2\cos t, 4\sin t + \cos t)$$

and

$$\mathbf{x}''(t) = (-2\cos t + 2\sin t, -4\cos t - 2\sin t, 4\cos t - \sin t)\ .$$

Then $\mathbf{x}'(t)\cdot\mathbf{x}'(t) = 9(4 - 3\cos^2 t)$ so that $v(t) = 3\sqrt{4 - 3\cos^2 t}$. Thus

$$\mathbf{T}(t) = \frac{1}{3\sqrt{4 - 3\cos^2 t}}(-2\sin t - 2\cos t, -4\sin t + 2\cos t, 4\sin t + \cos t)\ .$$

We next compute

$$\mathbf{x}'(t)\times\mathbf{x}''(t) = (12, 6, 12) = 6(2, 1, 2)\ .$$

But $\mathbf{x}'(t)\times\mathbf{x}''(t) = v^3(t)\kappa(t)\mathbf{B}(t)$ and so $v^3(t)\kappa(t) = 18$ and hence

$$\kappa(t) = \frac{2}{3(4 - 3\cos^2 t)^{3/2}}\ .$$

From this same computation, we see thar $\mathbf{B}(t)$ is constant, so that $\tau(t) = 0$.

(b) The curvature is maximal when $4 - 3\cos^2 t$ is minimal, which is when $\cos t = \pm 1$, so this means $t = k\pi$, k and integer.

(c) We have seen above that $\mathbf{x}'(0) = (-2, 2, 1)$, and so $\mathbf{T}(0) = \frac{1}{3}(-2, 2, 1)$. We have seen that $\mathbf{B}(0) = \frac{1}{3}(2, 1, 2)$. Finally, $\mathbf{N}(0) = \mathbf{B}(0)\times\mathbf{T}(0) = \frac{1}{3}(-1, -2, 2)$. Then

$$\mathbf{T}(0)\cdot\mathbf{x}(t) = 3\sin t\ ,\quad \mathbf{N}(0)\cdot\mathbf{x}(t) = 6\cos t\ ,\quad \mathbf{B}(0)\cdot\mathbf{x}(t) = 0\ ,$$

and hence

$$\mathbf{x}(t) = 3\sin t\mathbf{T}(0) + 6\cos t\mathbf{N}(0)\ .$$

The curve is an ellipse; a system of equations for the ellipse is

$$\begin{aligned} (2\mathbf{T}(0)\cdot\mathbf{x})^2 + (\mathbf{N}(0)\cdot\mathbf{x})^2 &= 36 \\ \mathbf{B}(0)\cdot\mathbf{x} &= 0\ . \end{aligned}$$

3: Let $f(x, y) = 2x^2y + y^2x - \dfrac{1}{2}x^4$.

(a) Find all of the critical points of f, and find the value of f at each of the critical points.

(b) Find all points on the curve given by $f(x,y) = -\frac{3}{2}$ that lie in the right half plane and at which the tangent line is vertical.

(c) Are there any points on the curve given by $f(x,y) = -\frac{3}{2}$ where this curve crosses itself? If so, find them. If not, explain why not.

(d) For which unit vector $\mathbf{u}$ is $\left.\frac{\mathrm{d}}{\mathrm{d}t} f\big((1,1) + t\mathbf{u}\big)\right|_{t=0}$ the largest?

SOLUTION (a) We compute

$$\nabla f(x,y) = (-2x^3 + 4xy + y^2, 2x^2 + 2xy) \ .$$

The critical points are the solutions of

$$\begin{aligned} -2x^3 + 4xy + y^2 &= 0 \\ 2x(x+y) &= 0 \ . \end{aligned}$$

From the second equation, we see that either $x = 0$ or $x + y = 0$. If $x = 0$, the first equation reduces to $y^2 = 0$, which gives the critical point $\mathbf{x}_1 := (0,0)$. If $y = -x$, the first equation becomes $-2x^3 - 4x^2 + x^2 = 0$, or $2x^3 + 3x^2 = 0$, and the non-zero root is $x = -3/2$. This give the critical point $\mathbf{x}_2 = (-3/2, 3/2)$. Then $f(\mathbf{x}_1) = 0$ and $f(\mathbf{x}_2) = \frac{27}{32}$.

