Solutions for the Even Numbered Exercises from Chapter 1

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SOLUTION:
1.2 Let $\mathbf{a} = (5,2)$, $\mathbf{b} = (2,-1)$ and $\mathbf{c} = (1,1)$. Express each of these three vectors as a linear combination of the other two.

SOLUTION: The equation $s\mathbf{b} + t\mathbf{c} = \mathbf{a}$ is equivalent to

$$\begin{aligned} 2s + t &= 5 \\ -s + t &= 2 \end{aligned}$$

The solution is $s = 1$ and $t = 3$.

The equation $s\mathbf{c} + t\mathbf{a} = \mathbf{b}$ is equivalent to

$$\begin{aligned} s + 5t &= 2 \\ s + 2t &= -1 \end{aligned}$$

The solution is $s = -3$ and $t = 1$.

The equation $s\mathbf{a} + t\mathbf{b} = \mathbf{c}$ is equivalent to

$$\begin{aligned} 5s + 2t &= 1 \\ 2s - t &= 1 \end{aligned}$$

The solution is $s = 1/3$ and $t = -1/3$.

1.4 Let $\mathbf{x} = (4,7,-4,1,2,-2)$ and $\mathbf{y} = (2,1,2,2,-1,-1)$. Compute the lengths of each of these vectors, and the angle between them.

SOLUTION: We compute

$$\mathbf{x}\cdot\mathbf{y} = 9 \qquad \mathbf{x}\cdot\mathbf{x} = 90 \quad \text{and} \quad \mathbf{y}\cdot\mathbf{y} = 15\ .$$

Thus, $\|\mathbf{x}\| = 3\sqrt{10}$ and $\|\mathbf{y}\| = \sqrt{15}$. Then θ, the angle between the $\mathbf{x}$ and $\mathbf{y}$ is

$$\theta = \arccos\left(\frac{9}{15\sqrt{6}}\right) = \arccos\left(\frac{3}{5\sqrt{6}}\right) \approx 1.323 \text{ radians} \approx 75.82^\circ$$

1.6 Let $\mathbf{x} = (-5, 2, -5)$ and $\mathbf{y} = (1, 2, 1)$. Is the angle between $\mathbf{x}$ and $\mathbf{y}$ acute or obtuse? Justify your answer.

SOLUTION: We compute

$$\mathbf{x} \cdot \mathbf{y} = -5 + 4 - 5 = -6 \ .$$

Since this is negative, $\dfrac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$ is negative, so

$$\theta = \arccos\left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right)$$

is greater than $\pi/2$. Thus, the angle is obtuse. (Note that we did not have to actually calculate the angle to answer this question.)

1.8 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be any three vectors in $\mathbb{R}^3$ with $\mathbf{a} \neq \mathbf{0}$. Show that $\mathbf{b} = \mathbf{c}$ if and only if

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \quad \text{and} \quad \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \ .$$

FIRST SOLUTION: We write the given equations in the form

$$\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0} \qquad \text{and} \qquad \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0 \ . \tag{0.1}$$

Since $\|\mathbf{a} \times (\mathbf{b} - \mathbf{c})\| = \|\mathbf{a}\|\|\mathbf{b} - \mathbf{c}\| \sin(\theta)$, and since $\mathbf{a} \neq \mathbf{0}$, the first equation says that $\mathbf{b} - \mathbf{c}$ is a multiple of $\mathbf{a}$; i.e.,

$$\mathbf{b} - \mathbf{c} = t\mathbf{a} \tag{0.2}$$

for some t. Using this in the second equation, we get

$$0 = \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot (t\mathbf{a}) = t\|\mathbf{a}\|^2 \ .$$

Since $\mathbf{a} \neq \mathbf{0}$, this says $t = 0$, and so (0.2) says that $\mathbf{b} - \mathbf{c} = \mathbf{0}$. That is, when (0.1) is true, then $\mathbf{b} = \mathbf{c}$, and the converse is obvious.

SECOND SOLUTION: We now give another solution, slightly longer, but illustrating useful points.

We begin by writing $\mathbf{b} = \mathbf{b}_{\|} + \mathbf{b}_{\perp}$ where $\mathbf{b}_{\|}$ is the component of $\mathbf{b}$ parallel to $\mathbf{a}$, and $\mathbf{b}_{\perp}$ is the component of $\mathbf{b}$ orthogonal to $\mathbf{a}$, and likewise for $\mathbf{c}$.

First of all,

$$\mathbf{b}_{\|} = \frac{1}{\|\mathbf{a}\|^2}(\mathbf{b} \cdot \mathbf{a})\mathbf{a} \qquad \text{and} \qquad \mathbf{c}_{\|} = \frac{1}{\|\mathbf{a}\|^2}(\mathbf{c} \cdot \mathbf{a})\mathbf{a} \ .$$

Thus,

$$\mathbf{b}_{\|} = \mathbf{c}_{\|} \quad \Longleftrightarrow \quad \mathbf{v} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \ .$$

Next, recall that

$$\mathbf{b}_{\perp} = -\frac{1}{\|\mathbf{a}\|^2}\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) \ ,$$

and likewise for $\mathbf{c}$.

