Solutions for Test II, Math 291 Fall 2017

November 28, 2017

1. Let $f(x,y) = x^3y + x^2y + x$.

(a) Find all of the critical points of f. Evaluate the Hessian matrix of f at each of these critical points, and determine where each is a local maximum, a local minimum, a saddle, or undecidable from the Hessian.

(b) Sketch a contour plot of f in the vicinity of each critical point. **Especially here, show the computations that lead to the plot to get credit**.

(c) Find all unit vectors $\mathbf{u}$ in $\mathbb{R}^2$ such that that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f((1,0) + t\mathbf{u})\bigg|_{t=0} = 0 \ .$$

SOLUTION For **(a)** we compute

$$\nabla f(x,y) = (3x^2y + 2xy + 1 \ , \ x^3 + x^2) \ .$$

The critical points are the solutions of the system

$$\begin{aligned} 3x^2y + 2xy + 1 &= 0 \\ x^3 + x^2 &= 0 \ . \end{aligned}$$

The second equation is satisfied if and only if $x = 0$ or $x = -1$. If $x = 0$, the first equation becomes $1 = 0$, and so there is no critical point with $x = 0$. If $x = -1$, the first equation becomes $y + 1 = 0$, and so $y = -1$. Hence there is exactly one critical point, $(-1, -1)$. We next compute that

$$\mathrm{Hess}_f(x,y) = \begin{bmatrix} 6xy + 2y & 3x^2 + 2x \\ 3x^2 + 2x & 0 \end{bmatrix} \ .$$

Let A denote the Hessian evaluated at $(-1, -1)$. We find

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} \ .$$

Then since $\det(A) = -1$, one eigenvalue is positive and one is negative, so $(-1, -1)$ is a saddle point.

For **(b)** we need the eigenvectors an eigenvalues of A. The characteristic polynomial is $\det(A - tI) = t^2 - 4t - 1$, and hence the eigenvalues are $\lambda_1 = 2 + \sqrt{5}$ and $\lambda_2 = 2 - \sqrt{5}$. Then

$$A - \lambda_1 I = \begin{bmatrix} 2-\sqrt{5} & 1 \\ 1 & -2-\sqrt{5} \end{bmatrix} .$$

Therefore $\mathbf{v}_1 = (1, \sqrt{5} - 2)$ is an eigenvector with eigenvalue λ_1, and $\mathbf{v}_2 = (2 - \sqrt{5}, 1)$ is an eigenvector with eigenvalue λ_2. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be the unit vectors obtained by normalizing these vectors:

$$\mathbf{u}_1 = \frac{1}{\sqrt{1 - -2\sqrt{5}}}(1, \sqrt{5} - 2) \qquad \text{and} \qquad \mathbf{u}_2 = \frac{1}{\sqrt{1 - -2\sqrt{5}}}(2 - \sqrt{5}, 1) \ .$$

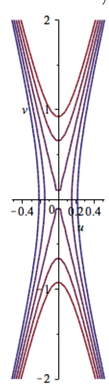
Then writing $\mathbf{x} = u\mathbf{u}_1 + v\mathbf{u}_2$, we have

$$\mathbf{x} \cdot \mathbf{x} = \lambda_1 u^2 + \lambda_2 v^2 = (2 + \sqrt{5})u^2 + (2 - \sqrt{5})v^2 \ .$$

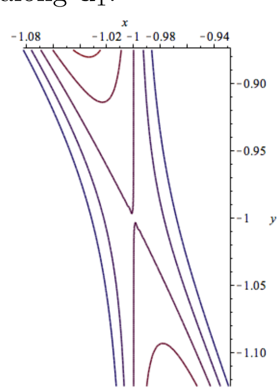
Setting the right hand side equal to any non-zero constant, we get the equation of an hyperbola. Setting the right hand side equal to zero gives us the asymptotes of this hyperbola. These are the lines

$$v = \pm\sqrt{\frac{\sqrt{5}+2}{\sqrt{5}-2}}u \approx \pm 4.24v \ .$$

Here is a plot showing a few contour curves in the u, v plane:



To draw the corresponding contour plot for f at $(-1, -1)$, we transport this plot to the x, y plane and recall that the u axis points along $\mathbf{u}_1$:



For **(c)**, let $B = \text{Hess}_f(1,0)$. We find $B = \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}$. Then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f((1,0) + t\mathbf{u})\bigg|_{t=0} = \mathbf{u} \cdot B\mathbf{u} \ .$$

If $\mathbf{u} = (u, v)$, then the right hand side is $10uv$ and this is zero if and only if either $u = 0$ or $v = 0$. Hence there are 4 such unit vectors, $\pm(0,1)$ and $\pm(1,0)$.