(b) The tangent line is vertical at points (x,y) where $\nabla f(x,y) = (c,0)$ for some c. From the computations in part (**a**), this can only happen if $x(x+y) = 0$. Since we wnat solutions in the open right half plane, $x = 0$ is inadmissible, we need only consider $y = -x$. For $y = -x$, the equation $f(x,y) = -\frac{3}{2}$ reduces to $-x^3 - \frac{1}{2}x^4 = -\frac{3}{2}$. This has one obvious root, namely $x = 1$. Let $p(x) := -x^3 - \frac{1}{2}x^4$. Then $p''(x) = -6(x + x^2)$ which is negative for $x > 0$. Hence $p(x)$ is concave on $x > 0$, and can only cross the x-axis once. Thus, $x = 1$ is the only solution, and since $y = -x$, the only such point is $(1,-1)$.

(c) There are no such points; the curve an only cross itself at critical points, and $-\frac{3}{2}$ is not the value of f at either critical point.

(d) Since $\nabla f(1,1) = (3,4)$, $\mathbf{u} = \|\nabla f(1,1)\|^{-1}\nabla f(1,1) = \frac{1}{5}(3,4)$.

Material from Part 2 of the Course:

4: Let $f(x,y) = 2x^2y + y^2x - \dfrac{1}{2}x^4$, as in problem **3.**

(a) Compute the Hessian of f at each critical point and determine whether each critical point is a local minimum, a local maximum, a saddle point, or if it cannot be classified through a computation of the Hessian.

(b) Sketch a contour plot of f near each critical point of determined type.

(c) For which unit vectors $\mathbf{u}$ is $\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2} f((-3/2, 3/2) + t\mathbf{u})\right|_{t=0}$ the largest? For which is it smallest?

SOLUTION (a) We compute $[\mathrm{Hess}_f(x,y)] = \begin{bmatrix} 4y - 6x^2 & 2y + 4x \\ 2y + 4x & 2x \end{bmatrix}$. We have already found that the critical points are $\mathbf{x}_1 = (0,0)$ and $\mathbf{x}_2 = (-3/2, 3/2)$. We then compute

$$[\mathrm{Hess}_f(0,0)] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad [\mathrm{Hess}_f(-3/2, 3/2)] = -3\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \ .$$

$[\mathrm{Hess}_f(0,0)]$ is degenerate, so the nature of this critical point is not determined by the Hessian. (For $(x,y) \approx (0,0)$, the cubic terms are dominant and the "best cubic approximation to f near $(0,0)$ is given by $h(x,y) = (2x+y)xy$. Note that $h(x,y) = 0$ all along the three lines $x = 0$, $y = 0$ and $2x + y = 0$. These three lines divide the plane into 6 sectors meeting at $(0,0)$, one being the usual posiitve quadrant. The function $h(x,y)$ is positive in the positive quadrant, and changes sign each time a line is crossed. Hence it is positive in 3 of the sectors and negative in the other three. The graph of $x = h(x,y)$ is called a "monkey saddle", since it curves downward in 3 directions, two for the two legs and one for the tail of the monkey. This was not a required part of the answer, but going beyond the best quadratic approximation, one can get a pretty good picture of what is happening at the this degenerate crtical point.)

(b) Let $A := [\mathrm{Hess}_f(-3/2,3/2)] = -3\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$. Then $\det(A) = 9(10-4) > 0$, and the upper-left entry is negative. Thus, by Sylvester's criterion, both eigenvalues are negative, and $(-3/2,3/2)$ is a local maximum.

(c) We only need to deal with $\mathbf{x}_2 = (-3/2,3/2)$. We compute that $\det([\mathrm{Hess}_f(-3/2,3/2)] - tI_{3\times 3}) = t^2 - \frac{21}{2}t + \frac{27}{2}$. Therefore, the eigenvalues are

$$\mu_1 = -\frac{3}{2} \qquad \text{and} \qquad \mu_2 = -9\ .$$

Then $[\mathrm{Hess}_f(-3/2,3/2)] + \frac{3}{2}I_{3\times 3} = -\frac{3}{2}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$. Therefore,

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}}(-1,2) \qquad \text{and} \qquad \mathbf{u}_2 = \frac{1}{\sqrt{5}}(2,1)\ .$$

It follows that for $(u,v) \approx (0,0)$,

$$f(\mathbf{x}_2 + u\mathbf{u}_1 + v\mathbf{u}_2) = \frac{27}{32} - \frac{3}{2}u^2 - 9v^2$$

The level curves of the right hand side are ellipses in the u,v-plane, and the major axis runs along the u-axis. The ratio of the major ot minor axis lengths is $\sqrt{6}$, and the ellipse *must* show this strong ecentricity to get any credit. The sketch should then show such ellipses centered on the point $(-3/2,3/2)$ and with hte major axis running along the line through the origin and $\mathbf{u}_1$; i.e., the line $y = \frac{1}{2}x$.