But if $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\mathbf{a} \times \mathbf{c})$ and so

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \quad \Longleftrightarrow \quad \mathbf{b}_{\perp} = \mathbf{c}_{\perp} \ .$$

Since

$$\mathbf{b} = \mathbf{c} \qquad \iff \qquad \mathbf{b}_{\|} = \mathbf{c}_{\|} \qquad \text{and} \qquad \mathbf{b}_{\perp} = \mathbf{c}_{\perp}\ ,$$

putting the pieces together,

$$\mathbf{b} = \mathbf{c} \qquad \iff \qquad \mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} \quad \text{and} \quad \mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c}\ .$$

1.10 (a) Let $\mathbf{a} = (-1, 1, 2)$ and $\mathbf{b} = (2, -1, 1)$. Find all vectors $\mathbf{x}$, if any exist, such that

$$\mathbf{a}\times\mathbf{x} = (-2, 4, -3) \qquad \text{and} \qquad \mathbf{b}\cdot\mathbf{x} = 2\ .$$

If none exist, explain why this is the case.

(b) Let $\mathbf{a} = (-1, 1, 2)$ and $\mathbf{b} = (2, -1, 1)$. Find all vectors $\mathbf{x}$, if any exist, such that

$$\mathbf{a}\times\mathbf{x} = (2, 4, 3) \qquad \text{and} \qquad \mathbf{b}\cdot\mathbf{x} = 2\ .$$

If none exist, explain why this is the case.

(c) Among all vectors $\mathbf{x}$ such that $(-1, 1, 2)\times\mathbf{x} = (-2, 4, -3)$, find the one that is closest to $(1, 1, 1)$.

SOLUTION: The vector $\mathbf{a}\times\mathbf{x}$ is orthogonal to $\mathbf{a}$, and hence $\mathbf{a}\times\mathbf{x} = \mathbf{c}$ cannot possibly have any solution unless $\mathbf{c}$ is orthogonal to $\mathbf{a}$.

On the other hand, if $\mathbf{a}\cdot\mathbf{c} = \mathbf{0}$, then

$$0 = \mathbf{a}\cdot(\mathbf{a}\times\mathbf{x}) = \mathbf{a}\cdot\mathbf{c}\ ,$$

and so, by the result of Exercise 1.8, $\mathbf{a}\times\mathbf{x} = \mathbf{c}$ if and only if

$$\mathbf{a}\times(\mathbf{a}\times\mathbf{x}) = \mathbf{a}\times\mathbf{c}\ .$$

But since

$$\mathbf{a}\times(\mathbf{a}\times\mathbf{x}) = -\|\mathbf{a}\|^2\mathbf{x}_{\perp}$$

this is equivalent to

$$\mathbf{x}_{\perp} = -\frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\times\mathbf{c}\ , \tag{0.3}$$

where $\mathbf{x}_{\perp}$ is the component of $\mathbf{x}$ orthogonal to $\mathbf{a}$.

Next, since since $\mathbf{a}\times\mathbf{x}_{\|} = \mathbf{0}$, $\mathbf{x}_{\|}$ can be *any* multiple of $\mathbf{a}$, and the equation will be satisfied, as long as $\mathbf{x}_{\perp}$ satisfies (0.3). Thus, the set of all vectors $\mathbf{x}$ satisfying $\mathbf{a}\times\mathbf{x} = \mathbf{c}$ is the empty set if $\mathbf{a}$ is not orthogonal to $\mathbf{c}$, and otherwise, it is the line parameterized by

$$\mathbf{x}(t) = -\frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\times\mathbf{c} + t\mathbf{a}\ .$$

In the case at hand, $\mathbf{a} = (-1, 1, 2)$ and $\mathbf{c} = (-2, 4, -3)$. We compute

$$\mathbf{a}\cdot\mathbf{c} = 0\ ,$$

so the set the set of all vectors $\mathbf{x}$ satisfying $(-1, 1, 2)\times\mathbf{x} = (-2, 4, -3)$ is the line line paramterized by

$$\mathbf{x}(t) = -\frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\times\mathbf{c} + t\mathbf{a} = \frac{1}{6}(11, 7, 2) + t(-1, 1, 2)\ . \tag{0.4}$$

Next, for part **(b)**, we compute $\mathbf{a}\cdot(2,4,3) = 8$. Thus, $\mathbf{a}$ and $(2,4,3)$ are not orthogonal, and there is no solution.

