1 Solutions for the even exercises from Chapter Two

2.2 Let $\mathbf{x}(t) = \big(t^{-2}\ ,\ 4/\sqrt{t}\ ,\ t\big)$ for $t > 0$.

(a) Compute $\mathbf{v}(t) = \mathbf{x}'(t)$ and $\mathbf{a}(t) = \mathbf{x}''(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Find the tangent line to this curve at $t = 1$.

SOLUTION: Differentiating, we find

$$\mathbf{v}(t) = \big(-2t^{-3}, -2t^{-3/2}, 1\big) \qquad \text{and} \qquad \mathbf{a}(t) = \big(6t^4, 3t^{-5/2}, 0\big) \ .$$

Then

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{1 + 4t^{-3} + 4t^{-6}} = 1 + 2t^{-3} \ ,$$

and so

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = \frac{1}{1+2t^{-3}}\big(-2t^{-3}, -2t^{-3/2}, 1\big) \ .$$

The tangent line is given by

$$\mathbf{x}(1) + t\mathbf{v}(1) = \big(1 - 2t, 4 - 2t, 1 + t\big) \ .$$

2.4 Let $\mathbf{x}(t) = \big(\cos(t)\ , \sin(t)\ , t/r\big)$ where $r > 0$. The curve $\mathbf{x}(t)$ is a helix in $\mathbb{R}^3$.

(a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$, as well as $\mathbf{N}(t)$ and $\mathbf{B}(t)$.

(d) Compute the Darboux vector $\boldsymbol{\omega}(t)$.

(e) Find the tangent line to this curve at $t = \pi/4$, and the equation of the osculating plane to the curve at $t = \pi/2$. Find the intersection of this line and plane.

SOLUTION: (We give here the solution of a slightly more general problem in which the third component of the curve it t/r instead of t/π. Simply replace r by π below to get the solution for the special case asked for in the text.) Differentiating, we find

$$\mathbf{v}(t) = \big(-\sin(t), \cos(t), 1/r\big) \qquad \text{and} \qquad \mathbf{a}(t) = \big(-\cos(t), -\sin(t), 0\big) \ .$$

Then

$$v(t) = \|\mathbf{v}(t)\| = \frac{\sqrt{r^2+1}}{r},$$

and so

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = \frac{r}{\sqrt{r^2+1}}\big(-\sin(t), \cos(t), 1/r\big) \ .$$

To find the curvature and $\mathbf{N}$, we first compute

$$\mathbf{a}_\perp := \mathbf{a} - (\mathbf{a}\cdot\mathbf{T})\mathbf{T} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \ .$$

By the computations made above, $\mathbf{a}\cdot\mathbf{v} = 0$, so all of the acceleration is orthogonal, and so *in this case*

$$\kappa(t) = \frac{\|\mathbf{a}(t)\|}{v(t)^2} = \frac{r^2}{r^2+1} \ ,$$

and

$$\mathbf{N}(t) = \frac{1}{\|\mathbf{a}_\perp(t)\|}\mathbf{a}_\perp(t) = \frac{1}{\|\mathbf{a}(t)\|}\mathbf{a}(t) = \mathbf{a}(t) = \big(-\cos(t), -\sin(t), 0\big) \ .$$

We next compute $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, and find

$$\mathbf{B}(t) = \frac{1}{\sqrt{r^2+1}}\big(\sin(t), -\cos(t), r\big) \ .$$

Since $\mathbf{B}$ is so simple, the easiest way to compute the torsion is to compute $\mathbf{B}'(t)$, and to use the definition $\mathbf{B}'(t) = -v(t)\tau(t)\mathbf{N}(t)$. We readily find

$$\tau(t) = \frac{r}{r^2+1} \ .$$

We are now ready to compute the Darboux vector, $\boldsymbol{\omega} = \tau\mathbf{T} + \kappa\mathbf{B}$. We find

$$\begin{aligned}\boldsymbol{\omega} &= \frac{r}{r^2+1}\frac{r}{\sqrt{r^2+1}}\big(-\sin(t), \cos(t), 1/r\big) + \frac{r^2}{r^2+1}\frac{1}{\sqrt{r^2+1}}\big(\sin(t), -\cos(t), r\big)\\ &= \frac{r}{\sqrt{r^2+1}}(0,0,1)\end{aligned}$$

