*Mathematical Research Letters* **8**, 1-10 (2001)

# INFINITE TOWERS OF TREE LATTICES

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## §0. Introduction

Let $X$ be a locally finite tree and let $G = Aut(X)$. Then $G$ is naturally a locally compact group ([BL], Ch. 3). A discrete subgroup $\Gamma \leq G$ is called an *$X$-lattice* if

$$Vol(\Gamma\backslash\backslash X) \quad := \quad \sum_{x \in V(\Gamma\backslash X)} \frac{1}{|\Gamma_x|} \tag{1}$$

is finite, and a *uniform $X$-lattice* if $\Gamma\backslash X$ is a finite graph, *non-uniform* otherwise ([BL], Ch. 3). Bass and Kulkarni have shown ([BK], (4.10)) that $G = Aut(X)$ contains a uniform $X$-lattice if and only if $X$ is the universal covering of a finite connected graph, or equivalently, that $G$ is unimodular, and $G\backslash X$ is finite. In this case, we call $X$ a *uniform* tree.

Following ([BL], (3.5)) we call $X$ *rigid* if $G$ itself is discrete, and we call $X$ *minimal* if $G$ acts minimally on $X$, that is, there is no proper $G$-invariant subtree. If $X$ is uniform then there is always a unique minimal $G$-invariant subtree $X_0 \subseteq X$ ([BL] (5.7), (5.11), (9.7)). We call $X$ *virtually rigid* if $X_0$ is rigid (*cf.* ([BL], (3.6)).

Let $X$ be a locally finite tree, and let $\Gamma \leq \Gamma'$ be an inclusion of $X$-lattices. Then by ([BL], (1.7)) we have:

$$Vol(\Gamma'\backslash\backslash X) \quad = \quad \frac{Vol(\Gamma\backslash\backslash X)}{[\Gamma' : \Gamma]}. \tag{2}$$

We call an infinite ascending chain

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \dots \tag{3}$$

of $X$-lattices an *infinite tower of $X$-lattices.* By (0.2), the lattice inclusions of (0.3) are of finite index, and $Vol(\Gamma_i\backslash\backslash X) \longrightarrow 0$ as $i \longrightarrow \infty$.

The first author was supported in part by NSF grant #DMS-9800604, and the second by NSF grant #DMS-9810750.

2000 Mathematics subject classification. Primary 20F32; secondary 22F50.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

The Kazhdan-Margulis property for lattices in Lie groups ([KM]) states that the co-volume of a lattice is bounded away from zero. Hence the existence of infinite towers of $X$-lattices in $G = Aut(X)$ shows that the Kazhdan-Margulis property is violated for $X$-lattices.

Bass and Kulkarni have given ([BK], (Sec. 7)) several examples of uniform trees such that $G = Aut(X)$ contains infinite towers of uniform $X$-lattices. The second author has extended the results and techniques of Bass-Kulkarni to all uniform trees that are not rigid ([R]).

Here our main result is that, with one exception (see §5), if $G = Aut(X)$ contains a non-uniform $X$-lattice, then $G$ contains an infinite tower of non-uniform $X$-lattices.

The authors would like to thank H. Bass for many helpful discussions and suggestions.

## §*1. The setting*

An *edge-indexed graph* $(A, i)$ consists of an underlying graph $A$ , and an assignment of a positive integer $i(e) > 0$ to each oriented edge $e \in EA$. Our underlying graph $A$ will always be understood to be locally finite. In [BK] and [BL] one allows $i(e)$ to be any positive cardinal, but our interest here is only in finite $i(e)$. If $i(e) > 1$, we call $e$ *ramified* and *unramified* otherwise.