(c) By what we have computed in part **(b)**, and by the meaning of the eigenvectors of the Hessian, $\mathbf{u} = \pm\mathbf{u}_1$ give the maximum, and $\mathbf{u} = \pm\mathbf{u}_2$ give the minimum.

5: (a) Let $f(x,y) = (x+y)^4 + (x-y)^2$. Find the minimum and maximum values of f on the unit circle $x^2 + y^2 = 1$, and all of the places on the circle at which f takes on these values.

(b) Let $\mathcal{S}$ be closed upper hemisphere of the unit sphere in $\mathbb{R}^3$. Let $f(x,y,z) = xyz$. Find the minimum and maximum values of f on $\mathcal{S}$, and all of the points at which f takes on these values. Explain how you are taking into account both of the constraints $x^2 + y^2 + z^2 = 1$ and $z \geq 0$.

SOLUTION (a) Define $g(x,y) = x^2 + y^2$, and then the constraint equation is $g(x,y) = 1$. Lagrange's method gives the equations

$$\begin{aligned} \det\left(\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix}\right) = 8(x+y)^3(y-x) + 2(x-y)(y+x) &= 0 \\ x^2 + y^2 &= 1 \end{aligned}$$

If $y = x$, then $x = y = \pm 1/\sqrt{2}$, where $f(x,y) = (2/\sqrt{2})^4 = 4$. If $y \neq x$, then we can divide by $y - x$ and 2 to get $4(x+y)^3 - (x+y) = 0$. If $y = -x$ then $y = -x = \pm 1/\sqrt{2}$, where $f(x,y) = (2/\sqrt{2})^2 = 2$.

If $y \neq -x$ we can divide further by $x + y$, to get $4(x+y)^2 - 1 = 0$. Then $x + y = \pm 1/2$, $x^2 + ((\pm 1/2) - x)^2 = 1$, $2x^2 \pm x - (3/4) = 0$, $x = (\pm 1 \pm \sqrt{7})/4$, $y = (\pm 1 \mp \sqrt{7})/4$, $x - y = \pm\sqrt{7}/2$, $f(x,y) = (1/16) + (7/4) = 29/16$.

The maximizers are $\pm(1/\sqrt{2}, 1/\sqrt{2})$ with the maximum value 4.

The minimizers are $\pm((1 \pm \sqrt{7})/4, (1 \mp \sqrt{7})/4)$ with the minimum value 29/16.

(b) The constraint set is a closed "inverted bowl". In particular, it is compact so there exist maximizers and minimizers. The Lagrange method applied to the hemispherical $x^2 + y^2 + z^2 = 1$ will give all possible maximizers and minimizers except possibly on the circular rim $x^2 + y^2 = 1$, $z = 0$, where a separate analysis should be used. The separate analysis is easy, since $xyz = 0$ on the circular rim. However, at suitable points of the bowl, xyz will positive, and at other points, xyz will be negative, so we can ignore the circular rim as well as any point where $xyz = 0$.

On the hemisphere we use Lagrange multipliers, and solve

$$\nabla(xyz) = \lambda \nabla(x^2 + y^2 + z^2) \text{ and } x^2 + y^2 + z^2 = 1,\ z > 0.$$

This yields

$$\begin{aligned} yz &= 2\lambda x \\ xz &= 2\lambda y \\ xy &= 2\lambda z \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

Since we can ignore points where x, y, or z is 0, we can divide by x, y, and z if we wish. This gives $yz/x = 2\lambda = xz/y = xy/z$. Therefore $y/x = x/y$ and $z/y = y/z$. So $x^2 = y^2 = z^2$. Substituting in the constraint equation, $x^2 = y^2 = z^2 = 1/3$, $z = 1/\sqrt{3}$. There are four candidates: $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, 1/\sqrt{3})$. The maximizers are $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and $(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, where $xyz = 1/3\sqrt{3}$. The other two candidates are minimizers, with $xyz = -1/3\sqrt{3}$.