Finally, we compute that tangent line to $\mathbf{x}(t)$ at $t = \pi/4$, finding it is parameterized by

$$\mathbf{x}(\pi/4) + t\mathbf{v}(\pi/4) = \big((1-t)/\sqrt{2}, (1+t)/\sqrt{2}, (\pi+4t)/4r\big) \ . \tag{1.1}$$

The equation of the osculating plane at $t = \pi/2$ is

$$\mathbf{B}(\pi/2)\cdot(\mathbf{x} - \mathbf{x}(\pi/2)) = 0 \ .$$

This simplifies to $\big(1, 0, r\big)\cdot\big(x, y-1, z-\pi/(2r)\big) = 0$ which then further simplifies to

$$x + rz = \pi/2 \ .$$

Plugging the parameterized line (**??**) into the equation of the plane and solving for t, we find that the intersection occurs at

$$t_0 = -\frac{2+\pi}{8-2\sqrt{2}} \ .$$

Plugging this value of t into (**??**) gives the intersection.

2.6: Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (t^{3/2}, 3t, 6t^{1/2})$$

for $t > 0$.

(a) what is the arc length along the curve between $\mathbf{x}(1)$ and $\mathbf{x}(4)$?

(b) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(c) Find an equation for the osculating plane at $t = 1$, and find a parameterization of the tangent line to the curve at $t = 1$.

SOLUTION: Differentiating, we find

$$\mathbf{v}(t) = 3(t^{1/2}/2, 1, t^{-1/2}) \qquad \text{and} \qquad \mathbf{a}(t) = \frac{3}{4}(t^{-1/2}, 0, 2t^{-3/2}) \ .$$

Then

$$v(t) = \|\mathbf{v}(t)\| = \frac{3}{2}\sqrt{4 + t + t/4} = \frac{3}{2}(t^{1/2} + 2t^{-1/2}) \ .$$

The arc length along the curve between $\mathbf{x}(1)$ and $\mathbf{x}(4)$ then is

$$\int_1^4 v(t)\mathrm{d}t = \frac{3}{2}\frac{26}{3} = 13 \ .$$

Computing the curvature and torsion, we find

$$\kappa(t) = \frac{2}{3}(t+2)^{-2} \qquad \text{and} \qquad \tau = -\frac{2}{3}(t+2)^{-2} \ .$$

Since $\mathbf{a} \times \mathbf{v}$ is a multiple of $\mathbf{B}$, the equation of the osculating plane at $t = 1$ is

$$(\mathbf{a}(1) \times \mathbf{v}(1) \cdot (\mathbf{x} - \mathbf{x}(1) = 0 \ .$$

Computing, this becomes

$$\frac{9}{4}(2, -2, 1) \cdot (x - 1, y - 3, z - 6) = 0 \ ,$$

and this further simplifies to

$$2x - 2y + z = 2 \ .$$

The tangent line is parameterized by

$$\mathbf{x}(1) + t\mathbf{v}(1) = (1 + 3t/2, 3 + 3t, 6 + 3t) \ .$$

2.8: Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (2t, t^2, t^3/3) \ .$$

(a) Compute the arc length $s(t)$ as a function of t, measured from the starting point $\mathbf{x}(0)$.

(b) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(c) Find equations for the osculating planes at time $t = 0$ and $t = 1$, and find a parameterization of the line formed by the intersection of these planes.

SOLUTION: Differentiating, we find

$$\mathbf{v}(t) = (2, 2t, t^2) \qquad \text{and} \qquad \mathbf{a}(t) = (0, 2, 2t) \ .$$

Then

$$v(t) = \|\mathbf{v}(t)\| = 2 + t^2 \ .$$

The arc length along the curve between $\mathbf{x}(0)$ and $\mathbf{x}(t)$ then is

$$\int_0^t v(r)\mathrm{d}r = \int_0^t (2+r^2)\mathrm{d}r = 2t + 26^3/3 \ .$$

Computing the curvature and torsion, we find

$$\kappa(t) = \frac{\sqrt{(2+t^2)^2+4}}{(2+t^2)^2}(t+2)^{-2} \qquad \text{and} \qquad \tau = -\frac{4}{(2+t^2)^2}\frac{1}{\sqrt{(2+t^2)^2+4}} \ .$$

