Solutions for the Odd Numbered Exercises 1-15 from Chapter 1

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1.1 Let $\mathbf{a} = (3, -1)$, $\mathbf{b} = (2, 1)$ and $\mathbf{c} = (1, 3)$. Express $\mathbf{a}$ as a linear combination of $\mathbf{b}$ and $\mathbf{c}$. That is, find numbers s and t so that $\mathbf{a} = s\mathbf{b} + t\mathbf{c}$.

SOLUTION: The equation $s\mathbf{b} + t\mathbf{c} = \mathbf{a}$ is equivalent to

$$\begin{aligned} 2s + t &= 3 \\ s + 3t &= -1 \end{aligned}$$

Subtracting twice the second equation form the first, we get $-5t = 5$ so $t = -1$. Then the first equation becomes $2s = 4$, so $s = 2$. Thus, $\mathbf{a} = 2\mathbf{b} - \mathbf{c}$.

The same procedure gives $\mathbf{b} = (1/2)\mathbf{a} + (1/2)\mathbf{c}$ and $\mathbf{c} = -\mathbf{a} + 2\mathbf{b}$.

SECOND SOLUTION: Here is another approach: Notice that for any vector $(a, b) \in \mathbb{R}^2$, the vector $(-b, a)$ is orthogonal to (a, b):

$$(-b, a)(a, b) = -ba + ab = 0 \ .$$

Let us denote this perpendicular vector by $(a, b)^{\perp}$. (Notice the the "perp" symbol is "upstairs" here; do not confuse this with the notation used to describe parallel and orthogonal components, $\mathbf{x}_{\|}$ and $\mathbf{x}_{\perp}$.)

Here is what this is good for: If $\mathbf{a} = s\mathbf{b} + t\mathbf{c}$, then taking the dot product of both sides with $\mathbf{b}^{\perp}$, and distributing, we get

$$\mathbf{a} \cdot \mathbf{b}^{\perp} = s\mathbf{b} \cdot \mathbf{b}^{\perp} + t\mathbf{c} \cdot \mathbf{b}^{\perp} \ .$$

The multiple of s is zero by orthogonality, and so s is eliminated, and we have

$$t = \frac{\mathbf{a} \cdot \mathbf{b}^{\perp}}{\mathbf{c} \cdot \mathbf{b}^{\perp}} \ ,$$

provided of course that the denominator is not zero. (Otherwise a solution will not exist unless $\mathbf{a}$ is a multiple of $\mathbf{b}$ so that the numerator is also zero.)

In this example, we find

$$t = \frac{(3, -1) \cdot (-1, 2)}{(1, 3) \cdot (-1, 2)} = \frac{-5}{5} = -1 \ .$$

The same approach can be used to find s,

Both solutions involve eliminating variables. Solving problems in algebra is largely a matter of eliminating variables. Orthogonality is important to us for many reasons, but one is that it helps us eliminate variables and cut through algebraic problems.

1.3 Let $\mathbf{x} = (1, 4, 8)$ and $\mathbf{y} = (1, 2, -2)$. Compute the lengths of each of these vectors, and the angle between them.

SOLUTION:

We compute

$$\mathbf{x} \cdot \mathbf{y} = -7 \qquad \mathbf{x} \cdot \mathbf{x} = 81 \quad \text{and} \quad \mathbf{y} \cdot \mathbf{y} = 9 \ .$$

Thus, $\|\mathbf{x}\| = 9$ and $\|\mathbf{y}\| = 3$. Then θ, the angle between the $\mathbf{x}$ and $\mathbf{y}$ is

$$\theta = \arccos\left(\frac{-7}{27}\right) \approx 1.833 \text{ radians} \approx 105.02^\circ$$

1.5 Let $\mathbf{x} = (4, 7, 4)$ and $\mathbf{y} = (2, 1, 2)$. Compute the lengths of each of these vectors, and the angle between them.