Next, for part **(c)**, using the result of part **(a)** we need to find the point on the line parameterized in (0.4) that is closest to $(1,1,1)$. This point is $-\frac{1}{\|\mathbf{a}\|^2}\mathbf{a}\times\mathbf{c}$ plus the component of

$$(1,1,1) - \frac{1}{6}(11,7,2) = \frac{1}{6}(-5,-1,4)$$

parallel to $\mathbf{a}$. This is

$$\begin{aligned}\frac{1}{6}(11,7,2) + \frac{1}{6}\frac{(-5,-1,4)\cdot(-1,1,2)}{(-1,1,2)\cdot(-1,1,2)}(-1,1,2) &= \frac{1}{6}(11,7,2) + \frac{1}{3}(-1,1,2)\\ &= \frac{1}{2}(3,3,2)\ .\end{aligned}$$

1.12 (a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors in $\mathbb{R}^3$. Define a transformation $\mathbf{f}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ by

$$\mathbf{f}(\mathbf{x}) = \mathbf{a}\times(\mathbf{b}\times(\mathbf{c}\times\mathbf{x}))\ .$$

Show that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}\in\mathbb{R}^3$ if and only if $\mathbf{b}$ is orthogonal to $\mathbf{c}$, and $\mathbf{a}$ is a multiple of $\mathbf{c}$. .

SOLUTION: By Lagrange's identity; i.e., $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$,

$$\mathbf{b}\times(\mathbf{c}\times\mathbf{x}) = (\mathbf{b}\cdot\mathbf{x})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{x}\ ,$$

and then taking the cross product of both sides with $\mathbf{a}$, we get

$$\mathbf{f}(\mathbf{x}) = (\mathbf{b}\cdot\mathbf{x})\mathbf{a}\times\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}\times\mathbf{x}\ . \tag{0.5}$$

If $\mathbf{b}$ is orthogonal to $\mathbf{c}$, and $\mathbf{a}$ is a multiple of $\mathbf{c}$, then $\mathbf{a}\times\mathbf{c} = \mathbf{0}$ and $\mathbf{b}\cdot\mathbf{c} = 0$, so in this case $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ no matter what $\mathbf{x}$ is.

We will now give two proofs for the converse. The first is short, but perhaps somewhat tricky. The second is longer, but is a more direct application of the ideas we used to prove Lagrange's identity in the first place. It is a good example of making a judicuous choice of coordinates to solve a problem

Here is the first approach to the converse: Let $\mathbf{u}$ be any unit vector that is ortogonal to both $\mathbf{a}$ and $\mathbf{c}$. Then

$$\mathbf{f}(\mathbf{u}) = (\mathbf{b}\cdot\mathbf{u})\mathbf{a}\times\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}\times\mathbf{u}\ .$$

Since $\mathbf{u}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{c}$, $\mathbf{a}\times\mathbf{c}$ is a multiple (possibly zero) of $\mathbf{u}$. On the other hand, $\mathbf{a}\times\mathbf{u}$ is orthogonal to $\mathbf{u}$. Hence $(\mathbf{b}\cdot\mathbf{u})\mathbf{a}\times\mathbf{c}$ and $(\mathbf{b}\cdot\mathbf{c})\mathbf{a}\times\mathbf{u}$ are orthogonal, and therefore

$$\|\mathbf{f}(\mathbf{u})\|^2 = \|(\mathbf{b}\cdot\mathbf{u})\mathbf{a}\times\mathbf{c}\|^2 + \|(\mathbf{b}\cdot\mathbf{c})\mathbf{a}\times\mathbf{u}\|^2\ .$$

Hence, assuming $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$, we have, in particular, that

$$(\mathbf{b}\cdot\mathbf{c})\mathbf{a}\times\mathbf{u} = \mathbf{0}\ .$$

Now since $\mathbf{a}$ and $\mathbf{u}$ are orthogonal and non-zero, $\mathbf{a}\times\mathbf{u}\neq\mathbf{0}$, and so it must be the case that $\mathbf{b}\cdot\mathbf{c} = 0$. This shows that when $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$, $\mathbf{b}$ is orthogonal to $\mathbf{c}$, and we are halfway there. Moreover, given this, (0.5) simplifies to

$$\mathbf{f}(\mathbf{x}) = (\mathbf{b}\cdot\mathbf{x})\mathbf{a}\times\mathbf{c}\ . \tag{0.6}$$

Now since $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$, $\mathbf{f}(\mathbf{b}) = \mathbf{0}$. But this means

$$\|\mathbf{b}\|^2 \mathbf{a} \times \mathbf{c} = \mathbf{0} \ .$$

Since $\mathbf{b} \neq \mathbf{0}$, this means $\mathbf{a} \times \mathbf{c} = \mathbf{0}$, and thus $\mathbf{a}$ is a multiple of $\mathbf{c}$. This completes the first approach.

We now give a second proof that when $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$, then $\mathbf{b}$ is orthogonal to $\mathbf{c}$, and $\mathbf{a}$ is a multiple of $\mathbf{c}$. This is based on the use of an adapted system of coordinates, as in the proof of Lagrange's identity.

