Solutions for Challenge Problem Set 5, Math 291 Fall 2017

Eric A. Carlen[1]
Rutgers University

December 20, 2017

This challenge problem set concerns surface area computations for parameterized surfaces and hypersurfaces in 4 and more dimensions.

0.1 Parameterized surfaces and hypersurfaces in 4 and more dimensions

A continuously differentiable parameterized surface $\mathcal{S}$ in $\mathbb{R}^n$, $n \geq 4$ is a continuously differentiable function $\mathbf{X}$ from an open set $U \subset \mathbb{R}^2$ to $\mathbb{R}^n$ such that $\text{rank}([D_{\mathbf{X}}(\mathbf{u})]) = 2$ for all $\mathbf{u} \in U$, so that at each $\mathbf{u}$, the columns of $[D_{\mathbf{X}}(\mathbf{u})]$ are linearly independent.

A continuously differentiable parameterized *hypersurface* $\mathcal{H}$ in $\mathbb{R}^n$, $n \geq 4$ is a continuously differentiable function $\mathbf{X}$ from an open set $U \subset \mathbb{R}^{n-1}$ to $\mathbb{R}^n$ such that $\text{rank}(D_{\mathbf{X}}(\mathbf{u})]) = n-1$ for all $\mathbf{u} \in U$, so that at each $\mathbf{u}$, the columns of $[D_{\mathbf{X}}(\mathbf{u})]$ are linearly independent.

Note that for $n = 3$, surfaces and hypersurfaces are the same thing, but otherwise that are not. We first discuss area computation for surfaces, and then turn to hypersurfaces.

0.2 Area computations for surfaces in $\mathbb{R}^n$, $n \geq 4$

.

We often denote elements $\mathbf{u}$ of the parameter set U by $\mathbf{u} = (u, v)$, and write $\mathbf{X}(u, v)$ to denote the value of $\mathbf{X}$ at (u, v).

The tangent plane to $\mathcal{S}$ at a point $\mathbf{p} \in \mathcal{S}$ is the space spanned by $\{\mathbf{X}_u(\mathbf{p}), \mathbf{X}_u(\mathbf{p})\}$, just as in the case $n = 3$.

In three dimensions, we have a formula for the surface area element in terms of the cross product of $\mathbf{X}_u$ and $\mathbf{X}_v$. However since for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2$, we can re-express this formula in terms of the dot product. And then it is true in every dimension.

Exercise 1: Let $\mathbf{p} \in \mathcal{S}$, where $\mathbf{p} = \mathbf{X}(u_0, v_0)$, and fix some small $h > 0$. Consider the little "tile" $T_{u_0,v_0,h}$ in $\mathcal{S}$ consisting of points of the form $\mathbf{X}(u, v)$ with $u_0 \leq u \leq u_0 + h$ and $v_0 \leq v \leq v_0 + h$. Explain why

$$\lim_{h \downarrow 0} \frac{1}{h^2}\text{area}(T_{u_0,v_0,h}) = \sqrt{\|\mathbf{X}_u(u_0, v_0)\|^2\|\mathbf{X}_v(u_0, v_0)\|^2 - (\mathbf{X}_u(u_0, v_0) \cdot \mathbf{X}_v(u_0, v_0))^2} \ .$$

SOLUTION Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors linearly independent in $\mathbb{R}^n$. Let $\{\mathbf{u}_1, \mathbf{u}_2\}$ be the orthonormal set obtained from $\{\mathbf{a}, \mathbf{b}\}$ by applying the Gram-Schmidt Algorithm. Then $\mathbf{a} = \|\mathbf{a}\|\mathbf{u}_1$

and $\mathbf{b} = (\mathbf{b}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{b}\cdot\mathbf{u}_2)\mathbf{u}_2$, with $\mathbf{b}\cdot\mathbf{u}_2 > 0$. In coordinates $u\mathbf{u}_1 + v\mathbf{u}_2$ in the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$, the tile in questions is a parallelogram with vertices $(0,0)$, $(\|\mathbf{a}\|,0)$, $(\mathbf{b}\cdot\mathbf{u}_1, \mathbf{b}\cdot\mathbf{u}_2)$ and $(\|\mathbf{a}\| + \mathbf{b}\cdot\mathbf{u}_1, \mathbf{b}\cdot\mathbf{u}_2)$. The area of this parallelogram, by the usual "base times height" formula is $\|\mathbf{a}\|(\mathbf{b}\cdot\mathbf{u}_2)$ and

$$\|\mathbf{a}\|(\mathbf{b}\cdot\mathbf{u}_2)^2 = \|\mathbf{a}\|^2(\|\mathbf{b}\|^2 - (\mathbf{b}\cdot\mathbf{u}_1)^2) = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \ .$$