SOLUTION:

We compute

$$\mathbf{x} \cdot \mathbf{y} = 23 \qquad \mathbf{x} \cdot \mathbf{x} = 81 \quad \text{and} \quad \mathbf{y} \cdot \mathbf{y} = 9 \ .$$

Thus, $\|\mathbf{x}\| = 9$ and $\|\mathbf{y}\| = 3$. Then θ, the angle between the $\mathbf{x}$ and $\mathbf{y}$ is

$$\theta = \arccos\left(\frac{23}{27}\right) \approx 0.551 \text{ radians} \approx 31.58^\circ$$

1.7 Let

$$\mathbf{u}_1 = \frac{1}{9}(1, -4, -8) \qquad \mathbf{u}_2 = \frac{1}{9}(8, 4, -1) \quad \text{and} \quad \mathbf{u}_3 = \frac{1}{9}(4, -7, 4) \ .$$

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Is it a right-handed orthonormal basis? Justify your answer.

(b) Find numbers y_1, y_2 and y_3 such that

$$y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + y_3\mathbf{u}_3 = (10, 11, -11) \ .$$

What are the lengths of the vectors $(10, 11, -11)$ and (y_1, y_2, y_3)? give calculations or an explanation in each case.

SOLUTION: We first compute the lengths of the vectors. Since $(1, -4, -8) \cdot (1, -4, -8) = 81 = 9^2$, $\mathbf{u}_1 \cdot \mathbf{u}_1 = 1$. So $\mathbf{u}_1$ is a unit vector. Since $\mathbf{u}_2$ has the same entries, up to signs and order, it too is a unit vector. Finally, $(4, -7, 4) \cdot (4, -7, 4) = 81$ again, so $\mathbf{u}_3 \cdot \mathbf{u}_3 = 1$, and $\mathbf{u}_3$ is a unit vector. This takes care of the "normal part"; all three vectors are unit vectors.

Now for the "ortho" part: We have to check whether the three vectors are mutually orthogonal. Here is a good way to do this we compute

$$(1, -4, -8) \times (8, 4, -1) = (36, -63, 36) \ .$$

Therefore,

$$\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3$$

Then from $\|\mathbf{u}_1 \times \mathbf{u}_2\| = \|\mathbf{u}_1\|\|\mathbf{u}_2\| \sin\theta$ and the fact that all three vectors are unit vectors we see $\sin\theta = 1$, so $\theta = \pi/2$, and $\mathbf{u}_1$ is orthogonal to $\mathbf{u}_2$. But since the cross product of two vectors is orthogonal to both vectors, $\mathbf{u}_3$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. This takes care of the "ortho" part, and so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. Finally, since $\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{u}_3$, it is right handed.

Note that in this approach we avoided explicitly working out $\mathbf{u}_1 \cdot \mathbf{u}_2$, $\mathbf{u}_1 \cdot \mathbf{u}_3$ and $\mathbf{u}_2 \cdot \mathbf{u}_3$, but you also could have taken care of the "ortho" part by computing each of these and showing they are all zero. But to answer the final part you need to compute $\mathbf{u}_1 \times \mathbf{u}_2$ anyway, and it is worth noticing that once you know the three vectors are unit vectors, this one computation tells you everything else.

For part **(b)** we use the facts that since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$, for any $\mathbf{x}$ in $\mathbb{R}^3$,

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x} \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{x} \cdot \mathbf{u}_3)\mathbf{u}_3 \qquad \text{and} \qquad \|y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + y_3\mathbf{u}_3\|^2 = y_1^2 + y_2^2 + y_3^2 \ .$$

So:

$$\begin{aligned}
y_1 &= \frac{1}{9}(10 - 44 + 88) = 6 \\
y_2 &= \frac{1}{9}(80 + 44 + 11) = 15 \\
y_3 &= \frac{1}{9}(40 - 77 - 44) = -9
\end{aligned}$$

By the second fact recalled above,

$$\|(10, 11, -11)\| = \|(y_1, y_2, y_3)\| \ .$$

We can compute either one: Let compute

$$\|(10, 11, -11)\| = \sqrt{100 + 121 + 121} = 3\sqrt{38} = \|(y_1, y_2, y_3)\| \ .$$

1.9 Let $\mathbf{a} = (1, 1, 1)$

(a) Find a vector $\mathbf{x}$ such that

$$\mathbf{x} \times \mathbf{a} = (-7, 2, 5) \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{a} = 0 \ .$$

(b) There is no vector $\mathbf{x}$ such that

$$\mathbf{x} \times \mathbf{a} = (1, 0, 0) \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{a} = 0 \ .$$

Show that no such vector exists.