Suppose that $\mathbf{b}$ is a multiple (necessarily non-zero) of $\mathbf{c}$, say $\mathbf{b} = \lambda\mathbf{c}$. Then

$$\mathbf{b} \times (\mathbf{c} \times \mathbf{x}) = -\lambda\|\mathbf{c}\|^2 \mathbf{x}_\perp$$

where $\mathbf{x}_\perp$ is the component of $\mathbf{x}$ that is orthogonal to $\mathbf{c}$. But then if $\mathbf{x}$ is orthogonal to both $\mathbf{c}$ and $\mathbf{a}$, then

$$\|\mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{x}))\| = \|\mathbf{a}\||\lambda|\|\mathbf{c}\|^2\|\mathbf{x}\| = \|\mathbf{a}\|\|\mathbf{b}\|\|\mathbf{c}\|\|\mathbf{x}\| \ .$$

Since there are non-zero vectors $\mathbf{x}$ orthogonal to any two vectors $\mathbf{a}$ and $\mathbf{c}$ in $\mathbb{R}^3$, we see that there are vectors $\mathbf{x}$ such that $\|\mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{x}))\| \neq 0$ if $\mathbf{b} = \lambda\mathbf{c}$.

Therefore, without loss of generality, we may assume that $\mathbf{b}$ is not a multiple of $\mathbf{c}$. Therefore, $\mathbf{b}_\perp$, the component of $\mathbf{b}$ orthogonal to $\mathbf{c}$, is not $\mathbf{0}$. We may then define an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ such that

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{c}\|}\mathbf{c} \qquad \mathbf{u}_2 = \frac{1}{\|\mathbf{b}_\perp\|}\mathbf{b}_\perp \quad \text{and} \quad \mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 \ .$$

Now let us write

$$\mathbf{x} = r\mathbf{u}_1 + s\mathbf{u}_2 + t\mathbf{u}_3 \ .$$

Since $\mathbf{c} = \|\mathbf{c}\|\mathbf{u}_1$, we then compute

$$\mathbf{c} \times \mathbf{x} = \|\mathbf{c}\|(s\mathbf{u}_3 - t\mathbf{u}_2) \ . \tag{0.7}$$

Now we write

$$\mathbf{b} = \mathbf{b}_\perp + \mathbf{b}_\| = \|\mathbf{b}_\perp\|\mathbf{u}_2 \pm \|\mathbf{b}_\|\|\mathbf{u}_1 \ , \tag{0.8}$$

where the plus sign is right if $\mathbf{b}_\|$ is a positive multiple of $\mathbf{c}$, and negative otherwise. (Of course, the sign is irrelevant if $\mathbf{b}_\| = \mathbf{0}$, as will turn out to be the case.)

Now we combine (0.7) and (0.8) to compute

$$\begin{aligned} \mathbf{b} \times (\mathbf{c} \times \mathbf{x}) &= (\|\mathbf{b}_\perp\|\mathbf{u}_2 \pm \|\mathbf{b}_\|\|\mathbf{u}_1) \times \|\mathbf{c}\|(s\mathbf{u}_3 - t\mathbf{u}_2) \\ &= s\|\mathbf{c}\|\|\mathbf{b}_\perp\|\mathbf{u}_1 \mp s\|\mathbf{c}\|\|\mathbf{b}_\|\|\mathbf{u}_2 \mp t\|\mathbf{c}\|\|\mathbf{b}_\|\|\mathbf{u}_3 \ . \end{aligned}$$

Finally then

$$s\|\mathbf{c}\|\|\mathbf{b}_\perp\|(\mathbf{a} \times \mathbf{u}_1) \mp s\|\mathbf{c}\|\|\mathbf{b}_\|\|(\mathbf{a} \times \mathbf{u}_2) \mp t\|\mathbf{c}\|\|\mathbf{b}_\|\|(\mathbf{a} \times \mathbf{u}_3) = \mathbf{0} \ , \tag{0.9}$$

and this must be true for *every* choice of s and t. Setting $t = 1$, and $s = 0$, we see that

$$\|\mathbf{c}\|\|\mathbf{b}_\|\|(\mathbf{a} \times \mathbf{u}_3) = \mathbf{0} \ .$$

Since $\|\mathbf{c}\| \neq 0$ by hypothesis, either $\|\mathbf{b}_{\|}\| = 0$, or $\mathbf{a} \times \mathbf{u}_3 = \mathbf{0}$. If the latter were true, it would be the case that $\mathbf{a} = \pm\|\mathbf{a}\|\mathbf{u}_3$.

Thus, supposing that $\|\mathbf{b}_{\|}\| \neq 0$, and setting $t = 0$, and $s = 1$ in (0.9), we would conclude that

$$\|\mathbf{c}\|\|\mathbf{b}_\perp\|\|\mathbf{a}\|\mathbf{u}_1 \pm \|\mathbf{c}\|\|\mathbf{b}_{\|}\|\|\mathbf{a}\|\mathbf{u}_2 = \mathbf{0} \ ,$$

and since $\{\mathbf{u}_1, \mathbf{u}_3\}$ is orthonormal, this means that $\|\mathbf{c}\|\|\mathbf{b}_\perp\|\|\mathbf{a}\| = 0$ and $\|\mathbf{c}\|\|\mathbf{b}_{\|}\|\|\mathbf{a}\| = 0$