6: (a) Let D be the set in the positive quadrant of $\mathbb{R}^2$ that bounded by

$$x^2 + 9y^2 = 10 \qquad \text{and} \qquad xy = 1\ .$$

Let $f(x,y) = xy$. Compute $\int_D f(x,y)\mathrm{d}A$.

(b) Let D be the set in $\mathbb{R}^2$ that is given by

$$x^2 \leq y \leq 2x^2 \qquad \text{and} \qquad x^3 \leq y \leq 2x^3\ .$$

Let $f(x,y) = \dfrac{x}{y}$. Compute $\int_D f(x,y)\mathrm{d}A$.

SOLUTION (a) Substituting $y = 1/x$ into $x^2 + 9y^2 = 10$ and multiplying through by x^2 gives $x^4 + 9 = 10x^2$, or $(x^2 - 5) = 25 - 9 = 16$. Hence $x^3 = 5 \pm 4$, and since $x > 0$, $x = 1$ or $x = 3$. Hence the regoing consists of the points (x,y) satisfying

$$1 \le x \le 3 \qquad \text{and} \qquad 1/x \le y \le \frac{1}{3}\sqrt{10 - x^2}\ .$$

Therefore,

$$\begin{aligned}
\int_D f(x,y)\mathrm{d}A &= \int_1^3 \left(\int_{1/x}^{\frac{1}{3}\sqrt{10-x^2}} y\mathrm{d}y \right) x\mathrm{d}x \\
&= \frac{1}{2}\int_1^3 \left(\frac{10x - x^3}{9} - 1/x \right) \mathrm{d}x \\
&= \frac{10}{9} - \frac{1}{2}\ln(3)\ .
\end{aligned}$$

(b) Introduce the variables $u = y/x^2$ and $v = y/x^3$. Then $x = u/v$ and $y = u^3/v^2$, and

$$[D_{\mathbf{x}}(u,v)] \begin{bmatrix} 1/v & -u/v^2 \\ 3u^2/v^2 & -2u^3/v^3 \end{bmatrix}$$

and hence

$$\mathrm{d}x\mathrm{d}y = \frac{u^3}{v^4}\mathrm{d}u\mathrm{d}v\ .$$

Then we have $f(x(u,v), y(u,v)) = v/u^2$ and $1 \le u, v \le 2$. Hence

$$\int_D f(x,y)\mathrm{d}A = \int_1^2 \left(\int_1^2 \frac{v}{u^2}\frac{u^3}{v^4}\mathrm{d}u \right) \mathrm{d}v = \left(\int_1^2 u\mathrm{d}u \right) \left(\int_1^2 \frac{1}{v^3}\mathrm{d}v \right) = \frac{3}{2}\frac{3}{8} = \frac{9}{16}\ .$$

7: Let $\mathcal{S}$ be the triangle in $\mathbb{R}^3$ with vertices $(0,4,1)$, $(1,0,2)$ and $(0,0,3)$.

(a) Find a parameterization of the surface $\mathcal{S}$.

(b) Let $f(x,y,z) = xyz$. Compute $\int_{\mathcal{S}} f\mathrm{d}S$.

SOLUTION (a) Taking $(0,0,3)$ as a base point, the the two edges of the triangle meeting at $(0,0,3)$ are the segments running from $(0,0,3)$ to $(0,4,1)$ and from $(0,0,3)$ to $(1,0,2)$ respectively. Let $\mathbf{x}_0 = (0,0,3)$, $\mathbf{v} = (0,4,-2)$ and $\mathbf{w} = (1,0,-1)$. Let $0 \le t \le 1$, and let $\mathbf{x}_1(t) = \mathbf{x}_0 + t\mathbf{v}$ and $\mathbf{x}_2(t) = \mathbf{x}_0 + t\mathbf{w}$. These segments trace out two sides of the tirangle. To get the triangle, joint them up with a segment parameterized by $0 \le s \le 1$:

$$\mathbf{X}(s,t) = (1-s)\mathbf{x}_1(t) + s\mathbf{x}_2(t) = \mathbf{x}_0 + (1-s)t\mathbf{v} + st\mathbf{w} = (st, 4(1-s)t, 3 - 2t + st)\ ,$$

with $0 \le s, t \le 1$.