2. Use the method of Lagrange multipliers in both parts of this problem.

(a) Let D be the region consisting of all points (x, y) satisfying

$$x^2 \le y \le 3 + 2x \ .$$

Let $f(x, y) = xy - 3x$. Find the minimum and maximum values of f on D, and find all minimizers and maximizers.

(b) Find the point on the paraboloid

$$z = 3 - \frac{1}{2}((x-1)^2 + (y-1)^2)$$

that is closest to the origin. **Hint:** The cubic polynomial $t^3 - 3t^2 + t + 2$ has a simple integer root.

SOLUTION For **(a)**, we first find the intersection of the bounding parabola and bounding line. At the intersection, $x^2 - 2x = 3$, or $(x-1)^2 = 4$, so that $x = -1$ and $x = 3$ are the two solutions. The points of intersection are then $(-1, 1)$ and $(3, 9)$.

Next, we introduce the constraint equations $g_1(x, y) = 0$ and $g_2(x, y) = 0$ where $g_1(x, y) = y - x^2$ and $g_2(x, y) = 3 + 2x - y$. We then compute

$$\nabla f(x, y) = (y - 3, x) \ , \quad \nabla g_1(x, y) = (-2x, 1) \quad \text{and} \quad \nabla g_2(x, y) = (2, -1) \ .$$

We first look for solutions of Lagrange's equation along the parabola.

$$\det\left(\begin{bmatrix} \nabla f(x, y) \\ \nabla g_1(x, y) \end{bmatrix}\right) = \det\left(\begin{bmatrix} y - 3 & x \\ -2x & 1 \end{bmatrix}\right) = 2x^2 + y - 3 \ .$$

Setting this equal to zero, and using the constraint equation $g_1(x, y) = 0$ to eliminate y, we obtain $x^2 = 1$. Hence there are two solutions of this system on the part of the parabola that bounds D, namely $\mathbf{x}_1 = (-1, 1)$ and $\mathbf{x}_2 = (1, 1)$.

We next look for solutions of Lagrange's equation along the line.

$$\det\left(\begin{bmatrix} \nabla f(x, y) \\ \nabla g_2(x, y) \end{bmatrix}\right) = \det\left(\begin{bmatrix} y - 3 & x \\ 2 & -1 \end{bmatrix}\right) = 3 - y - 2x \ .$$

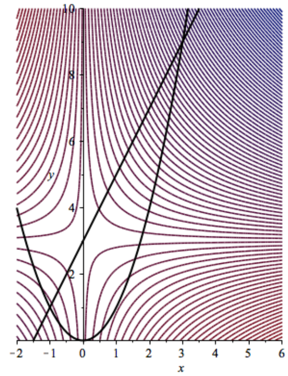
Setting this equal to zero, and using the constraint equation $g_2(x, y) = 0$ to eliminate y, we obtain $x = 0$. Hence there is one solution of this system on the part of the line that bounds D, namely $\mathbf{x}_3 = (0, 3)$.

The critical point of f is $(0, 3)$, but this is $\mathbf{x}_3$ which we have already taken into account. Finally, we must consider the other corner point that is not already on our list, namely $\mathbf{x}_4 = (3, 9)$.

We now compute

$$f(\mathbf{x}_1) = 2 \ , \quad f(\mathbf{x}_2) = -2 \ , \quad f(\mathbf{x}_3) = 0 \ , \quad \text{and} \quad f(\mathbf{x}_4) = 18 \ .$$

Hence the maximum value is 18, and the maximizer is $\mathbf{x}_4 = (3, 9)$, while the minimum value is -2, and the minimizers is $\mathbf{x}_2 = (1, 1)$. Here is a contour plot of f together with the two constraint curves; note that the points of tangency appear where we calculated them to be:



For **(b)**, define $f(x, y, z) = x^2 + y^2 + z^2$, which is the *squared* distance from (x, y, z) to the origin. We also define $g(x, y, z) = z + ((x-1)^2 + (y-1)^2)/2 - 3$ so that the constraint equation is $g(x, y, z) = 0$. By Lagrange's Theorem, the point on the paraboloid that is closest to the origin satisfied $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ for some number λ. This means that $\nabla f(x, y, z) \times \nabla g(x, y, z) = (0, 0, 0)$. Computing the gradients and cross product, we find

$$2(x, y, z) \times (x - 1, y - 1, 1) = (y - z(y - 1), z(x - 1) - x, y - x) = (0, 0, 0) \ .$$