SOLUTION: Let $\mathbf{x} = \mathbf{x}_\perp + \mathbf{x}_\|$ be the decomposition of $\mathbf{x}$ into its parts orthogonal and parallel to $\mathbf{a}$. Then the equation $\mathbf{x} \cdot \mathbf{a} = 0$ tells us that $\mathbf{x}_\| = \mathbf{0}$.

Next, we use the formula

$$(\mathbf{x} \times \mathbf{a}) \times \mathbf{a} = -\|\mathbf{a}\|^2 \mathbf{x}_\perp \ .$$

Since we are given $\mathbf{x}\times\mathbf{a} = (1,0,0)$, we have

$$(\mathbf{x}\times\mathbf{a})\times\mathbf{a} = (\,-7,2,5)\times(1,1,1) = (\,-3,12,-9)\ .$$

Then since $\|\mathbf{a}\|^2 = 3$, we have

$$\mathbf{x} = \mathbf{x}_\perp = (1,-4,3)\ .$$

This is the unique solution of the system of equations.

The vector $\mathbf{a}\times\mathbf{x}$ is orthogonal to $\mathbf{a}$, and hence $\mathbf{a}\times\mathbf{x} = \mathbf{c}$ cannot possibly have any solution unless $\mathbf{c}$ is orthogonal to $\mathbf{a}$. Since $(1,0,0)$ is not orthogonal to $\mathbf{a}$, there is no solution.

1.11 (a) Let $\mathbf{a}$ and $\mathbf{b}$ be orthogonal vectors. Define a sequence of vectors $\{\mathbf{b}_n\}$ by

$$\mathbf{b}_{n+1} = \mathbf{a}\times\mathbf{b}_n \qquad \text{and} \qquad \mathbf{b}_0 = \mathbf{b}\ .$$

Show that for all positive integers m

$$\mathbf{b}_{2m+2} = (-1)^m\|\mathbf{a}\|^{2m}\mathbf{b}\ .$$

How do you have to adjust the formula if the hypothesis that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal is dropped?

SOLUTION: We use the formula

$$\mathbf{a}\times(\mathbf{a}\times\mathbf{b}) = -\|\mathbf{a}\|^2\mathbf{b}_\perp$$

where $\mathbf{b}_\perp$ is the component of $\mathbf{b}$ orthogonal to $\mathbf{a}$. Since $\mathbf{b}_\perp = \mathbf{b}$ by hypothesis, this tells us that

$$\mathbf{b}_2 = -\|\mathbf{a}\|^2\mathbf{b}$$

. Next,

$$\mathbf{b}_4 = \mathbf{a}\times(\mathbf{a}\times\mathbf{b}_2) = -\|\mathbf{a}\|^2\mathbf{a}\times(\mathbf{a}\times\mathbf{b}) = \|\mathbf{a}\|^4\mathbf{b}\ .$$

Now you see the pattern. To give an inductive proof, we assume that we have shown that

$$\mathbf{b}_{2m} = (-1)^{m-1}\|\mathbf{a}\|^{2m-2}\mathbf{b}\ .$$

Then

$$\mathbf{b}_{2m+2} = \mathbf{a}\times(\mathbf{a}\times\mathbf{b}_{2m}) = (-1)^{m-1}\|\mathbf{a}\|^{2m-2}\mathbf{a}\times(\mathbf{a}\times\mathbf{b}) == (-1)^m\|\mathbf{a}\|^{2m}\mathbf{b}\ .$$

Thus be induction, the formula is valid for all m.