But by hypothesis $\|\mathbf{a}\|, \|\mathbf{c}\| \neq, \|\mathbf{b}_{\|}\| \neq 0$, and we have seen in the first part of the argument that $\|\mathbf{b}_\perp\| \neq 0$. Thus we have a contradiction, and it must be the case that our supposition that $\|\mathbf{b}_{\|}\| \neq 0$ is false. Hence, $\|\mathbf{b}_{\|}\| - 0$, and thus $\mathbf{b}$ is orthogonal to $\mathbf{c}$ as was to be shown,

Moreover, armed with this knowledge, (0.9) simplifies to

$$\|\mathbf{c}\|\|\mathbf{b}\|(\mathbf{a} \times \mathbf{u}_1) = \mathbf{0} \ .$$

Since $\|\mathbf{c}\|, \|\mathbf{b}\| \neq 0$, $\mathbf{a} \times \mathbf{u}_1 = \mathbf{0}$ which means that $\mathbf{a}$ is a multiple of $\mathbf{u}_1$, and hence $\mathbf{c}$, as was to be shown.

Altogether, if $\mathbf{a} \times (\mathbf{b} \times (\mathbf{c} \times \mathbf{x})) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^3$ then $\mathbf{a}$ is a multiple of $\mathbf{c}$, and $\mathbf{b}$ is orthogonal $\mathbf{c}$. This completes the second approach.

1.14 Let P_1 the plane through the three points $\mathbf{a}_1 = (1, 2, 1)$ $\mathbf{a}_2 = (-1, 2, -3)$ and $\mathbf{a}_3 = (2, -3, -2)$. Let P_2 denote the plane through the three points $\mathbf{b}_1 = (1, 1, 0)$ $\mathbf{b}_2 = (1, 0, 1)$ and $\mathbf{b}_3 = (0, 1, 1)$.

(a) Find equations for the planes P_1 and P_2.

(b) Parameterize the line given by $P_1 \cap P_2$, and find the distance between this line and the point $\mathbf{a}_1$.

(c) Consider the line through $\mathbf{b}_1$ and $\mathbf{b}_2$. Determine the point of intersection of this line with the plane P_1.

SOLUTION: The plane contains $\mathbf{a}_1$, and its normal vector is $(\mathbf{a}_2 - \mathbf{a}_1) \times (\mathbf{a}_3 - \mathbf{a}_1) = (-20, -10, 10)$. Hence an equation for P_1 is

$$(-2, -1, 1) \cdot (x - 1, y - 2, z - 1) = -2x - y + z = -3 \ .$$

You can check this: Each of the three given points do satisfy this equation.

In the same way, or by inspection, using the symmetry of the points specifying P_2, we find that an equation for it is

$$x + y + z = 2 \ .$$

To parameterize the line $P_1 \cap P_2$, we need a base point. Let us look for one with $z = 0$, and then we must have

$$\begin{aligned} -2x - y &= -3 \\ x + y &= 2 \end{aligned}$$

. Hence, $-3x = -5$, so $x = 3/5$ and $y = 7/5$. Our base point is $\mathbf{x}_0 = (3/5, 7/5, 0)$.

The direction vector is

$$\mathbf{v} = (-2, -1, 1) \times (1, 1, 1) = (-2, 3, -1) \ .$$

Hence the line is

$$\mathbf{x}(t) = (3/5 - 2t, 7/5 + 3t, -t) \ .$$

The distance between this line and $\mathbf{a}_1$ is the length of the component of $\mathbf{a}_1 - \mathbf{x}_0$ that is orthogonal to $\mathbf{v}$; i.e.,

$$(\mathbf{a}_1 - \mathbf{x}_0)_\perp = (\mathbf{a}_1 - \mathbf{x}_0) - \frac{1}{\|\mathbf{v}\|^2}[(\mathbf{a}_1 - \mathbf{x}_0)\cdot\mathbf{v}]\mathbf{v} \ .$$

Since

$$\mathbf{a}_1 - \mathbf{x}_0 = (1,2,1) - (3/5, 7/5, 0) = (2/5, 3/5, 1) \ ,$$

which is orthogonal to $\mathbf{v}$, one has

$$(\mathbf{a}_1 - \mathbf{x}_0)_\perp = (2/5, 3/5, 1) \ ,$$

and so the distance is

$$\frac{1}{5}\sqrt{38} \ .$$

Finally, for part **(c)**, The line through $\mathbf{b_1}$ and $\mathbf{b_2}$ is parameterized by

$$\mathbf{x}(t) = \mathbf{b_1} + t(\mathbf{b_2} - \mathbf{b_1}) = (1, 1-t, t) \ .$$

Plugging this into our equation for P_1, we have

$$-2(1) - (1-t) + t = -3$$

which means $t = 0$. Indeed, you can see that $\mathbf{b}_1$ does satisfy the equation for P_1. Had you noticed this at the outset, you could have used $\mathbf{b}_1$ as the base point for the line in part **(b)**.