Now apply this with $\mathbf{a} = \mathbf{X}(u_0+h, v_0) - \mathbf{X}(u_0,v_0)$ and $\mathbf{b} = \mathbf{X}(u_0, v_0+h) - \mathbf{X}(u_0,v_0)$, and use the definition of the derivative.

Exercise 2. Show that if $\mathcal{S}$ is parameterized by $\mathbf{X}(u,v)$, $(u,v)\in U$, then the surface area of $\mathcal{S}$ is given by

$$\mathrm{d}S = \int_U \sqrt{\|\mathbf{X}_u(u_0,v_0)\|^2\|\mathbf{X}_v(u_0,v_0)\|^2 - (\mathbf{X}_u(u_0,v_0)\cdot\mathbf{X}_v(u_0,v_0))^2}\mathrm{d}u\mathrm{d}v \ .$$

SOLUTION This follows immediately from the previous exercise using the definition of the surface area element.

Now let's consider a specific surface $\mathcal{S}$, that we will generate in exactly the same way that we generate the usual torus in $\mathbb{R}^3$. Recall that we can do this by taking a circle of radius $r > 0$ in the x,z plane, centered at $(R,0,0)$ on the x-axis, $R > r$, and then rotate the circle around the z-axis to sweep out the torus.

The circle of radius r in the x,z plane centered at $(R,0,0)$ is parameterized by

$$\mathbf{x}(v) = (R + r\cos(v), 0, r\sin v) \ .$$

The rotation about the z-axis through the angle u is given by

$$R_u := \begin{bmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{bmatrix} \ .$$

we then define $\mathbf{X}(u,v) = R_u\mathbf{x}(v)$.

We can do something similar in 4 (or more) dimensions. Now let $\mathbf{x}(v)$ the continuously differentiable closed curve in $\mathbb{R}^3$ given by

$$\mathbf{x}(v) = (\cos v, \sqrt{2}\sin v, \cos v) \ , \quad v\in[0,2\pi) \ .$$

Embedding $\mathbb{R}^3$ into $\mathbb{R}^4$ by sending (x,y,z) to $(x,y,z,0)$, we obtain a curve, still denoted $\mathbf{x}(v)$, in $\mathbb{R}^4$:

$$\mathbf{x}(v) = (\cos v, \sqrt{2}\sin v, \cos v.0) \ , \quad v\in[0,2\pi) \ .$$

Now consider the 4-dimensional rotation matrix R_u defined by

$$R_u := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos u & -\sin u \\ 0 & 0 & \sin u & \cos u \end{bmatrix} \ .$$

Finally define

$$\mathbf{X}(u,v) = R_u\mathbf{x}(v) \ ,$$

and let $\mathcal{S}$ denote the surface parameterized by $\mathbf{X}(u,v)$ as (u,v) ranges over $[0,2\pi)\times[0,2\pi)$.

Exercise 3: Compute the surface area of $\mathcal{S}$.

SOLUTION Doing the multiplication explicitly, we find

$$\mathbf{X}(u,v) = R_u\mathbf{x}(v) = (\cos(v), \sqrt(2)\sin(v), \cos(v)\cos(2u), +\cos(v)\sin(2u)) \ .$$

Differentiating,

$$\mathbf{X}_u(u,v) = (0,0,-2\cos(v)\sin(2u), 2\cos(v)\cos(2u))$$

and

$$\mathbf{X}_v(u,v) = (-\sin(v), \sqrt(2)\cos(v), -\sin(v)\cos(2u), -\sin(v)\sin(2u)) \ .$$

