Challenge Problem Set 1, Math 291 Fall 2017

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This challenge problem set concerns the Gram-Schmidt algorithm and finding the distance between lines and planes in five dimensional space.

0.1 Applying the Gram-Schmidt Algorithm in $\mathbb{R}^5$

Define the vectors

$$\begin{aligned}
\mathbf{v}_1 &= (1,2,0,2,0) \\
\mathbf{v}_2 &= (2,1,1,1,1) \\
\mathbf{v}_3 &= (0,1,-1,1,-1) \\
\mathbf{v}_4 &= (-1,-1,0,-3,0) \\
\mathbf{v}_5 &= (1,2,1,2,-1)
\end{aligned}$$

Exercise 1: Apply the Gram-Schmidt Algorithm to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ in this order to produce an orthonormal set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$. (You will find that all 5 vectors are pivotal.)

Exercise 2: Let $\mathbf{x}_0 = (1,1,1,0,0)$ and consider the plane parameterized by

$$\mathbf{x}_1(s,t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2 \ .$$

Show that this plane is exactly the solution set of the system of equations

$$\begin{aligned}
\mathbf{u}_3 \cdot (\mathbf{x} - \mathbf{x}_0) &= 0 \qquad (0.1) \\
\mathbf{u}_4 \cdot (\mathbf{x} - \mathbf{x}_0) &= 0 \qquad (0.2) \\
\mathbf{u}_5 \cdot (\mathbf{x} - \mathbf{x}_0) &= 0 \qquad (0.3)
\end{aligned}$$

Also, show that an alternative parameterization of this same plane is given by

$$\widetilde{\mathbf{x}}_1(s,t) = \mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 \ .$$

We now consider another plane in $\mathbb{R}^5$, this time the one parameterized by

$$\mathbf{x}_2(u, v) = u\mathbf{v}_3 + v\mathbf{v}_4 \ .$$

(Notice that this plane passes through $\mathbf{0}$; it is a two dimensional subspace of $\mathbb{R}^5$.

We wish to find the distance between this plane and the plane considered in Exercise 2, and we wish to find the pairs of the closest points, if any. It will be convenient to use the second parameterization of the plane from Exercise 2; that is, the one in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$. Then we seek to find parameter values s, t, u and v that minimize

$$\|\widetilde{\mathbf{x}}_1(s,t) - \mathbf{x}_2(u,v)\|^2 = \|\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4\|^2 \ .$$

By the Pythagorean Theorem,

$$\|\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4\|^2 = \sum_{j=1}^{5}((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4) \cdot \mathbf{u}_j)^2 \ .$$

Exercise 3: Show that $((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4) \cdot \mathbf{u}_5)^2$ is independent of s, t, u, v, and compute its value.

Exercise 4: Show that $((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4) \cdot \mathbf{u}_4)^2$ depends only on v, and there is a choice v_0 of this parameter such that $((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v_0\mathbf{v}_4) \cdot \mathbf{u}_4)^2 = 0$.

Exercise 5: Show that $((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4) \cdot \mathbf{u}_3)^2$ depends only on u and v, and there is a choice u_0 of u, so that with $v = v_0$ from the previous exercise, $((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u_0\mathbf{v}_3 - v_0\mathbf{v}_4) \cdot \mathbf{u}_3)^2 = 0$.

Exercise 6: Show that $((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4) \cdot \mathbf{u}_2)^2$ depends only on t, u and v, and there is a choice t_0 of t, so that with $u = u_0$ and $v = v_0$ from Exercise 5, $((\mathbf{x}_0 + s\mathbf{u}_1 + t_0\mathbf{u}_2 - u_0\mathbf{v}_3 - v_0\mathbf{v}_4) \cdot \mathbf{u}_2)^2 = 0$.

Exercise 7: Show that $((\mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2 - u\mathbf{v}_3 - v\mathbf{v}_4) \cdot \mathbf{u}_1)^2$ depends only on s, u and v, and there is a choice s_0 of s, so that with $u = u_0$ and $v = v_0$ from Exercise 5, $((\mathbf{x}_0 + s_0\mathbf{u}_1 + t\mathbf{u}_2 - u_0\mathbf{v}_3 - v_0\mathbf{v}_4) \cdot \mathbf{u}_1)^2 = 0$.

Exercise 8: Find the distance between the two planes, and the points $\mathbf{p}$ on the first plane and $\mathbf{q}$ on the second plane such that $\|\mathbf{p} - \mathbf{q}\|$ is equal to the distance.

In the final exercise, we wish to find a system of equations for the second plane. One way to do this is to find an orthonormal basis $\{\widetilde{\mathbf{u}_1}, \widetilde{\mathbf{u}_2}, \widetilde{\mathbf{u}_3}, \widetilde{\mathbf{u}_1}, \widetilde{\mathbf{u}_4}\}$ in which the span of the first two vectors is the same as $\mathrm{Span}(\{\mathbf{v}_3, \mathbf{v}_4\})$, which is the plane in question. Once we have this, we can proceed just as in Exercise 2.

Exercise 9: Apply the Gram-Schmidt Algorithm to $\{\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$ in this order to produce an orthonormal set $\{\widetilde{\mathbf{u}_1}, \widetilde{\mathbf{u}_2}, \widetilde{\mathbf{u}_3}, \widetilde{\mathbf{u}_1}, \widetilde{\mathbf{u}_4}\}$. (This one is slightly messier than the one in Exercise 1.) Then find a system of equations for $\mathrm{Span}(\{\mathbf{v}_3, \mathbf{v}_4\})$.