1.16 Consider the vector $\mathbf{a} = (1,4,3)$ and $\mathbf{b} = (3,2,1)$. Find an orthonormal basis of $\mathbb{R}^3$ whose third vector is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

SOLUTION: We compute $\mathbf{a}\times\mathbf{b} = (-2, 8, -10)$ and $\|\mathbf{a}\times\mathbf{b}\| = 2\sqrt{42}$. Thus we take

$$\mathbf{u}_3 := \frac{1}{2\sqrt{42}}(-2, 8, -10) \ .$$

We now choose $\mathbf{u}_2$ to be any unit vectors that is orthogonal to $\mathbf{u}_1$. Since $(-b, a, 0)$ is always orthogonal to (a, b, c), no matter the values of a, b and c, the vector $(8,2,0))$ is orthogonal to $\mathbf{u}_1$. Normalizing it we find

$$\mathbf{u}_1 = \frac{1}{\sqrt{17}}(4,1,0) \ .$$

Finally,

$$\mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1 = \frac{1}{\sqrt{714}}(5, -20, -17) \ .$$

1.18 Consider the plane passing through the three points

$$\mathbf{p}_1 = (-1,-3,0) \qquad \mathbf{p}_2 = (5,1,2) \qquad \text{and} \qquad \mathbf{p}_3 = (0,-3,4)$$

and the line passing through

$$\mathbf{z}_0 = (1,1,-1) \qquad \text{and} \qquad \mathbf{z}_1 = (1,-2,2)$$

(a) Find a parametric representation $\mathbf{x}(s,t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$ for the plane.

(b) Find a parametric representation $\mathbf{z}(u) = \mathbf{z}_0 + u\mathbf{w}$ for the line.

(c) Find an equation for the plane.

(d) Find a system of equations for the line.

(e) Find the points, if any, where the line intersects the plane.

(f) Find the distance from $\mathbf{p}_1$ to the line.

(g) Find the distance from $\mathbf{z}_0$ to the plane.

SOLUTION: For **(a)**, we form

$$\mathbf{v}_1 := \mathbf{p}_2 - \mathbf{p}_1 = (6,4,2) \qquad \text{and} \qquad \mathbf{v}_2 := \mathbf{p}_3 - \mathbf{p}_1 = (1,0,4)\ .$$

We take $\mathbf{p}_1$ as the base point, and then the parameterization is

$$\mathbf{x}(s,t) = (-1,-3,0) + s(6,4,2) + t(1,0,4) = (-1+6s+t, -3+4s, 2s+4t)\ .$$

For **(b)**, we form $\mathbf{w} = \mathbf{z}_1 - \mathbf{z}_0 = (0,-3,3)$. We then take $\mathbf{z}_0$ as the base point, and the parameterization is

$$\mathbf{z}(u) = \mathbf{z}_0 + u\mathbf{w} = (1,1,-1) + u(0,-3,3) = (1, 1-3u, -1+3u)\ .$$

For **(c)**, we compute $\mathbf{v}_1 \times \mathbf{v}_2 - (16,-22,-4)$. We can use any non-zero multiple of this as the normal vector. Discarding a common factor of 2, we define

$$\mathbf{a} := (8,-11,-2)\ .$$

The equation then is $\mathbf{a}\cdot(\mathbf{x} - \mathbf{p}_0) = 0$, which simplifies to

$$8x - 11y - 2z = 25\ .$$

For **(d)**, we note that $\mathbf{x} = \mathbf{z}_0 + u\mathbf{w}$ for some u if and only if $\mathbf{w}\times\mathbf{x} = \mathbf{w}\times\mathbf{z}_0$. Doing the two cross products, this works out to

$$(3z+3y, -3x, -3x)) = ((0,-3,-3)\ .$$

We get two independent equations by equating components of these vectors:

$$\begin{aligned} y + z &= 0 \\ x &= 1 \end{aligned}$$

and you can easily check that this is consistent with the answer to part **(b)**.

For **(e)**, we substitute $\mathbf{z}(u) = (1, 1-3u, -1+3u)$ into $8x - 11y - 2z = 25$, and solve for u to find $u = 26/27$. Thus the point of intersection is

$$\mathbf{z}(26/27) = \frac{1}{9}(9,-17,17)\ .$$

For **(f)**, the distance is the length of $(\mathbf{p}_1 - \mathbf{z}_0)_\perp$, the component of $\mathbf{p}_1 - \mathbf{z}_0$ orthogonal to $\mathbf{w}$. This is given by

$$\frac{1}{\|\mathbf{w}\|}\|\mathbf{w} \times (\mathbf{p}_1 - \mathbf{z}_0)\| = \frac{1}{3\sqrt{2}}\|(0,3,-3) \times (\,-2,-4,1)\| = \frac{1}{3\sqrt{2}}9 = \frac{3}{\sqrt{2}} \ .$$

For **(g)**, the distance is the length of $(\mathbf{p}_1 - \mathbf{z}_0)_\|$, the component of $\mathbf{p}_1 - \mathbf{z}_0$ orthogonal to $\mathbf{a}$. This is given by

$$\frac{1}{\|\mathbf{a}\|}\|\mathbf{a} \cdot (\mathbf{p}_1 - \mathbf{z}_0)\| = \frac{26}{3\sqrt{21}}.$$

1.20 Consider the plane given by

$$2x - y + 3z = 4 \ .$$

Let $\mathbf{p} = (-1,-3,0)$. What is the distance from $\mathbf{p}$ to the plane?

