Challenge Problem Set 3, Math 291 Fall 2017

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November 5, 2017

This challenge problem set illustrates "realistic" use of Lagrange Multiplies and Newton's Method to solve an interesting optimization problem. It can be done with efficiently a calculator, but suggestions for using Maple to do the Newton iteration will also be provided in class.

The problem to be investigated concerns the effect of friction on ballistic motion: Suppose you are standing at the origin in $\mathbb{R}^3$ and throw a ball with an initial velocity $\mathbf{v}_0$ that we assume to be of the form

$$\mathbf{v}_0 = v_0(\cos\theta, 0, \sin\theta) \tag{0.1}$$

for some angle θ with $0 \le \theta \le \pi/2$. That is, the ball is thrown up and out, along the direction of the positive x-axis.

If you can throw the ball at a maximum speed of v_0 meters per second, what angle should you choose so that the ball travels the farthest before hitting the ground? If friction is neglected, the answer is $\theta = \pi/4$ no matter what v_0 is.

Now let us take friction into account. There are various ways to model the effects of friction, depending on the problem at hand, but one way that is suitable for the present problem is to suppose that the frictive force has the direction opposite that of the velocity, with a magnitude proportional to the magnitude of the velocity. That is, friction decelerates, and the greater the velocity, the stronger the frictive force. Thus, for some constant $\alpha > 0$, we suppose the frictive force to be

$$\mathbf{F}_{\text{friction}} = -\alpha\mathbf{v}$$

while the gravitational force is

$$\mathbf{F}_{\text{gravity}} = -mg(0,0,1)\ .$$

Newton's second law then gives

$$m\mathbf{v}'(t) = \mathbf{F}_{\text{friction}} + \mathbf{F}_{\text{gravity}} = -\alpha\mathbf{v}(t) - mg(0,0,1)\ .$$

Let us work in MKS (meter, kilograms and seconds) units. Then we take $g = 9.8m/s^2$. Let us take $m = 0.25kg$, and $\alpha = 0.0125kg/s$. Finally, in (0.1) let us take $v_0 = 40m/s$, which is a bit under 90 miles an hour.

By Newton's Second Law, $\mathbf{v}' = \frac{1}{m}\mathbf{F}$ then becomes

$$\mathbf{v}'(t) + \frac{1}{20}\mathbf{v}(t) = -9.8(0,0,1)\ . \tag{0.2}$$

1: Multiply both sides of (0.2) by $e^{t/20}$, to obtain

$$e^{t/20}\left(\mathbf{v}'(t)+\frac{1}{20}\mathbf{v}(t)\right) = -e^{t/20}9.8(0,0,1)\ . \tag{0.3}$$

Note that the left hand side is the derivative of $e^{t/20}\mathbf{v}(t)$. Apply the Fundamental Theorem of Calculus; integrate both sides to find an explicit formula for $\mathbf{v}(t)$ in terms of the initial data $\mathbf{v}_0$ as given by (0.1), with $v_0 = 40m/s$.

2: Integrate $\mathbf{v}(t) = \mathbf{x}'(t)$ to find and explicit formula for $\mathbf{x}(t)$, recalling that we have chosen $\mathbf{x}(0) = (0,0,0)$ and $\mathbf{v}(0) = 3(\cos\theta, 0, \sin\theta)$ The position at time t also depends on θ, the angle at which the ball is thrown. So you will get a function of two variables, t, and θ: $\mathbf{x}(t,\theta) = (x(t,\theta), 0, z(t,\theta))$.

Show that

$$x(t,\theta) = 800(1-e^{-t/20})\cos\theta\ ,$$

and that

$$z(t) = 20(1-e^{-t/20})(196+40\sin\theta) - 196t\ .$$

The ball hits the ground when $z(t,\theta) = 0$. We want to choose θ in (0.1) to maximize the distance the ball travels before it hits the ground.

• *That is, we seek to maximize $x(t,\theta)$ subject to the constraint $z(t,\theta) = 0$. This is a Lagrange multipliers problem, and we know how to solve it. Let us do so.*

3: Using Lagrange's Theorem, write down the system of equations to be solved to find the maximizers of $x(t,\theta)$ subject to $y(t,\theta) = 0$, using the equivalent formulation of $y(t,\theta) = 0$ from the previous problem. (Note that in this problem, the independent variables are t and θ, while the dependent variables are x and y, not f and g.)

4: We know the optimal t and θ for the zero friction case. (With zero friction $\theta = \pi/4$, and in the text it is explained how to find the time t when the ball hits the ground given the initial velcoity.) Use these as a starting point, and use Newton's method to compute the optimal angle to three decimal places of accuracy.