Since $\mathbf{a} \times \mathbf{v}$ is a multiple of $\mathbf{B}$, the equation of the osculating plane at $t = 1$ is

$$\mathbf{a}(1) \times \mathbf{v}(1) \cdot (\mathbf{x} - \mathbf{x}(1) = 0 \ .$$

Computing, this becomes

$$\frac{9}{4}(2, -2, 1) \cdot (x-1, y-3, z-6) = 0 \ ,$$

and this further simplifies to

$$2x - 2y + z = 2 \ .$$

The tangent line is parameterized by

$$\mathbf{x}(1) + t\mathbf{v}(1) = (1 + 3t/2, 3 + 3t, 6 + 3t) \ .$$

Finally since $\mathbf{a}(t) \times \mathbf{v}(t)$ is a multiple of $\mathbf{B}(t)$ (as is $\mathbf{a}(t) \times \mathbf{v}(t)$), we can write an equation for the normal plane at time t as

$$[\mathbf{a}(t) \times \mathbf{v}(t)] \cdot [\mathbf{x} - \mathbf{x}(t)] = 0 \ .$$

Doing this for $t = 0$, we find $(0, 0, -3) \cdot (x, y, z) = 0$, which reduces to

$$z = 0 \ .$$

Doing this for $t = 0$, we find $(\,-2, 4, -4) \cdot (x-2, y-1, z-1/3) = 0$, which reduces to

$$2x - 4y + 4z = \frac{4}{3} \ .$$

Plugging $z = 0$ (from the first equation) into the second equation, we find that the line is the line lies in the x, y plane, where is is given by

$$y = \frac{x}{2} - \frac{1}{3}$$

This line can also be prameterized as

$$\mathbf{x}(t) = (t, t/2 - 1/3, 0) \ .$$

2.10 Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = (t \ , \ \sqrt{2}\ln(t) \ , \ 1/t)$ for $t > 0$.

(a) Find the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(3)$.

(b) Find the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(t)$ as a function of t.

(c) Find the arc length parameterization $\mathbf{x}(s)$ of this curve.

SOLUTION: We compute

$$\mathbf{v}(t) = \left(1, \sqrt{2}/t, -1/t^2\right)\ ,$$

and hence

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}} = \sqrt{(1+t^{-2})^2} = 1 + t^{-2}\ .$$

Then the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(3)$ is

$$\int_1^3 v(t)\mathrm{d}t = \int_1^3 (1 + t^{-2})\mathrm{d}t = \frac{8}{3}\ .$$

Likewise, the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(t)$ is, for $t \geq 1$,

$$\int_1^t v(r)\mathrm{d}r = \int_1^t (1 + r^{-2})\mathrm{d}r = t - 1/t\ .$$

For $0 < t < 1$, it is

$$\int_t^1 v(r)\mathrm{d}r = \int_t^1 (1 + r^{-2})\mathrm{d}r = -(t - 1/t)\ .$$

Combining the formulas, the along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(t)$ is $|t - 1/t|$.

To find the arc length parameterization, we compute

$$s(t) = \int_1^t (1 + r^{-2})\mathrm{d}r = t - 1/t\ ,$$

for all $t > 0$, and then solve

$$s = t - 1/t$$

to determine $t(s)$. We find $ts = t^2 - 1$, or $t^2 - st = 1$ or $(t - s/2)^2 = 1 + s^2/4$, so

$$t(s) = \sqrt{1 + s^2/4} - s/2\ .$$

Then

$$\mathbf{x}(s) = \mathbf{x}(t(s)) = \left(t\sqrt{1+s^2/4} - s/2\ ,\ \sqrt{2}\ln\left(\sqrt{1+s^2/4} - s/2\right)\ ,\ 1/\left(\sqrt{1+s^2/4} - s/2\right)\right)\ .$$

2.12 Find the arc length parameterization of the curve given by $\mathbf{x}(t) = \left(t^{-2}\ ,\ 4/\sqrt{t}\ ,\ t\right)$ for $t > 0$. What is the arc length along the segment of the curve joining $\mathbf{x}(1)$ and $\mathbf{x}(4)$?

SOLUTION: We compute

$$\mathbf{v}(t) = \left(-2/t^3, -2/t^{3/2}, 1\right)\ ,$$

and hence

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{1 + \frac{4}{t^3} + \frac{4}{t^6}} = \sqrt{(1+2t^{-3})^2} = 1 + 2t^{-3}\ .$$

We then pick some time, say $t = 1$ as a refrence starting time, and define

$$s(t) = \int_1^t v(r)\mathrm{d}r = \int_1^t (1 + 2r^{-3}\mathrm{d}r = t - t^{-2}\ .$$

We must then solve

$$t - t^{-2} = s \tag{1.2}$$

to find $t(s)$, We then plug $t(s)$ in to $\mathbf{x}(t)$ to find $\mathbf{x}(s)$. Solving (**??**), we find

$$t(s) = \frac{1}{6}\left(108 + 8s^2 + 12\sqrt{81 + 12s^3}\right)^{1/3} + \frac{2}{3}\frac{s^2}{\left(108 + 8s^2 + 12\sqrt{81 + 12s^3}\right)^{1/3}} + \frac{1}{3}s \ .$$

Plugging this in, one has the (rather messy) answer.