The argument shows that if we drop the hypothesis that $\mathbf{b}$ is orthogonal to $\mathbf{a}$, the formula simply becomes

$$\mathbf{b}_{2m+2} = (-1)^m\|\mathbf{a}\|^{2m}\mathbf{b}_\perp\ .$$

1.13 (a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors in $\mathbb{R}^3$. Show that

$$|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| \le \|\mathbf{a}\|\|\mathbf{b}\|\|\mathbf{c}\|$$

and there is equality if and only if $\left\{\dfrac{1}{\|\mathbf{a}\|}\mathbf{a}, \dfrac{1}{\|\mathbf{b}\|}\mathbf{b}, \dfrac{1}{\|\mathbf{c}\|}\mathbf{c}\right\}$ is orthonormal.

SOLUTION: By the Cauchy-Schwarz inequality,

$$|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| \le \|\mathbf{a}\|\|\mathbf{b}\times\mathbf{c}\|$$

with equality if and only if $\mathbf{a}$ is a multiple of $\mathbf{b}\times\mathbf{c}$, since since $\mathbf{b}\times\mathbf{c}$ is orthogonal to both $\mathbf{b}$ and $\mathbf{c}$, there is equality in this first inequality if and only if $\mathbf{a}$ is orthogonal to both $\mathbf{b}$ and $\mathbf{c}$.

Next,

$$\|\mathbf{b}\times\mathbf{c}\| = \|\mathbf{b}\|\|\mathbf{c}\|\sin\theta \le \|\mathbf{b}\|\|\mathbf{c}\| \ ,$$

with equality exactly when $\theta = \pi/2$; i.e., when $\mathbf{b}$ and $\mathbf{c}$ are orthogonal. Putting the two inequalities together, we have $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| \le \|\mathbf{a}\|\|\mathbf{b}\|\|\mathbf{c}\|$, and we have seen along the way that the inequality is strict unless $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is orthogonal.

1.15 Consider the vector $\mathbf{v} = (1,4,3)$. Find an orthonormal basis of $\mathbb{R}^3$ whose third vector is a multiple of $\mathbf{v}$.

SOLUTION: We normalize $\mathbf{v}$ to get $\mathbf{u}_3$: Since $\|\mathbf{v}\| = \sqrt{26}$,

$$\mathbf{u}_3 = \frac{1}{26}(1,4,3) \ .$$

You could have used minus this vector, bat that is the only other choice,

Next, we have a lot of choice. Let us choose $\mathbf{u}_2$. We must choose it to a unit vector that is orthogonal to $\mathbf{u}-3$. That is, we must find x, y and z so that

$$(x,y,z)\cdot(1,4,3) = 0 \qquad \text{and} \qquad x^2+y^2+z^2 = 1 \ .$$

We can always normalize any non-zero solution of the first equation, so let us pay attention to it. This is one equation in three variables. Let us set $x = 0$. Then $(x,y,z)\cdot(1,4,3) = 0$ reduces to $4y+3z = 0$, we can solve this with $y = -3$ and $z = 4$. So our orthogonal vector is

$$(0,-3,4) \ .$$

Normalizing this, we get

$$\mathbf{u}_2 = \frac{1}{5}(0,-3,4) \ .$$

Finally, we get

$$\mathbf{u}_1 = \mathbf{u}_2\times\mathbf{u}_3 = \frac{1}{5\sqrt{26}}(-25,4,3) \ .$$

Note: There is another approach to finding the second vector $\mathbf{u}_2$*: The get a non-zero vector orthogonal to* $\mathbf{v}$*, and hence to* $\mathbf{u}_3$*, compute*

$$\mathbf{e}_1\times\mathbf{v} = (1,0,0)\times(1,4,3) = (0,-3,4) \ .$$

This is guaranteed to be orthogonal to $\mathbf{v}$ *by the properties of the cross product. The choice of* $\mathbf{e}_1$ *is somewhat arbitrary: Anything that is not a multiple of* $\mathbf{v}$ *would do, and we might as well keep it simple. The choice* $\mathbf{e}_1$ *fits the bill.*