Let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups, with underlying graph $A$, vertex groups $(\mathcal{A}_a)_{a\in VA}$, edge groups $(\mathcal{A}_e = \mathcal{A}_{\overline{e}})_{e\in EA}$ and monomorphisms $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}$. A graph of groups $\mathbb{A}$ naturally gives rise to an edge-indexed graph $I(\mathbb{A}) = (A, i)$ whose indices are the indices of the edge-groups as subgroups of the adjacent vertex groups: that is, $i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e \mathcal{A}_e]$, which we assume to be finite, for all $e \in EA$.

Given an edge-indexed graph $(A, i)$, a graph of groups $\mathbb{A}$ such that $I(\mathbb{A}) = (A, i)$, is called a *grouping* of $(A, i)$. We call $\mathbb{A}$ a *finite grouping* if the vertex groups $\mathcal{A}_a$ are finite and a *faithful grouping* if $\mathbb{A}$ is a faithful graph of groups, that is if $\pi_1(\mathbb{A}, a)$ acts faithfully on $X = \widetilde{(\mathbb{A}, a)}$.

Let $\mathbb{A}'$ and $\mathbb{A}$ be groupings of $(A, i)$. Then $\mathbb{A}' = (A, \mathcal{A}')$ is called a *full graph of subgroups* of $\mathbb{A} = (A, \mathcal{A})$ (as in ([B], (1.14)) if $\mathcal{A}'_a \leq \mathcal{A}_a$ for $a \in A'$, and for $e \in EA'$, $\mathcal{A}'_e \leq \mathcal{A}_e$, and $\alpha'_e = \alpha_e|_{\mathcal{A}'_e}$. We further assume that for $e \in EA'$, with $\partial_0 e = a$, $\mathcal{A}'_a \cap \alpha_e \mathcal{A}_e = \alpha_e \mathcal{A}'_e$, that is $\mathcal{A}'_a/\alpha_e \mathcal{A}'_e \longrightarrow \mathcal{A}_a/\alpha_e \mathcal{A}_e$ is injective, and hence bijective. This assumption implies that $I(\mathbb{A}') = (A, i)$, and that $\pi_1(\mathbb{A}', a') \leq \pi_1(\mathbb{A}, a)$ ([B], (1.14)).

Let $(A, i)$ be an edge-indexed graph. A *tower of groupings* on $(A, i)$ is a semi-infinite sequence $(\mathbb{A}_i)_{i\in\mathbb{Z}_{>0}}$ of groupings of $(A, i)$ such that each $\mathbb{A}_i$ is a full graph of proper subgroups of $\mathbb{A}_{i+1}$. A tower of faithful groupings induces an infinite ascending chain of fundamental groups:

$$\pi_1(\mathbb{A}_1, a_0) \leq \pi_1(\mathbb{A}_2, a_0) \leq \pi_1(\mathbb{A}_3, a_0) \leq \dots \tag{1}$$

For an edge $e \in EA$, define:

$$\Delta(e) = \frac{i(\overline{e})}{i(e)}. \tag{2}$$

If $\gamma = (e_1, \dots, e_n)$ is a path, set:

$$\Delta(\gamma) = \Delta(e_1)\dots\Delta(e_n).$$

**Definition.** *An indexed graph* $(A, i)$ *is unimodular if* $\Delta(\gamma) = 1$ *for all closed paths* $\gamma$ *in* $A$*.*

Now assume that $(A, i)$ is unimodular. Pick a base point $a_0 \in VA$, and define, for $a \in VA$,

$$N_{a_0}(a) = \frac{\Delta a}{\Delta a_0} (= \Delta(\gamma) \text{ for any path } \gamma \text{ from } a_0 \text{ to } a) \in \mathbb{Q}_{>0}. \tag{3}$$

For $e \in EA$, put

$$N_{a_0}(e) \quad := \quad \frac{N_{a_0}(\partial_0(e))}{i(e)}.$$

Following ([BL], (2.6)), we say that $(A, i)$ has *bounded denominators* if

$$\{N_{a_0}(e) \mid e \in EA\}$$

has bounded denominators, that is, if for some integer $D > 0$, $D \cdot N_{a_0}$ takes only integer values on edges. Since

$$N_{a_1} \quad = \quad \frac{\Delta a_0}{\Delta a_1} N_{a_0},$$

this condition is independant of $a_0 \in VA$.