Then

$$\|\mathbf{X}_u(u,v)\|^2 = 4\cos^2(v) \ , \qquad \|\mathbf{X}_v(u,v)\|^2 = 2 \quad \text{and} \quad \mathbf{X}_u(u,v)\cdot\mathbf{X}_v(u,v) = 0 \ .$$

Therefore,

$$\mathrm{d}S = 2\sqrt{2}|\cos(v)|\mathrm{d}u\mathrm{d}v \ .$$

(Remember to take the positive square root!) Then since $\int_0^{2\pi}|\cos(v)|\mathrm{d}v = 4$, the surface area is

$$16\sqrt{2}\pi \ .$$

Exercise 4: Compute the vectors $\mathbf{X}_u(0,0)$, $\mathbf{X}_v(0,0)$, $\mathbf{X}_u(\pi/8,\pi/6)$ and $\mathbf{X}_v(\pi/8,\pi/6)$. Could it be that the surface $\mathcal{S}$ lies in a 3-dimensional hyperplane in $\mathbb{R}^4$? In other words does the exist a non-zero vector $\mathbf{A}\in\mathbb{R}^4$ and a vector $\mathbf{x}_0\in\mathbb{R}^4$ so that

$$\mathbf{A}\cdot(\mathbf{X}(u,v)-\mathbf{X}_0) = \mathbf{0}$$

for all u,v? Justify your answer.

SOLUTION We compute $\mathbf{X}_u(0,0) = (0,0,0,2)$, $\mathbf{X}_v(0,0) = (0,\sqrt{2},0,0)$, $\mathbf{X}_u(\pi/8,\pi/6) = \sqrt{3/2}(0,0,-1,1)$ and $\mathbf{X}_v(\pi/8,\pi/6) = \frac{1}{4}(-2,2\sqrt{6},-\sqrt{2},-\sqrt{2})$. The vectors evidently span $\mathbb{R}^4$ and so the surface cannot lie in a 3 dimensional subspace.

0.3 Hypersurfaces in $\mathbb{R}^n$, $n\geq 4$.

Consider a continuously differentiable parameterized hypersurface $\mathcal{H}$ in $\mathbb{R}^n$, $n\geq 4$, given by $\mathbf{X}(\mathbf{u})$, $\mathbf{u}\in U\subset\mathbb{R}^{n-1}$. By hypothesis on the rank of $D_{\mathbf{X}}(\mathbf{u}_0)$, $\{\mathbf{X}_{u_1}(\mathbf{u}_0),\dots,\mathbf{X}_{u_{n-1}}(\mathbf{u}_0)\}$ spans an $n-1$ dimensional subspace of $\mathbb{R}^n$, which is called the tangent space of $\mathcal{H}$ at the point $\mathbf{p} := \mathbf{X}(\mathbf{u}_0)$. The orthogonal complement of the Tangent space is a one-dimensional subspace called the *normal line*. Orthonormal bases for the tangent space at its complement can be found, in the usual way, by applying the Gram-Schmidt Algorithm to $\{\mathbf{X}_{u_1}(\mathbf{u}_0),\dots,\mathbf{X}_{u_{n-1}}(\mathbf{u}_0)\}$, and then proceeding with the vectors in, for example, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, until one finds the nth vector, which is a unit normal to the hypersurface.

Let $\mathbf{N}(\mathbf{u}_0)$ be such a normal vector. Fix $h > 0$, and consider the parallelepiped P_h consisting of the sets of points of the form

$$\mathbf{p} + \sum_{j=1}^{n-1} t_j \mathbf{X}_{u_j}(\mathbf{u}_0) + t_n \mathbf{N}(\mathbf{u}_0) \ , \quad 0 \le t_1, \ldots, t_n \le h \ .$$

We have seen that the n-dimensional volume of this parallelepiped is

$$\mathrm{vol}(P_h) = h^n |\det([\mathbf{X}_{u_1}(\mathbf{u}_0), \ldots, \mathbf{X}_{u_{n-1}}(\mathbf{u}_0), \mathbf{N}(\mathbf{u}_0)])| \ .$$

