Challenge Problem Set 2, Math 291 Fall 2017

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This challenge problem set concerns the generalization of the Frenet-Serret formulas to higher dimension.

0.1 Gram-Schimdt orthonormalization and the Frenet-Seret basis

Consider a smooth curve $\mathbf{x}(t) \in \mathbb{R}^3$; i.e., a curve with at least 3 continuous derivatives. The Frenet-Seret basis $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ is what you get, up to the sign of $\mathbf{B}(t)$ by applying the Gram-Schmidt Algorithm to $\{\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}'''(t)\}$.

Indeed, by definition $\mathbf{T}(t) = \|\mathbf{x}'(t)\|^{-1}\mathbf{x}'(t)$, and $\mathbf{N}(t) = \|\mathbf{x}''(t)_\perp\|^{-1}\mathbf{x}''(t)_\perp$, and these are the first two vectors produced in applying the Gram-Schmidt Algorithm to $\{\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}'''(t)\}$. The third one will be $\pm\mathbf{B}(t)$, where the sign is given by the sign of $\mathbf{x}'(t) \times \mathbf{x}''(t) \cdot \mathbf{x}'''(t)$.

Let $\mathbf{x}(t)$ be a thrice differentiable parametrized curve in $\mathbb{R}^3$. Then for each t we have a natural set of three vectors associated to the motion describe by the parametrized curve, namely $\{\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}'''(t)\}$.

Exercise 1: Consider the helix

$$\mathbf{x}(t) := (r\cos t, r\sin t, bt)$$

where $r > 0$ but where b may be either positive or negative. Show that the Gram-Schmidt orthonormalization procedure applied to $\mathbf{x}'(t)$, $\mathbf{x}''(t)$ and $\mathbf{x}'''(t)$ yields $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ if $b > 0$, and yields $\{\mathbf{T}(t), \mathbf{N}(t), -\mathbf{B}(t)\}$ if $b < 0$.

0.2 Differentiating time dependent orthonormal bases

Let $\mathbf{u}_j(t)$ be continuously differentiable in $\mathbb{R}^3$ on (a,b) for $j = 1,2,3$. Suppose that $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is orthonormal for each t.

Since $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is an orthonormal basis, we may expand $\mathbf{u}_i'(t)$ as a linear combination of these basis vectors for each $i = 1,2,3$:

$$\mathbf{u}_i'(t) = \sum_{j=1}^{3}[\mathbf{u}_i'(t) \cdot \mathbf{u}_j(t)]\mathbf{u}_j(t)\ .$$

There is a convenient way to express this conclusion: First, define

$$A_{i,j}(t) := \mathbf{u}_i'(t) \cdot \mathbf{u}_j(t) \qquad \text{for } 1 \le i,j \le 3\ . \tag{0.1}$$

We can then arrange the 9 functions $A_{i,j}$, $1 \le i,j \le 3$ in a *matrix*; i.e., a rectangular array in which we place $A_{i,j}$ in the ith row and the jth column. Call this matrix $A(t)$. By definition we have

$$A(t) := \begin{bmatrix} A_{1,1}(t) & A_{1,2}(t) & A_{1,3}(t) \\ A_{2,1}(t) & A_{2,2}(t) & A_{2,3}(t) \\ A_{3,1}(t) & A_{3,2}(t) & A_{3,3}(t) \end{bmatrix}\ .$$

Then we can write

$$\mathbf{u}_i'(t) = \sum_{j=1}^{3} A_{i,j}(t)\mathbf{u}_j(t) \qquad j = 1,2,3\ .$$

It turns out that the matrix $A(t)$ has a relatively simple form:

$$A(t) = \begin{bmatrix} 0 & A_{1,2}(t) & A_{1,3}(t) \\ -A_{1,2}(t) & 0 & A_{2,3}(t) \\ -A_{1,3}(t) & -A_{2,3}(t) & 0 \end{bmatrix}\ . \tag{0.2}$$

That is, all of the diagonal elements are zero, and the matrix is *antisymmetric*, meaning that

$$A_{i,j}(t) = -A_{j,i}(t) \qquad \text{for all } 1 \le i,j \le 3\ . \tag{0.3}$$

Exercise 2: Since $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is orthonormal for all $t \in (a,b)$,

$$\mathbf{u}_i(t) \cdot \mathbf{u}_j(t)$$

is constant: The value is 1 if $i = j$ and 0 if $i \ne j$.