SOLUTION: We first need to find a base point $\mathbf{x}_0$ in the plane. Looking for a solution with $y = z = 0$, we see that $x = 2$. Thus, we take $\mathbf{x}_0 = (2,0,0)$. Then, since $\mathbf{a} := (2,-1,3)$ is orthogonal to the plane, Then the distance is

$$\frac{|(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{a}|}{\|\mathbf{a}\|} = -\frac{3}{\sqrt{14}} \ .$$

1.22 Consider the line ℓ given by

$$\begin{aligned} 2x - y + 3z &= 4 \\ x + y + z &= 2 \ . \end{aligned}$$

Find a parametric representation of the line obtained by reflecting this line through the plane $x + 3y - z = 1$. That is; the outgoing line should have as its base point the intersection of the incoming line and the plane, and its direction vector should be $\mathbf{h}_\mathbf{u}(\mathbf{v})$ where $\mathbf{v}$ is the incoming direction vector, and $\mathbf{u}$ is a unit normal vector to the plane.

SOLUTION: First, we parameterize the line. It is easy to observe that $(2,0,0)$ satisfies both equations, so we take this as our base point $\mathbf{x}_0$.

The direction vector $\mathbf{v}$ of the lines is given by the cross product of the normals of the two intersecting planes:

$$\mathbf{v} = (2,-1,3) \times (1,1,1) = (-3,-1,4)$$

So the line is parameterized by

$$\mathbf{x}(t) = (2,0,0) + t(-3,-1,4) = (2 - 3t, -t, 4t) \ .$$

Plugging this into the equation of the plane $x + 3y - z = 1$, we find

$$(2 - 3t) - 3t - 4t = 1$$

or $t = 1/10$. Thus the point where the line intersects the plane, which I will call $\mathbf{z}_0$ is

$$\mathbf{z}_0 = \frac{1}{10}(17,-1,4) \ .$$

As you can check, this does satisify the equation for the plane.

Let $\mathbf{w}$ be the outgoing direction vector. One gets this by reflecting the incoming one about the plane. The unit vector normal to the plane is

$$\mathbf{u} = \frac{1}{\sqrt{11}}(1, 3, -1) ,$$

and so

$$\mathbf{w} = \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{u})\mathbf{u} .$$

We know $\mathbf{u}$ and we know $\mathbf{v}$, so we compute

$$\mathbf{w} = \frac{1}{11}(- 31, -5, 46) .$$

Using this, one computes $\mathbf{w}$, and the the outgoing line is parameterized by

$$\mathbf{z}_0 + t\mathbf{w} = \frac{1}{10}(17, -1, 4) + t\frac{1}{11}(- 31, -5, 46) .$$

1.24 Let $\mathbf{x} = (5, 2, 4, 2)$. Let $\mathbf{u}$ be a unit vector such that $\mathbf{h}_{\mathbf{u}}(\mathbf{x})$ is a multiple of $\mathbf{e}_1$. What are the possible values of this multiple? Find four such unit vectors $\mathbf{u}$.

SOLUTION: We compute $\mathbf{x} \cdot \mathbf{x}$, so that $\|\mathbf{x}\| = 7$. Thus, the only two multiples of $\mathbf{e}_1$ that Householder reflections of $\mathbf{x}$ for some $\mathbf{u}$ are $\pm(7, 0, 0, 0)$.

Taking the multiple of $\mathbf{e}_1$ to be $(7, 0, 0, 0)$, we have

$$\mathbf{u} = \pm\frac{1}{\|\mathbf{x} - 7\mathbf{e}_1\|}(\mathbf{x} - 7\mathbf{e}_1) = \pm\frac{1}{\|(- 2, 2, 4, 2)\|}(- 2, 2, 4, 2) = \pm\frac{1}{\sqrt{7}}(- 1, 1, 2, 1) .$$

Taking the multiple of $\mathbf{e}_1$ to be $-(7, 0, 0, 0)$, we have

$$\mathbf{u} = \pm\frac{1}{\|\mathbf{x} + 7\mathbf{e}_1\|}(\mathbf{x} + 7\mathbf{e}_1) = \pm\frac{1}{\|(12, 2, 4, 2)\|}(12, 2, 4, 2) = \pm\frac{1}{\sqrt{42}}(6, 1, 2, 1) .$$

1.26 Consider two lines in $\mathbb{R}^3$ given parametrically by $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$ where

$$\mathbf{x}_1 = (1, 2, 3) \qquad \mathbf{x}_2 = (2, 0, 2) \qquad \mathbf{v}_1 = (1, 2, 2) \qquad \text{and} \qquad \mathbf{v}_2 = (-2, 1, 1) .$$

Compute the distance between these two lines.