We can also get a parameterization from the equation of the plane that contains the triangle. Since $\mathbf{v} \times \mathbf{w} = -2(2,1,2)$, the equation is $(2,1,2) \cdot (x,y,z-3) = 0$, which is $2x + y + 2z = 6$, or $z = 3 - x - \frac{y}{2}$. The triangle is the graph of $z = 3 - x - \frac{y}{2}$ for x, y belonging to D, the triangle in the x, y plane that has vertices $(0,4)$, $(1,0)$, and $(0,0)$. The bounding lines are $x = 0$, $y = 0$ and $y = 4 - 4x$, and D is givne by

$$0 \le x \le 1 \qquad \text{and} \qquad 0 \le y \le 4(1-x) \ .$$

(b) The limits are simplest for the first paprameterization, so we use this first: Then we have $0 \le s, t \le 1$, and

$$\mathbf{X}_s(s,t) = (t, -4t, t) \qquad \text{and} \qquad \mathbf{X}_t(s,t) = (s, 4(1-s), s-2) \ .$$

Therefore,

$$\mathbf{X}_s \times \mathbf{X}_t = 2t(2,1,2) \ ,$$

and hence $\|\mathbf{X}_s \times \mathbf{X}_t\| = 6t$, and then $\mathrm{d}S = 6\mathrm{d}s\mathrm{d}t$. Finally, $f(\mathbf{X}(s,t)) = 4s(1-s)t^2(3-2t+st)$. Hence

$$\int_{\mathcal{S}} f\mathrm{d}S = 6\int_0^1 \left(\int_0^1 4s(1-s)t^3(3-2t+st)\mathrm{d}s \right) \mathrm{d}t = \int_0^1 \left(12t^3 - 6t^4\right) \mathrm{d}t = \frac{9}{5} \ .$$

We now do the integral using the second paprameterization $\mathbf{X}(x,y) = (x, y, 3 - x - \frac{y}{2})$. We obtain

$$\mathbf{X}_x \times \mathbf{X}_y = (1, 0, -1) \times (0, 1, -\tfrac{1}{2}) = (1, \tfrac{1}{2}, 1)$$

so $\|\mathbf{X}_x \times \mathbf{X}_y\| = \frac{3}{2}$ is constant. Then

$$\begin{aligned} \int_{\mathcal{S}} xyz\,\mathrm{d}S = \int_D xy\left(3 - x - \frac{y}{2}\right)\frac{3}{2}\mathrm{d}A \ &= \ \int_0^1 \left(\int_0^{4-4x} \frac{18xy - 6x^2y - 3xy^2}{4}\mathrm{d}y \right) \mathrm{d}x \\ &= \ 4\int_0^1 \left(x(x+5)(x-1)^2 \right) \mathrm{d}x = \frac{9}{5} \ , \end{aligned}$$

as we found before.

Material from Part 3 of the Course:

8: Consider the two vector fields

$$\mathbf{F} = (2xyz - y^2, x^2z - 2xy, x^2y) \qquad \text{and} \qquad \mathbf{G} = (2yz - y^2, x^2z - 2x, x^2y) \ .$$

(a) Compute the divergence and curl of $\mathbf{F}$ and $\mathbf{G}$.

(b) Let $\mathcal{S}$ be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the x, y plane. Compute

$$\int_{\mathcal{S}} \mathbf{G} \cdot \mathbf{N}\mathrm{d}S \ .$$

(c) One of the vector fields $\mathbf{F}$ and $\mathbf{G}$ is equal to $\nabla\varphi$ for some potential function φ. Which one is it? Find such a potential function.

(d) One of the vector fields $\mathbf{F}$ and $\mathbf{G}$ is equal to curl$\mathbf{A}$ for some vector potential $\mathbf{A}$. Which one is it? Find such a vector potential.

(e) Let C be the curve that is parametrized by

$$\mathbf{x}(t) = (t^3 - 2t^2, t - 4t^2, t + t^3) \qquad \text{for} \qquad 0 \le t \le 1 \ .$$

Compute $\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s$, giving the *exact numerical value.*

SOLUTION (a) We compute

$$\mathrm{curl}\mathbf{F}(x,y,z) = \mathbf{0} \qquad \text{and} \qquad \mathrm{div}\mathbf{F}(x,y,z) = 2yz - 2x \ ,$$

and also

$$\mathrm{curl}\mathbf{G}(x,y,z) = (0, -2xy + 2y, 2xz + 2y - 2z - 2) \qquad \text{and} \qquad \mathrm{div}\mathbf{G}(x,y,z) = 0,$$