Therefore

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{u}_i(t) \cdot \mathbf{u}_j(t)]\ .$$

Use this to prove (0.3) and then (0.2) for all $t \in (a,b)$.

So far, the time dependent orthonormal basis $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ was arbitrary, apart from the requirement that each $\mathbf{u}_i(t)$ be differentiable.

Now let us suppose we are given a thrice continuously differentiable curve $\mathbf{x}(t)$ defined on (a,b), and let us suppose that that $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is obtained by applying the Gram-Schmidt Algorithm to $\{\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}'''(t)\}$ for all $t \in (a,b)$, In particular, we are assuming that for no $t \in (a,b)$ do all three vectors in $\{\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}'''(t)\}$ lie in a common plane through the origin. In this case, there is further simplification:

The way the Gram-Schmidt Algorithm works,

$$\begin{aligned} \mathbf{u}_1(t) &= a_1(t)\mathbf{x}'(t) \\ \mathbf{u}_2(t) &= b_2(t)\mathbf{x}'(t) + a_2(t)\mathbf{x}''(t) \\ \mathbf{u}_3(t) &= c_3(t)\mathbf{x}'(t) + b_3(t)\mathbf{x}''(t) = a_3(t)\mathbf{x}'''(t) \end{aligned}$$

where $a_1(t) > 0$, $a_2(t) > 0$ and $a_3(t) > 0$. This is part *(3)* of Theorem 16, which gives the key properties of the Gram-Schmidt Algorithm.

Exercise 3: Show that when $\{\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t)\}$ is obtained by applying the Gram-Schmidt Algorithm to $\{\mathbf{x}'(t), \mathbf{x}''(t), \mathbf{x}'''(t)\}$, then

$$\mathbf{u}_1'(t)\cdot\mathbf{u}_2(t) > 0 \ , \quad \mathbf{u}_2'(t)\cdot\mathbf{u}_3(t) > 0 \quad \text{and} \quad \mathbf{u}_1'(t)\cdot\mathbf{u}_3(t) = 0$$

for all t.

Now define

$$\kappa_1(t) := \frac{1}{v(t)}A_{1,2}(t) \qquad \text{and} \qquad \kappa_2(t) := \frac{1}{v(t)}A_{2,3}(t) \ ,$$

By the previous exercise, $\kappa_1(t) > 0$ and $\kappa_2(t) > 0$ for all t, and we have

$$A(t) = v(t)\begin{bmatrix} 0 & \kappa_1(t) & 0 \\ -\kappa_1(t) & 0 & \kappa_2(t) \\ 0 & -\kappa_2(t) & 0 \end{bmatrix} \ ,$$

and therefore that

$$\begin{aligned} \mathbf{u}_1'(t) &= v(t)\kappa_1(t)\mathbf{u}_2(t) \\ \mathbf{u}_2'(t) &= -v(t)\kappa_1(t)\mathbf{u}_1(t) + v(t)\kappa_2(t)\mathbf{u}_3(t) \\ \mathbf{u}_3'(t) &= -v(t)\kappa_2(t)\mathbf{u}_2(t) \ . \end{aligned}$$

As we have seen above, if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is right handed, then

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\} \ ,$$

and (0.4) reduces to the usual Frenet-Serret equations with $\kappa_1(t) = \kappa(t)$ and $\kappa_2(t) = \tau_2$.

However, if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is left handed, then

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{T}(t), \mathbf{N}(t), -\mathbf{B}(t)\} \ ,$$

and in this case, comparison between (0.4) and the usual Frenet-Serret equations shows that $\kappa_1(t) = \kappa(t)$ and $\kappa_2(t) = -\tau(t)$.