We see that at the minimizer, $y = x$. Using this to eliminate y form the constraint equation, we see that $z + (x - 1)^2 = 3$. Using this to eliminate z from the equation $x = z(x - 1)$, we see that $x = (3 - (x - 1)^2)(x - 1)$. This cubic equation simplifies to

$$x^3 - 3x^2 + x + 2 = 0 \ .$$

It is easy to see that $x = 2$ is a root, and then we factor $x^3 - 3x^2 + x + 2 = (x - 2)(x^2 - x - 1)$ and so the other two roots are $x = (1 + \sqrt{5})/2$ and $x = (1 - \sqrt{5})/2$. We get the corresponding candidate points by substituting these values of x into $y = x$ and $z = 3 - (x = 1)^2$. We find the three candidates:

$$\mathbf{x}_1 = (2, 2, 2) \ , \quad \mathbf{x}_2 = \frac{1}{2}(1 + \sqrt{5}, 1 + \sqrt{5}, 3 + \sqrt{5}) \quad \text{and} \quad \mathbf{x}_3 = \frac{1}{2}(1 - \sqrt{5}, 1 - \sqrt{5}, 3 - \sqrt{5}) \ .$$

We then compute

$$f(\mathbf{x}_1) = 12 \ , \quad f(\mathbf{x}_2) = \frac{1}{2}(13 + 5\sqrt{5}) \quad \text{and} \quad f(\mathbf{x}_3) = \frac{1}{2}(13 - 5\sqrt{5}) \ .$$

Therefore $\mathbf{x}_3$ is the minimizer, and the minimum distance is $\sqrt{f(\mathbf{x}_3)}$, which is

$$\sqrt{\frac{13 - 5\sqrt{5}}{2}} \ .$$

3. (a) Let D be the set in the positive quadrant of $\mathbb{R}^2$ that is bounded by

$$x^2 + y^2 = 4 \qquad \text{and} \qquad xy = 1\ .$$

Let $f(x,y) = \sqrt{1+x^2+y^2}$. Compute $\int_D f(x,y)\mathrm{d}A$.

(b) Let D be the set in $\mathbb{R}^2$ that is given by

$$x^2 \le y \le 2x^2 \qquad \text{and} \qquad x^3 \le y \le 2x^3\ .$$

Let $f(x,y) = \dfrac{x}{y}$. Compute $\int_D f(x,y)\mathrm{d}A$.

SOLUTION For **(a)**, it is simplest to use polar coordinates. In polar variables, the bounding equations are $r = 2$ and $r^2\cos\theta\sin\theta = 1$. Using the first equation to eliminate r from the second, we find $\sin(2\theta) = 1/2$. This has the solutions $2\theta = \pi/6$ and $2\theta = 5\pi/6$. Therefore, D lies in the sector

$$\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}\ .$$

Then since the hyperbola is the inner boundary of D,

$$\sqrt{\frac{2}{\sin(2\theta)}} \le r \le 3\ .$$

The area element in polar coordinates is $r\mathrm{d}r\mathrm{d}\theta$, and so

$$\begin{aligned}\int_D f(x,y)\mathrm{d}A &= \int_{\pi/12}^{5\pi/12}\left(\int_{\sqrt{2/\sin(2\theta)}}^{2} r\sqrt{1+r^2}\mathrm{d}r\right)\mathrm{d}\theta\\ &= \frac{1}{3}\int_{\pi/12}^{5\pi/12}\left(5^{3/2} - \left(1+\frac{2}{\sin(2\theta)}\right)^{3/2}\right)\mathrm{d}\theta\ .\end{aligned}$$

An advantage of polar coordinates is that the factor of r in the area element combines with the integrand $f(r\cos\theta, r\sin\theta) = \sqrt{1+r^2}$ to give the function $r\sqrt{1+r^2}$ with has the simple antiderivative $\frac{1}{3}(1+r^2)^{3/2}$. One can also set up the integral in Cartesian coordinates, but then one does not have this advantage.

To set the integral up in Cartesian coordinates, we first find the intersection of the bounding curve. Using $y = 1/x$ to eliminate y in $x^2+y^2 = 4$, we find $x^4 - 4x^2 = -1$, so that $(x^2-2)^2 = 3$. Thus, $x^2 = 2 \pm \sqrt{3}$. Thus, the region D lies in the strip

$$\sqrt{2-\sqrt{3}} \le x \le \sqrt{2+\sqrt{3}}\ .$$

The hyperbola $xy = 1$ bounds D from below, and the circle $x^2+y^2 = 4$ bounds D from above, and so

$$\frac{1}{x} \le y \le \sqrt{4-x^2}\ .$$

Therefore

$$\int_D f(x,y)\mathrm{d}A = \int_{\sqrt{2-\sqrt{3}}}^{\sqrt{2+\sqrt{3}}}\left(\int_{1/x}^{\sqrt{4-x^2}}\sqrt{1+x^2+y^2}\mathrm{d}y\right)\mathrm{d}x$$

anyone getting to here would get most of the credit. To do the one integration we use that fact that for any $a > 0$, the antiderivative of $\sqrt{a+y^2}$ is

$$\frac{1}{2}\left(y\sqrt{a+y^2} + a\ln(y+\sqrt{a+y^2})\right) \ .$$

Using this with $a = 1 + x^2$, one can do the y integral.