Note that the "base" of this parallelepiped in the tangent space is the $n-1$ dimensional "tile" T_h spanned by $\{\mathbf{X}_{u_1}(\mathbf{u}_0), \ldots, \mathbf{X}_{u_{n-1}}(\mathbf{u}_0)\}$ using coefficients in $(0, h)$. The height is exactly h, and from the "volume = base area time height" we get a formula for the (hyper) area of the tile T_h, namely,

$$\mathrm{area}(T_h) = h^{n-1} |\det([\mathbf{X}_{u_1}(\mathbf{u}_0), \ldots, \mathbf{X}_{u_{n-1}}(\mathbf{u}_0), \mathbf{N}(\mathbf{u}_0)])| \ . \tag{0.1}$$

(Think about how to justify "volume = base area time height" in higher dimension using the definition of volume in terms of cube packings and cube coverings.)

We now apply this to compute the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$, $n \ge 4$. (Perhaps hypersurface area would be better terminology, but for the merit of brevity.)

We recursively construct parameterizations in terms of $n-1$ angles. Let $\mathbf{u}_1(\theta) = (\cos\theta, \sin\theta)$, $\theta \in (0, 2\pi)$ be the standard parameterization of the unit circle in $\mathbb{R}^2$.

For $\phi_1 \in (0, \pi)$, define

$$\mathbf{u}_2(\theta, \phi_1) = (\sin\phi_1 \mathbf{u}_1(\theta), \cos\phi_1) \ ,$$

where we use the notation in which for $\mathbf{x} \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$, $(\mathbf{x}, t)$ denotes the vector $(x_1, \ldots, x_{n-1}, t) \in \mathbb{R}^n$. Notice that $\mathbf{u}_2(\theta, \phi_1)$ is the "usual" parameterization of S^2.

Now go on to define, for $\phi_2 \in (0, \pi)$,

$$\mathbf{u}_3(\theta, \phi_1, \phi_2) = (\sin\phi_2 \mathbf{u}_2(\theta, \phi_1), \cos\phi_2)] \ .$$

Notice that for all θ, ϕ_1 and ϕ_2, $\|\mathbf{u}_3(\theta, \phi_1, \phi_2)\| = 1$ so that indeed $\mathbf{u}_3(\theta, \phi_1, \phi_2)$ belongs to S^3. It is easy to see that this formula parameterizes S^3, apart from some inconsequential missing bits that we could include by letting our angles run over closed intervals (but the the parameterization would not be one-to-one onto the new points). The missing points are negligible with regard to 3 dimensional hypersurface area, just as the missing poles and line of longitude are negligible as far as area goes for S^2.

Exercise 5: Show that for $\mathbf{p} = \mathbf{u}_3(\theta, \phi_1, \phi_2)$, the vector $\mathbf{u}_3(\theta, \phi_1, \phi_2)$ itself is normal to the tangent plane, and hence we may take

$$\mathbf{N}(\theta, \phi_1, \phi_2) := \mathbf{u}_3(\theta, \phi_1, \phi_2)$$

in the formula (0.1).

SOLUTION The equation for S^3 is $f(\mathbf{x}) = 0$. where $f(\mathbf{x}) := \|\mathbf{x}\|^2 - 1$. The gradient of f is normal to the solutions set of $f(\mathbf{x}) = 0$, and $\nabla f(\mathbf{x}) = 2\mathbf{x}$. Normalizing we find,

$$\mathbf{N}(\mathbf{x}) = \|\nabla f(\mathbf{x})\|^{-1} \nabla f(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}\mathbf{x} = \mathbf{x}$$

when $\mathbf{x} \in S^3$. Taking $\mathbf{x} = \mathbf{u}_3(\theta, \phi_1, \phi_2)$, we have $\mathbf{N}(\theta, \phi_1, \phi_2) := \mathbf{u}_3(\theta, \phi_1, \phi_2)$, and this is evidently the outward normal.