(b) Let $\widetilde{\mathcal{S}}$ be the unit disk in the x,y plane. Then $\mathcal{S}\cup\widetilde{\mathcal{S}}$ bound $\mathcal{V}$, the set of points (x,y,z) that lies above the x,y plane and be low the grapho of $z = 1 - x^2 - y^2$. By the Divergence Theorem,

$$\int_{\mathcal{V}} \mathrm{div}\mathbf{G}(\mathbf{x})\mathrm{d}V = \int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}(\mathbf{x})\mathrm{d}S + \int_{\widetilde{\mathcal{S}}} \mathbf{G}\cdot\mathbf{N}(\mathbf{x})\mathrm{d}S \ .$$

Since $\mathrm{div}\mathbf{G}(x,y,z) = 0$,

$$\int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}(\mathbf{x})\mathrm{d}S = -\int_{\widetilde{\mathcal{S}}} \mathbf{G}\cdot\mathbf{N}(\mathbf{x})\mathrm{d}S \ .$$

On $\widetilde{\mathcal{S}}$, $z = 0$ and hence $\mathbf{G} = (-y^2, -2x, x^2y)$. Morover, on $\widetilde{\mathcal{S}}$, the outward norm $\mathbf{N}$ is the constant vector $(0,0,-1)$. Hence on $\widetilde{\mathcal{S}}$, $\mathbf{G}\cdot\mathbf{N}(\mathbf{x}) = -x^2y$. Therefore, with D denoting the unit disk in the x,y plane,

$$\int_{\widetilde{\mathcal{S}}} \mathbf{G}\cdot\mathbf{N}(\mathbf{x})\mathrm{d}S = -\int_D x^2y\mathrm{d}A = 0$$

by symmetry – the integrand is an odd function of y. We conclude that $\int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S = 0$.

(c) Since $\mathrm{curl}\mathbf{F} = \mathbf{0}$, it is $\mathbf{F}$. Integrating, we find $\mathbf{F}(x,y,z) = \nabla\varphi(x,y,z)$ where $\varphi(x,y,z) = x^2yz - y^2x$.

(d) Since $\mathrm{div}\mathbf{G} = 0$, it is $\mathbf{G}$. Integrating, we find $\mathbf{G}(x,y,z) = \mathrm{curl}\mathbf{A}(x,y,z)$ where $\mathbf{A}(x,y,z) = (F9x,y,z),0H(x,y,z))$ and

$$\begin{aligned}
F(x,y,z) &= -\int_0^y R(x,t,z)\mathrm{d}t + \int_0^z Q(x,0,t)\mathrm{d}t \\
&= -x^2\int_0^y t\mathrm{d}t + \int_0^z (x^2t - 2x)\mathrm{d}t = \frac{x^2}{2}(z^2 - y^2) - 2xz \\
H(x,y,z) &= \int_0^y P(x,t,z)\mathrm{d}t = \int_0^y (2zt - t^2)\mathrm{d}t = y^2z - \frac{1}{3}y^3 \ .
\end{aligned}$$

. Therefore,

$$\mathbf{A}(x,y,z) = (\tfrac{1}{2}x^2(z^2 - y^2) - 2xz, 0, y^2z - \tfrac{1}{3}y^3) \ .$$

(d) Since $\mathbf{F} = \nabla\varphi$ where $\varphi(x,y,z) = x^2yz - y^2x$,

$$\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s = \varphi(\mathbf{x}(1)) - \varphi(\mathbf{x}(0)) = \varphi(-1,-3,2) = 3 \ .$$

9: (a) Let $\mathcal{S}$ be the part of the surface in $\mathbb{R}^3$ given by $\sqrt{x^2+y^2} = 10 - z^3$ that lies inside the cylinder $x^2+y^2 = 4$. Let $\mathbf{F} = (2yz - y^2, x^2z - 2x, z)$. Compute the flux

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S\ ,$$

where $\mathbf{N}$ is taken to point upward.

(b) Let C be the contour that runs from $(0,0,0)$ to $(0,1,2)$, and from there to $(2,3,2)$, and from there back to $(0,0,0)$. Let $\mathbf{G} = (z,x,y)$. Compute the total circulation

$$\int_C \mathbf{G}\cdot\mathbf{T}\mathrm{d}s\ .$$

SOLUTION (a) We compute $\mathrm{div}\mathbf{F}(x,y,z) = 1$. Since this is quite simple, we can avoid parameterizing the surface.