Thus, if we define

$$\tau(t) = \mathbf{u}_2'(t)\cdot\mathbf{u}_3(t) \ ,$$

$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ satisfies the Frenet-Serret equations *whether or not* the basis produced by the Gram-Schmidt Algorithm is right handed. However, if the basis is left handed, τ will have the opposite sign from the usual definition in which we give preference to right handed bases.

0.3 Higher dimensions

We have not used any cross products at any point in our analysis so far. It has all been based on the dot product, and thus it extends to arbitrary dimensions.

Exercise 4: Let $\mathbf{x}(t)$ be an n times continuously differentiable curve in $\mathbb{R}^n$ defined on (a, b). Suppose that on this interval, the Gram-Schmidt orthonormalization procedure produces an orthonormal basis $\{\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)\}$ out of $\{\mathbf{x}'(t), \ldots, \mathbf{x}^{(n)}(t)\}$. Define

$$A_{i,j} = \mathbf{u}_i'(t) \cdot \mathbf{u}_j(t) \qquad \text{for } 1 \leq i, j \leq n \ .$$

Show that for all t,

(1) $A_{i,j}(t) = -A_{j,i}(t)$ for all $1 \leq i, j \leq n$,

(2) $A_{i,i+1}(t) > 0$ for all $1 \leq i \leq n-1$
and

(3) $A_{i,j}(t) = 0$ for all $j > i + 1$.

Then show that if $A(t)$ is the $n \times n$ matrix whose entry in the ith row and jth column is $A_{i,j}(t)$,

$$A(t) = \begin{bmatrix} 0 & A_{1,2}(t) & 0 & 0 & 0 & \cdots \\ -A_{1,2}(t) & 0 & A_{2,3} & 0 & 0 & \cdots \\ 0 & -A_{2,3}(t) & 0 & A_{3,4}(t) & 0 & \cdots \\ 0 & 0 & -A_{3,4}(t) & 0 & A_{4,5}(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

Exercise 5: Continuing with the previous exercise, define the $n - 1$ *generalized curvatures* κ_j, $j = 1, \ldots, n-1$ by

$$\kappa_j(t) = \frac{1}{v(t)} \mathbf{u}_j'(t) \cdot \mathbf{u}_{j+1}(t) > 0 \ ,$$

where $v(t) = \|\mathbf{x}'(t)\|$. Define

$$\kappa_0(t) = \kappa_n(t) = 0 \ .$$

Likewise define $\mathbf{u}_0(t) = \mathbf{u}_{n+1}(t) = \mathbf{0}$.

Show that for $i = 1, \ldots n$,

$$\mathbf{u}_i'(t) = v(t)[-\kappa_{i-1}(t)\mathbf{u}_{i-1}(t) + \kappa_i(t)\mathbf{u}_{i+1}(t)] \ . \tag{0.4}$$

These are the *general Frenet-Seret formulas.*

In particular, when $n = 4$, we have 3 generalized curvatures, and

$$A(t) = v(t) \begin{bmatrix} 0 & \kappa_1(t) & 0 & 0 \\ -\kappa_1(t) & 0 & \kappa_2(t) & 0 \\ 0 & -\kappa_2(t) & 0 & \kappa_3(t) \\ 0 & 0 & -\kappa_3(t) & 0 \end{bmatrix}$$

Exercise 6: Consider the 4-dimensional curve

$$\mathbf{x}(t) = (\, \sin(2t) + 4\sin(t) \ , \ \sqrt{2}\cos(2t) + 2\sqrt{2}\cos(t) \ , \ -\sqrt{2}\sin(2t) + 2\sqrt{2}\sin(t) \ , \ -\cos(2t) + 4\cos(t) \,) \ .$$

Compute the generalized curvatures $\kappa_1(t)$, $\kappa_2(t)$ and $\kappa_3(t)$. You will find that these are constant in time and non-zero. Hence this curve is an example of the 4-dimensional analog of a helix. Notice however, that unlike a helix in 3 dimensions with non-zero torsion, this curve is bounded, In fact, $\|\mathbf{x}(t)\|^2 = 27$ for all t, so that this is also a spherical curve in 4 dimensions.