Exercise 6: The 3-dimensional hypersurface area element of S^3 for the present parameterization is, by Exercise 5 and by what we have explained above,

$$\mathrm{d}S = |\det([\mathbf{u}_3, (\mathbf{u}_3)_\theta, (\mathbf{u}_3)_{\phi_1}, (\mathbf{u}_3)_{\phi_2}])| \mathrm{d}\theta \mathrm{d}\phi_1 \mathrm{d}\phi_2 \ ,$$

with $(\mathbf{u}_3)_\theta$ denoting the θ partial derivative of $\mathbf{u}_3(\theta, \phi_1, \phi_2)$, etc., and with all four column vectors evaluated at (θ, ϕ_1, ϕ_2). Show that the matrix $[\mathbf{u}_3, (\mathbf{u}_3)_\theta, (\mathbf{u}_3)_{\phi_1}, (\mathbf{u}_3)_{\phi_2}]$ has the following "block structure":

$$[\mathbf{u}_3, (\mathbf{u}_3)_\theta, (\mathbf{u}_3)_{\phi_1}, (\mathbf{u}_3)_{\phi_2}] = \left[\begin{array}{cc} \sin\phi_2 A & \cos\phi_2 \mathbf{u}_2 \\ (\cos\phi_2, \mathbf{0}) & -\sin\phi_2 \end{array} \right] \ ,$$

where $(\cos\phi_1, \mathbf{0})$ is the row vector $(\cos\phi_1, 0, 0)$ in $\mathbb{R}^3$, A is the 3×3 matrix

$$A(\theta, \phi_1) = [\mathbf{u}_2(\theta, \phi_1), (\mathbf{u}_2)_\theta(\theta, \phi_1), (\mathbf{u}_2)_{\phi_1}(\theta, \phi_1)] \ . \tag{0.2}$$

Notice that only the first and last entries in the final row can be non-zero. This makes it easy to recursively compute the determinant.

SOLUTION Differentiating

$$\mathbf{u}_3(\theta, \phi_1, \phi_2) = (\sin\phi_2 \mathbf{u}_2(\theta, \phi_1), \cos\phi_2)]$$

with respect to θ ,ϕ_1 and ϕ_2 in this order allows us to express the Jacobian of $\mathbf{u}_3$ in terms of the Jacobian of $\mathbf{u}_2$, and this is the result we get.

Exercise 7: Let B be an $n \times n$ matrix such that $B_{n,j} = 0$ unless $j = 1$ or $j = n$. Let C_1 be the $(n-1) \times (n-1)$ matrix that you get by crossing out the first column and last row of B. Let C_n be the $(n-1) \times (n-1)$ matrix that you get by crossing out the last column and last row of B. Show that

$$\det(B) = (-1)^{n+1} B_{n,1} \det(C_1) + B_{n,n} \det(C_n) \ .$$

You may use either the determinant formula or the Laplace row expansion method if you are familiar with that. (You can easily deduce the Laplace row expansion method from the determinant formula.) Then apply this to the result form Exercise 7 to deduce that

$$|\det([\mathbf{u}_3, (\mathbf{u}_3)_\theta, (\mathbf{u}_3)_{\phi_1}, (\mathbf{u}_3)_{\phi_2}])| = (\cos^2\phi_2 + \sin^2\phi_2)\sin^2\phi_2 |\det(A(\theta, \phi_1))| = \sin^2\phi_2 |\det(A(\theta, \phi_1))|$$

where $A(\theta, \phi_1)$ is given by (0.2). Then show $|\det(A(\theta, \phi_1))| = \sin\phi_1$ so that altogether,

$$\mathrm{d}S = \sin^2\phi_2 \sin\phi_1 \mathrm{d}\theta \mathrm{d}\phi_1 \mathrm{d}\phi_2 \ ,$$

and, finally, compute the surface are of S^3.

SOLUTION Since the last row of $[\mathbf{u}_3, (\mathbf{u}_3)_\theta, (\mathbf{u}_3)_{\phi_1}, (\mathbf{u}_3)_{\phi_2}]$ has only two non-zero entries, we can expand in the last row using the linearity of the determinant, and we get a sum of multiples of two 3×3 determinants.