**Theorem ([BK], (2.4)).** *An indexed graph* $(A, i)$ *admits a finite grouping if and only if* $(A, i)$ *is unimodular and has bounded denominators. The grouping can further be taken to be faithful.*

We define the *volume* of an indexed graph $(A, i)$ at a basepoint $a_0 \in VA$:

$$Vol_{a_0}(A, i) \quad := \quad \sum_{a \in VA} \frac{1}{(\frac{\Delta a}{\Delta a_0})} \quad = \quad \sum_{a \in VA} (\frac{\Delta a_0}{\Delta a}). \tag{4}$$

Then

$$Vol_{a_1}(A, i) \quad = \quad \frac{\Delta a_0}{\Delta a_1} Vol_{a_0}(A, i),$$

as in ([BL], Ch. 2) We write $Vol(A, i) < \infty$ if $Vol_a(A, i) < \infty$ for some, and hence every $a \in VA$.

If $\mathbb{A}$ is a finite grouping of $(A, i)$, then we have ([BL], (2.6.15)):

$$Vol(\mathbb{A}) \quad = \quad \frac{1}{|\mathcal{A}_a|} Vol_a(A, i), \tag{5}$$

which is automatically finite if $Vol(A,i) < \infty$.

We now describe a method for constructing $X$-lattices which follows naturally from the fundamental theory of Bass-Serre ([B], [S]), and was first suggested in ([BK]). We begin with an edge-indexed graph $(A,i)$. Then $(A,i)$ determines $X = \widetilde{(A,i,a_0)}$ up to isomorphism ([BL], Ch. 2).

We say that $(A,i)$ *admits a lattice* if $(A,i)$ admits a grouping $\mathbb{A}$ such that $\pi_1(\mathbb{A},a_0)$ is an $X$-lattice. This happens if and only if $(A,i)$ satisfies:

(U) $(A,i)$ is unimodular, and
(BD) $(A,i)$ has bounded denominators, and
(FV) $(A,i)$ has finite volume.

Assume that $(A,i)$ is unimodular and has bounded denominators (which is automatic if $A$ if finite). By ([BK], (2.4)) we can find a finite faithful grouping $\mathbb{A}$ of $(A,i)$ and a group $\Gamma = \pi_1(\mathbb{A},a_0)$ acting faithfully on $X$. Then

(a) $\Gamma$ is *discrete*, since $\mathbb{A}$ is a graph of *finite* groups.
(b) $\Gamma$ is a *uniform $X$-lattice* if and only if $A$ is finite.
(c) $\Gamma$ is a *non-uniform $X$-lattice* if and only if $A$ is infinite, and

$$(6) \qquad Vol(\Gamma\backslash\backslash X) = Vol(\mathbb{A})(:= \sum_{a\in VA} \frac{1}{|\mathcal{A}_a|} = \frac{1}{|\mathcal{A}_a|} Vol_a(A,i)) < \infty.$$

Our task is the following: *given an edge-indexed graph $(A,i)$ of finite volume, construct an infinite tower of finite faithful groupings of $(A,i)$*. This induces an infinite tower

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \dots$$

of $X$-lattices in $Aut(X)$, for $X = \widetilde{(A,i)}$, with $\Gamma_i\backslash X = A$, $i = 1,2,\dots$.

An edge $e \in EA$ is called *separating* if $A - \{e,\overline{e}\}$ has two connected components $A_0(e)$ and $A_1(e)$, where $A_0(e)$ and $A_1(e)$ contain $\partial_0(e)$ and $\partial_1(e)$ respectively.