On the other hand, the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(4)$ is

$$\int_1^4 v(t)\mathrm{d}t = \int_1^3 (1 + 2t^{-3})\mathrm{d}t = \frac{63}{16} \ .$$

Moral of the story: *Arc length parameterization, while useful conceptually, is very messy in practice. Computing arc length between two points may be very much simpler.*

2.14 Let $\mathbf{a} = \frac{1}{3}(2, 1, 2)$. Let $\mathbf{u} = \frac{1}{3}(1, 2, -2)$, and note that this is a unit vector orthogonal to $\mathbf{a}$. Find a unit vector $\mathbf{v}$ so that $\mathbf{f}(\mathbf{x}) := \mathbf{h_v}(\mathbf{h_u}(\mathbf{x}))$ is the rotation of $\mathbf{x}$ through the angle $\theta = \pi/3$ about the axis along $\mathbf{a}$, and then compute $\mathbf{f}((1, 1, 1))$.

SOLUTION: If $\mathbf{u}$ and $\mathbf{v}$ are two unit vectors in $\mathbb{R}^3$ such that the angle between $\mathbf{u}$ and $\mathbf{v}$ is Θ, then $\mathbf{h_v} \circ \mathbf{h_u}$ is the rotation through the angle 2Θ about the axis along $\frac{1}{\|\mathbf{u}\times\mathbf{v}\|}\mathbf{u}\times\mathbf{v}$.

Thus, we seek a unit vector $\mathbf{v}$ so that

$$\mathbf{u}\times\mathbf{v} = \sin(\pi/6)\mathbf{a} = \frac{1}{2}\mathbf{a} \ , \tag{1.3}$$

and

$$\mathbf{u}\cdot\mathbf{v} = \cos(\pi/3) = \frac{1}{2} \ . \tag{1.4}$$

We know from previous exercises that there is one and only one vector $\mathbf{v}$ for which both (**??**) and (**??**) are true. Writing $\mathbf{v} = \mathbf{v}_\| + \mathbf{v}_\perp$, that is, breaking $\mathbf{v}$ up into its components parallel and orthogonal to $\mathbf{u}$, we have

$$\mathbf{v}_\| = (\mathbf{v}\cdot\mathbf{u})\mathbf{u} = \frac{1}{6}(1, 2, -2) \ .$$

Since $\mathbf{u}\cdot\mathbf{v} = \cos(\pi/3)$, $\|\mathbf{u}\times\mathbf{v}\| = \sin(\pi/3) = \sqrt{3}/2$. Therefore,

$$\mathbf{v}_\perp = -\mathbf{u}\times(\mathbf{u}\times\mathbf{v}) = \|\mathbf{u}\times\mathbf{v}\|\mathbf{u}\times\mathbf{a} = \frac{\sqrt{3}}{2}\frac{1}{9}(1, 2, -2)\times(2, 1, 2)$$

Computing the cross product, we find:

$$\mathbf{v}_\perp = \frac{\sqrt{3}}{2}\frac{1}{3}(2, -2, -1) \ .$$

Thus

$$\begin{aligned} \mathbf{v} = \mathbf{v}_\| + \mathbf{v}_\perp &= \frac{1}{6}(1, 2, -2) + \frac{\sqrt{3}}{6}(2, -2, -1) \\ &= \frac{1}{6}(1 + 2\sqrt{3}, 2 - 2\sqrt{3}, -2 - \sqrt{3}) \ . \end{aligned}$$

As you can (and should) check, the right hand side is indeed a unit vector, as it should be.
We then compute

$$\mathbf{h_u}((1,1,1)) = \frac{1}{9}((7,5,13)) \ ,$$

and then

$$\frac{1}{9}\mathbf{h_u}((7,5,13)) = \frac{1}{6}(7+\sqrt{3}, 2, 7-\sqrt{3}) \ .$$

Thus,

$$\mathbf{f}((1,1,1)) = \frac{1}{6}(7+\sqrt{3}, 2, 7-\sqrt{3}) \ .$$