Let C_1 be the upper-left 3×3 block. As we have seen above, this is

$$C_1 := [\sin\phi_2 \mathbf{u}_2(\theta,\phi_1), \sin\phi_2(\mathbf{u}_2)_\theta(\theta,\phi_1), \sin\phi_2(\mathbf{u}_2)_{\phi_1}(\theta,\phi_1)]$$

The upper-right 3×3 block is

$$C_4 := [\sin\phi_2(\mathbf{u}_2)_\theta(\theta,\phi_1), \sin\phi_2(\mathbf{u}_2)_{\phi_1}(\theta,\phi_1), \cos\phi_2 \mathbf{u}_2(\theta,\phi_1)] \ ,$$

which is almost the same as C_1 except for the order of the columns, and now one column is multiplied by $\cos\phi_2$ instead of $\sin\phi_2$. Since the columns of C_1 are the columns of $A(\theta,\phi_1)$ multiplied by $\sin\phi_2$,

$$\det(C_1) = \sin^3\phi_2 \det(A(\theta,\phi_1)) \ .$$

Since C_4 results from $A(\theta,\phi_1)$ using 2 row swaps – an even number – and multiply two columns by $\sin\phi_2$ and one by $\cos\phi_2$,

$$\det(C_4) = \cos\phi_2 \sin^2\phi_2 \det(A(\theta,\phi_1)) \ .$$

Finally, since in this application, $(-1)^{4+1}B_{11} = -\cos\phi_2$ and $B_{4,4} = -\sin\phi_2$,

$$(-1)^{n+1}B_{4,1}\det(C_1) + B_{4,4}\det(C_4) = -(\cos^2\phi_2 + \sin^2\phi_2)\sin^2\phi_2 \det(A(\theta,\phi_1))$$

We have seen that $|\det(A(\theta,\phi_1))| = \sin\phi_1$ in our computation of the Jacobian for spherical coordinates, and so

$$\mathrm{d}S = \sin^2\phi_2 \sin\phi_2 \mathrm{d}\theta\mathrm{d}\phi_1\mathrm{d}\phi_2 \ ,$$

It now follows that the 3-dimensional area of S^3 is $\int_0^\pi \sin^2\phi_2\mathrm{d}\phi_2 = \pi/2$ times the area of S^2, which is 4π. Hence the area of 3-dimensional area S^3 is $\pi^2/2$

Exercise 8 (Extra Credit): Generalize what you did in the last few exercises to show that if $\mathrm{d}S_{n-1}$ denotes the surface element on S^{n-1} in the coordinates being used here, for all $n > 2$,

$$\mathrm{d}S_n = \sin^{n-1}\phi_{n-1}\mathrm{d}S_{n-1}\mathrm{d}\phi_{n-1} \ . \tag{$*$}$$

then go on to prove that

$$\lim_{n\to\infty} \mathrm{area}(S^{n-1}) = 0$$

and find the smallest value of n such that

$$\mathrm{area}(S^n) < \mathrm{area}(S^{n-1}) \ .$$

This is not hard once you have gotten this far!

SOLUTION The formula $(*)$ shows that with coordinates $\{\theta,\phi_1,\dots,\phi_{n-1}\}$ for S^n, $\theta \in (0,2\pi)$, $\phi_j \in (0,\pi)$ for all j,

$$\mathrm{area}(S^n) = \int_0^\pi \sin^{n-1}\phi_{n-1}\mathrm{d}\phi_{n-1}\mathrm{area}(S^n) \ .$$

Since $0 < \sin(t) < 1$ for all $t \in (0,\pi)$, it is evident that

$$\int_0^\pi \sin^{n-1}\phi_{n-1}\mathrm{d}\phi_{n-1}$$

is monotone decreasing in n. Taking $n = 7$, we find

$$\int_0^\pi \sin^6 t \mathrm{d}t = \frac{5}{16}\pi = 0.981747704...$$

while

$$\int_0^\pi \sin^5 t \mathrm{d}t = \frac{16}{15} \ .$$

Hence the area of S^n is increasing as a function of n up until it reaches its maximum at $n = 6$, where the values is exactly π^3, and then it decreases rapidly from there. Indeed,

$$\int_0^\pi \sin^{26} t \mathrm{d}t < \frac{1}{2} \ ,$$

(and this is the smallest such power), so that for all $n > 27$, the area of S^n is-less than half the area of s^{n-1}, and therefore it goes to zero at least exponentially fast.