Let $(A,i)$ be any connected edge-indexed graph. A subset $\beta \subset EA$ of $n \geq 2$ (oriented) edges is called an *arithmetic bridge* for $(A,i)$ (as in ([C1], Sec. 4)) if:

(1) $\beta \cap \overline{\beta} = \varnothing$, $A - (\beta \cup \overline{\beta})$ has two connected components, $A_0$ and $A_1$,
(2) For every $e \in \beta$, $\partial_0 e \in A_0$ and $\partial_1 e \in A_1$,
(3) There exists an integer $d > 1$ such that $d \mid i(e)$ for every $e \in \beta$.

Following [BT] we say that $(A,i)$ is *discretely ramified* if for $e \in EA$

$$i(e) > 1 \implies i(e) = 2,\ e \text{ is separating, and } (A_0(e),i) \text{ is an unramified tree.}$$

We call $(A,i)$ a *dominant rooted edge-indexed tree* if $A$ is a tree, and there exists an $a \in VA$ such that $i(e) = 1$ for all edges $e \in EA$ directed towards $a$. Let $(A,i)$ be an edge-indexed graph. We say that $(A,i)$ is *restricted* if $(A,i)$ satisfies any one of the following conditions:

(DR) $(A,i)$ is discretely ramified, or
(F) $(A,i)$ is a dominant rooted edge-indexed tree, or
(GS) $A$ is a tree, and $(A,i)$ contains a prime-prime interval (see [R]) and no other ramified edges.

We say that $(A,i)$ is *permissible* if $(A,i)$ admits a lattice and if $(A,i)$ is not restricted. We note that an infinite edge-indexed graph $(A,i)$ with finite volume is automatically non-discretely ramified, is not a dominant rooted tree ([CR2]), is obviously not (GS) as above, and hence is not restricted.

### *§2. Rooted products of graphs of groups*

Given rooted graphs of groups $\mathbb{A} = (A, \mathcal{A}, a_0)$, $a_0 \in VA$, and $\mathbb{B} = (B, \mathcal{B}, b_0)$, $b_0 \in VB$, we construct a rooted graph of groups $\mathbb{C} = (C, \mathcal{C}, c_0) = \mathbb{A} \times_{a_0=b_0} \mathbb{B}$ as follows: we set

$$C \quad := \quad A \sqcup B/(a_0 = b_0 = c_0).$$

For $a \in VA$, $e \in EA$, we set

$$\mathcal{C}_a := \mathcal{A}_a \times \mathcal{B}_{b_0}, \quad \mathcal{C}_e := \mathcal{A}_e \times \mathcal{B}_{b_0},$$

and if $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}$, we set

$$\gamma_e \quad := \quad \alpha_e \times Id_{B_{b_0}}.$$

Similarly, for $b \in VB$, $e \in EB$, we set

$$\mathcal{C}_b := \mathcal{A}_{a_0} \times \mathcal{B}_b, \quad \mathcal{C}_e := \mathcal{A}_{a_0} \times \mathcal{B}_e,$$

and if $\beta_e : \mathcal{B}_e \hookrightarrow \mathcal{B}_{\partial_0 e}$, we set

$$\gamma_e \quad := \quad Id_{A_{a_0}} \times \beta_e.$$

If we set $(A, i^A) = I(\mathbb{A})$ and $(B, i^B) = I(\mathbb{B})$, then we have the 'rooted union of edge-indexed graphs':

$$(C, i^C) \quad := \quad (A, i^A) \sqcup (B, i^B)/(a_0 = b_0 = c_0),$$

for $c_0 \in VC$, and clearly $\mathbb{C}$ is a grouping of $(C, i^C)$.

**(2.1) Remarks.**

(1) The graph of groups $\mathbb{C}$ is faithful if and only if $\mathbb{A}$ and $\mathbb{B}$ are faithful. In fact, if $N_{\mathbb{A}}$ is the maximal normal subgroup of $\mathbb{A}$, and $N_{\mathbb{B}}$ is the maximal normal subgroup of $\mathbb{B}$, then the maximal normal subgroup of $\mathbb{C}$ is $N_{\mathbb{A}} \times N_{\mathbb{B}}$.