SOLUTION: This is given by

$$\frac{|(\mathbf{x}_1 - \mathbf{x}_2) \cdot (\mathbf{v}_1 \times \mathbf{v}_2)|}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} = \frac{1}{\sqrt{2}} .$$

1.28 Consider two lines in $\mathbb{R}^3$ given parametrically by $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$ where

$$\mathbf{x}_1 = (1, 2, -1) \qquad \mathbf{x}_2 = (2, 1, -5) \qquad \mathbf{v}_1 = (1, -4, -2) \qquad \text{and} \qquad \mathbf{v}_2 = (1, 1, -2).$$

Find the point on the first line that is closest to the second line, the point on the second line that is closest to the first line, and the distance between these two lines.

SOLUTION: We first compute the good orthonormal basis for this task:

$$\mathbf{u}_1 := \frac{1}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}|\mathbf{v}_1 \times \mathbf{v}_2 \qquad \mathbf{u}_2 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 \times \mathbf{u}_1 \quad \text{and} \quad \mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2 \ .$$

We find

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}}(2,0,1) \qquad \mathbf{u}_2 = \frac{1}{\sqrt{105}}(-4,-5,8) \quad \text{and} \quad \mathbf{u}_3 = \frac{1}{\sqrt{21}}(1,-4,-2) \ .$$

Next, let

$$\mathbf{b} = \mathbf{x}_1 - \mathbf{x}_2 = (-1,1,4) \ .$$

Then, as explained at the end of Chapter One, the optimal choice for t is

$$t = \frac{b \cdot \mathbf{u}_2}{\mathbf{v}_2 \cdot \mathbf{u}_2} = -\frac{31}{25} \ ,$$

and the optimal choice for s is

$$s = \frac{t(\mathbf{v}_2 \cdot \mathbf{u}_3 - \mathbf{b} \cdot \mathbf{u}_3}{vv_1 \cdot \mathbf{u}_3} = \frac{14}{25} \ .$$

Thus, the point on the fist line that is closest to the second is

$$\mathbf{x}_1(s) = \frac{1}{25}(39,-6,-53) \ ,$$

and the point on the second line that is closest to the first is

$$\mathbf{x}_2(t) = \frac{1}{25}(19,-6,-63) \ ,$$

Thus

$$\mathbf{x}_1(s) - \mathbf{x}_2(t) = \frac{1}{5}(4,0,2) \ ,$$

and so the distance between the two lines is $\|\mathbf{x}_1(s) - \mathbf{x}_2(t)\| = 2/\sqrt{5}$.

1.30 Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be given by

$$\mathbf{u}_1 = \frac{1}{9}(1,4,8) \qquad \mathbf{u}_2 = \frac{1}{9}(8,-4,1) \qquad \text{and} \qquad \mathbf{u}_3 = \frac{1}{9}(4,7,-4)) \ .$$

(a) Verify whether $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is, or is not, an orthonormal basis of $\mathbb{R}^3$.

(b) Find a unit vector $\mathbf{u}$ so that $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_1) = \mathbf{e}_1$.

(c) With this same choice of $\mathbf{u}$, compute $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_2)$ and $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_3)$.

SOLUTION: We compute that

$$\|\mathbf{u}_1\|^2 = \frac{1+16+64}{81} = 1 \qquad \text{and} \qquad \|\mathbf{u}_2\|^2 = \frac{64+16+1}{81} = 1 \ ,$$

so both $\mathbf{u}_1$ and $\mathbf{u}_2$ are unit vectors.

We next compute

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \frac{8-16+8}{81} = 0 \qquad \text{and} \qquad \mathbf{u}_1 \times \mathbf{u}_2 = \frac{1}{81}(36,63,-36) = \mathbf{u}_3 \ .$$

This shows that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a right handed orthonormal basis.

For **(b)**, we take

$$\mathbf{u} = \frac{1}{\|\mathbf{u}_1 - \mathbf{e}_1\|}(\mathbf{u}_1 - \mathbf{e}_1) = \frac{1}{\|(-8,4,8)\|}(-8,4,8) = \frac{1}{12}(-8,4,8) \ .$$

We then compute

$$\begin{aligned} \mathbf{h}_{\mathbf{u}}(\mathbf{u}_1) &= \mathbf{u}_1 - 2(\mathbf{u}_1 \cdot \mathbf{u})\mathbf{u} = (1,0,0) \\ \mathbf{h}_{\mathbf{u}}(\mathbf{u}_2) &= \mathbf{u}_2 - 2(\mathbf{u}_2 \cdot \mathbf{u})\mathbf{u} = (0,0,1) \\ \mathbf{h}_{\mathbf{u}}(\mathbf{u}_3) &= \mathbf{u}_3 - 2(\mathbf{u}_3 \cdot \mathbf{u})\mathbf{u} = (0,1,0) \end{aligned}$$

Notice that this Householder transformation has transformed a right handed orthonormal basis into a left handed orthonormal basis. As we shall see in Chapter 2, this always happens, and so the result for $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_3)$ is exactly what we must have, given this fact, and our results for $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_1)$ and $\mathbf{h}_{\mathbf{u}}(\mathbf{u}_2)$.