For **(b)**, we introduce new variables $u = y/x^2$ and $v = y/x^3$. Then $x = u/v$, and $y = x^2u = u^3/v^2$. Thus we have

$$\mathbf{x}(u,v) = (u/v, u^3/v) \ .$$

Computing the Jacobian matrix

$$[D_{\mathbf{x}}(u,v)] = \begin{bmatrix} 1/v & -u/v^2 \\ 3u^2/v^2 & -2u^3/v^3 \end{bmatrix} = u^3/v^4 \ .$$

Next, $f(x(u,v), y(u,v)) = (u/v)/(u^3/v^2) = v/u^2$. Therefore,

$$\begin{aligned} \int_D f(x,y)\mathrm{d}A &= \int_1^2 \left(\int_1^1 \frac{u}{v^3}\mathrm{d}u\right)\mathrm{d}v \\ &= \left(\int_1^2 \frac{1}{v^3}\mathrm{d}v\right)\left(\int_1^2 u\mathrm{d}v\right) \\ &= \left(\frac{3}{8}\right)\left(\frac{3}{2}\right) = \frac{9}{16} \ . \end{aligned}$$

4. (a) Let $\mathcal{S}$ be the part of the graph of $z = xy$ that lies above the graph of $z = x^2 + y^2 - 4$. Compute the area of $\mathcal{S}$.

SOLUTION The graphs of $z = xy$ and $z = x^2 + y^2$ meet above the curve in the x, y plane given by $x^2 + y^2 - xy = 4$. Writing this in polar coordinates, we find $r^2(1 - \cos\theta\sin\theta) = 4$, or,

$$r = \sqrt{\frac{8}{2 - \sin(2\theta)}} \ .$$

The surface is then parameterized by

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, r^2\sin\theta\cos\theta) \ .$$

We compute

$$\mathbf{X}_r(r,\theta) = (\cos\theta, \sin\theta, 2r\sin\theta\cos\theta) \qquad \text{and} \qquad \mathbf{X}_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, r^2(\cos^2\theta - \sin^2\theta)) \ .$$

We then compute

$$\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta) = r(-r\sin\theta, -r\cos\theta, 1) \ .$$

Therefore,

$$\mathrm{d}S = \|\mathbf{X}_r \times \mathbf{X}_\theta(r,\theta)\|\mathrm{d}r\mathrm{d}\theta = r\sqrt{1+r^2}\mathrm{d}r\mathrm{d}\theta \ .$$

Hence the surface area is given by

$$\int_0^{2\pi}\left(\int_0^{\sqrt{8/(2-\sin(2\theta))}} r\sqrt{1+r^2}\mathrm{d}r\right)\mathrm{d}\theta$$

Since the antiderivative of $r\sqrt{1+r^2}$ is $\frac{1}{3}(1+r^2)^{3/2}$, this reduces to

$$\frac{1}{3}\int_0^{2\pi}\left(\left(1+\frac{8}{1-\sin(2\theta)}\right)^{3/2}-1\right)\mathrm{d}\theta\ .$$

This is the exact answer. The integral cannot be evaluated in terms of elementary functions, but it is easily evaluated numerically, and the value is 26.8709032....

Extra Credit: The function $f(x,y,z) = 2x^3y^2 - 3x^2y - 3y + x^2 + y^2 + z^2$ has $(1,1,1)$ as one of its critical points. Is this a local minimum, a local maximum, a saddle, on a critical point whose nature cannot be decided by analysis of the Hessian of f? Does the function have either minimum or a maximum on all of $\mathbb{R}^3$?

SOLUTION We compute the Hessian of f finding

$$\mathrm{Hess}_f(x,y,z) = \begin{bmatrix} 12xy^2-6y+2 & 12x^2y-6x & 0 \\ 12x^2y-6x & 4x^3+2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\ .$$

Evaluating this at $(1,1,1)$, we find

$$\mathrm{Hess}_f(1,1,1) = \begin{bmatrix} 8 & 6 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}\ .$$

The determinant of this 3×3 matrix is 24. The determinant of the upper left 2×2 block is 12. The upper left entry is 8. Since all of these are positive, Sylvester's Theorem tells us that all three eigenvalues are positive, and so $(1,1,1)$ is a local minimum.

Notice that for all $x\in\mathbb{R}$, $f(x,x,0) = 2x^5 - 3x^3 + 2x^2 - 3x$. Since the highest power of x is odd, this is unbounded above and below. Hence f does not have either a maximum or am minimum on all of $\mathbb{R}^3$.