(2) We have

$$\pi_1(\mathbb{C}, c_0) \quad = \quad (\pi_1(\mathbb{A}, a_0) \times \mathcal{B}_{b_0}) \quad *_{(\mathcal{A}_{a_0} \times \mathcal{B}_{b_0})} \quad (\mathcal{A}_{a_0} \times \pi_1(\mathbb{B}, b_0)).$$

(3) If $\mathbb{A}$ and $\mathbb{B}$ are graphs of finite groups, then so also is $\mathbb{C}$, and

$$Vol(\mathbb{C}) \quad = \quad \frac{1}{|\mathcal{B}_{b_0}|} Vol(\mathbb{A}) \quad + \quad \frac{1}{|\mathcal{A}_{a_0}|} Vol(\mathbb{B}) \quad - \quad \frac{1}{|\mathcal{A}_{a_0}||\mathcal{B}_{b_0}|}.$$

**(2.2) Functoriality.**

Suppose that we have groupings $\mathbb{A} \leq \mathbb{A}'$ of an edge-indexed graph $(A, i^A)$ and $\mathbb{B} \leq \mathbb{B}'$ of an edge-indexed graph $(B, i^B)$, then we get groupings

$$\mathbb{C} = \mathbb{A} \times_{a_0 = b_0} \mathbb{B} \quad \leq \quad \mathbb{C}' = \mathbb{A}' \times_{a_0 = b_0} \mathbb{B}'$$

for $a_0 \in VA$, $b_0 \in VB$ of the edge-indexed graph $(C, i^C) = (A, i^A) \sqcup (B, i^B)/(a_0 = b_0 = c_0)$, for $c_0 \in VC$. In particular, for an edge $e \in VA$ with initial vertex $a \in VA$,

$$\begin{array}{ccc}
\mathcal{A}'_e \times \mathcal{B}'_{b_0} & \overset{\alpha'_e \times Id_{B'_{b_0}}}{\longrightarrow} & \mathcal{A}'_a \times \mathcal{B}'_{b_0} \\
\leq \quad \leq & & \leq \quad \leq \\
\mathcal{A}_e \times \mathcal{B}_{b_0} & \overset{\alpha_e \times Id_{B_{b_0}}}{\longrightarrow} & \mathcal{A}_a \times \mathcal{B}_{b_0}
\end{array}$$

commutes, and similarly in $B$.

**(2.3) Corollary.** *A tower* $\mathbb{A}_1 \leq \mathbb{A}_2 \leq \mathbb{A}_3 \leq \dots$ *yields a tower*

$$\mathbb{A}_1 \times_{a_0 = b_0} \mathbb{B} \quad \leq \quad \mathbb{A}_2 \times_{a_0 = b_0} \mathbb{B} \quad \leq \quad \mathbb{A}_3 \times_{a_0 = b_0} \mathbb{B} \quad \leq \quad \dots$$

Since a unimodular edge-indexed graph with bounded denominators admits a finite faithful grouping, we can apply the above corollary repeatedly to obtain the following lemma.

**(2.4) Lemma.** *Let $(A, i)$ be an edge-indexed graph and $(A_0, i)$ a core subgraph such that $(A, i)$ is obtained from $(A_0, i)$ by attaching to finitely many vertices $a_1, \dots, a_n \in VA_0$, rooted edge-indexed graphs $(A_j, i_j, a_j)$, $j = 1, \dots, n$. Suppose that $(A_0, i)$ admits an infinite ascending chain of finite faithful groupings of finite volume. Suppose that $(A_j, i_j)$, $j = 1, \dots, n$ are unimodular, have finite volume and bounded denominators.Then $(A, i)$ admits an infinite tower of finite faithful groupings of finite volume.*

## §3. *Infinite towers of uniform tree lattices*

In [R], the second author proved the following:

**(3.1) Theorem ([R]).** *Let $(A, i)$ be a finite permissible edge-indexed graph. Then $(A, i)$ admits an infinite tower of finite faithful groupings.*

The proof of Theorem (3.1) generalizes the techniques of Bass-Kulkarni ([BK]) for constructing towers of groupings on certain fundamental examples, and uses certain constructions with edge-indexed graphs to extend to a more general setting.