The surface meets the cylinder when $\sqrt{4} = 10 - z^3$, so that $z = 2$. Let $\mathcal{V}$ be the region lying below the graph of $\sqrt{x^2+y^2} = 10 - z^3$ and above the plane $z = 2$. Let $\widetilde{\mathcal{S}}$ be the part of this plane lying inside the cyliner, equipped with the downward unit notmal $\mathbf{N}$. Then $\mathcal{S}\cup\widetilde{\mathcal{S}}$ is the boundary of $\mathcal{V}$. By the Divergence Theorem and the fact that $\mathrm{div}\mathbf{F}(x,y,z) = 1$,

$$\int_{\mathcal{V}} \mathrm{d}V = \int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S + \int_{\widetilde{\mathcal{S}}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S\ .$$

To do the volume integral, note that in cylindrical coordinates,

$$0 \le \theta \le 2\pi\ , \quad 0 < r < 2 \quad \text{and} \quad 2 \le z \le (10-r)^{1/3}\ .$$

Hence

$$\int_{\mathcal{V}} \mathrm{d}V = 2\pi\int_0^2 ((10-r)^{1/3} - 2)r\mathrm{d}r = 2\pi\int_0^2 (10-r)^{1/3} r\mathrm{d}r - 8\pi\ .$$

Next, everyhere on $\widetilde{\mathcal{S}}$, $\mathbf{N} = (0,0,-1)$ and $z = 2$. Therefore, everywhere on $\widetilde{\mathcal{S}}$, $\mathbf{F}\cdot\mathbf{N} = -2$. Since $\widetilde{\mathcal{S}}$ is a disk of radius 2,

$$\int_{\widetilde{\mathcal{S}}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = -8\pi\ .$$

Hence

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = 8\pi + 2\pi\int_0^2 (10-r)^{1/3} r\mathrm{d}r - 8\pi = 2\pi\int_0^2 (10-r)^{1/3} r\mathrm{d}r = \frac{2\pi}{7}(10^{1/3}225 - 456)\ .$$

If one does this directly, the surface is parameterized by

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, (10-r)^{1/3})\ , \quad 0 \le \theta \le 2\pi\ , \quad 0 < r < 2\ .$$

Then

$$\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta) = \frac{1}{3}(10-r)^{-2/3}(r\cos\theta, r\sin\theta, 3r(10-r)^{2/3})\ .$$

Then

$$\mathbf{F}(\mathbf{X}(r,\theta)) = (2r\sin\theta(10-r)^{1/3} - r^2\sin^2\theta, r^2\cos^2\theta(10-r)^{1/3} - 2r\cos\theta, (10-r)^{1/3})\ .$$

Then, on $\mathcal{S}$,

$$\mathbf{F}\cdot\mathbf{N}\mathrm{d}S = \mathbf{F}(\mathbf{X}(r,\theta))\cdot\mathbf{X}_r\times\mathbf{X}_\theta(r,\theta)\mathrm{d}r\mathrm{d}\theta\ ,$$

and this is integrated over $0\le\theta\le 2\pi\ ,\quad 0<r<2$. The integration in θ will eliminate all the terms that explicitly depend on θ since, e.g., $\int_0^{2\pi}\sin\theta\cos\theta\mathrm{d}\theta = 0$. We are then left with

$$\mathbf{F}\cdot\mathbf{N}\mathrm{d}S = 2\pi\int_0^2(10-r)^{1/3}r\mathrm{d}r = \frac{2\pi}{7}(10^{1/3}225-456)\ ,$$

exactly as we found above.

(b) We compute curl$\mathbf{G}(x,y,z) = (1,1,1)$. The curve bounds a triangle whose normal in the plane through the origin that is normal to $(0,1,2)\times(2,3,2) = (-4,4,-2)$. The curve goes around the triangle clockwise when viewed from above, so the normal that is consistent with this orientation is the downward normal, which is

$$\mathbf{N} = \frac{1}{3}(-2,2,-1)\ .$$

Hence curl$G\cdot\mathbf{N} = (1,1,1)\cdot\frac{1}{3}(-2,2,-1) = -\frac{1}{3}$ everywhere on the tirangle, and since the area of the triangle is $\frac{1}{2}\|(0,1,2)\times(2,3,2)\| = \|-2,2,-1\| = 3$, we have by Stokes' Theorem that

$$\int_C \mathbf{G}\cdot\mathbf{T}\mathrm{d}s = -1\ .$$