Theorem (3.1) yields the following:

**(3.2) Theorem ([R]).** *Let $X$ be a locally finite tree. The following conditions are equivalent:*

(a) *$X$ is uniform and not rigid.*
(b) *$X$ is the universal cover of a finite permissible edge-indexed graph.*
(c) *$Aut(X)$ contains an infinite ascending chain*

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots$$

*of uniform $X$-lattices.*
(d) *The set of uniform covolumes*

$$\{Vol(\Gamma\backslash\backslash X) \mid \Gamma \text{ is a uniform } X\text{-lattice}\} \subset \mathbb{Q}_{>0}$$

*is not bounded away from zero.*

This generalizes Theorem 7.1(a) of [BK] which states the result for homogeneous trees.

## *§4. Infinite towers of non-uniform $X$-lattices with quotient a tree*

The techniques described in §3 extend to certain infinite edge-indexed graphs. We have the following:

**(4.1) Theorem.** *Let $(A, i)$ be an edge-indexed graph that admits a lattice, and which is infinite, hence permissible. Assume that $(A, i)$ is a tree, but is not dominant-end-rooted. Then $(A, i)$ admits an infinite tower of finite faithful groupings.*

Except for the case that $(A, i)$ is a *dominant-end-rooted edge-indexed tree* (see [CR2]), the assumption that $(A, i)$ is an infinite permissible tree implies the existence of a *finite* permissible 'core' graph $(A_0, i)$ which is an edge-indexed path of length $n \geq 1$. By Theorem (3.1), $(A_0, i)$ admits an infinite tower of finite faithful groupings, and we may then apply Lemma (2.4) to extend the tower of groupings to $(A, i)$.

If $(A, i)$ is a dominant end-rooted edge-indexed tree, then $(A, i)$ does not contain a finite permissible core. We know that in this case, the set of covolumes of non-uniform lattices in $Aut(X)$, $X = \widetilde{(A, i)}$, is not bounded away from zero, however our techniques do not suffice to produce a tower of groupings on $(A, i)$.

**(4.2) Theorem.** *Let $(A, i)$ be as in Theorem (4.1). Let $X = \widetilde{(A, i)}$. Then there is an infinite ascending chain*

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots$$

*of non-uniform $X$-lattices in $Aut(X)$. Hence $Vol(\Gamma_i\backslash\backslash X) \longrightarrow 0$ as $i \longrightarrow \infty$.*

In Theorems (4.1), and (4.2), the covering tree $X = \widetilde{(A, i)}$ may be uniform or not.

## §5. *Infinite towers of non-uniform $X$-lattices*

We have the following:

**(5.1) Theorem ([R]).** *Let $(A, i)$ be a permissible edge-indexed graph. Suppose $(A, i)$ contains an arithmetic bridge with $n \geq 2$ edges. Then $(A, i)$ admits an infinite tower of finite faithful groupings.*

Concerning existence of arithmetic bridges, we have the following:

**(5.2) Theorem ([C1], [CR1]).** *Let $(A, i)$ be a unimodular edge-indexed graph. Let $e \in EA$ be a ramified edge such that $\Delta(e)$ is not an integer. If $e$ is not separating, then $e$ is contained in an arithmetic bridge with $n \geq 2$ edges.*

Combining the results of §2, §4 and the above, we have:

**(5.3) Theorem.** *Let $(A, i)$ be a permissible edge-indexed graph that is not a dominant-end-rooted edge-indexed tree. Then $(A, i)$ admits an infinite tower of finite faithful groupings.*

A corollary of Theorem (5.3) is the following:

**(5.4) Theorem.** *Let $X$ be a locally finite tree. If $Aut(X)$ contains a non-uniform $X$-lattice $\Gamma$, and $X$ is not the universal cover of a dominant-end-rooted edge-indexed tree, then $Aut(X)$ contains an infinite tower*

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots$$

*of non-uniform $X$-lattices. Hence $Vol(\Gamma_i \backslash\backslash X) \longrightarrow 0$ as $i \longrightarrow \infty$.*

## §6. *Existence of non-uniform $X$-lattices*

By Theorem (5.4), the question of existence of infinite towers of non-uniform $X$-lattices reduces to the question of existence of non-uniform $X$-lattices.

To outline the results on existence of non-uniform $X$-lattices, we make the following definition. Let $X$ be a locally finite tree, $G = Aut(X)$, and let $\mu$ be a (left) Haar measure on $G$. Suppose that $G$ is unimodular. Then $\mu(G_x)$ is constant on $G$-orbits, so we can define ([BL], (1.5)):

$$\mu(G\backslash\backslash X) := \sum_{x \in V(G\backslash X)} \frac{1}{\mu(G_x)}.$$

We have the 'Lattice existence theorem':

**(6.1) Theorem ([BCR], (0.2)).** *Let $X$ be a locally finite tree, let $G = Aut(X)$, and let $\mu$ be a (left) Haar measure on $G$. The following conditions are equivalent:*

(a) *$G$ contains an $X$-lattice $\Gamma$.*

(b) *(U)* *$G$ is unimodular, and*
*(FV)* $\mu(G\backslash\backslash X) < \infty$.

In particular, we have the following theorem, which together with Bass-Kulkarni's 'Uniform existence theorem' ([BK], (4.10)) gives Theorem (6.1):

**(6.2) Theorem ([BCR], (0.5)).** *Let $X$ be a locally finite tree, let $G = Aut(X)$, and let $\mu$ be a (left) Haar measure on $G$. Assume that:*

*(U)* $G$ *is unimodular,*

*(FV)* $\mu(G\backslash\backslash X) < \infty$, *and*

*(INF)* $G\backslash X$ *is infinite.*

*Then $G$ contains a (necessarily non-uniform) $X$-lattice $\Gamma$.*

For uniform trees, we have the following:

**(6.3) Theorem ([C1], [C2]).** *If $X$ is uniform and not virtually rigid then $G$ contains a non-uniform $X$-lattice $\Gamma$.*

## References


[B] Bass, H, *Covering theory for graphs of groups*, Journal of Pure and Applied Algebra **89** (1993), 3-47.

[BCR] Bass, H, Carbone L, and Rosenberg, G, *The existence theorem for tree lattices*, Appendix [BCR], '*Tree Lattices*' by Hyman Bass and Alex Lubotzky (2000), Progress in Mathematics 176, Birkhauser, Boston.

[BK] Bass, H and Kulkarni, R, *Uniform tree lattices*, Journal of the Amer Math Society **3 (4)** (1990).

[BL] Bass, H and Lubotzky, A, *Tree lattices*, Progress in Mathematics 176, Birkhauser, Boston (2000).

[BT] Bass, H and Tits, J, *A Discreteness Criterion for Certain Tree Automorphism Groups*, Appendix [BT], '*Tree Lattices*' by Hyman Bass and Alex Lubotzky (2000), Progress in Mathematics 176, Birkhauser, Boston.

[C1] Carbone, L, *Non-uniform lattices on uniform trees*, Memoirs of the AMS, vol. 152, no. 724 (July 2001).

[C2] Carbone, L, *Non-minimal tree actions and the existence of non-uniform tree lattices*, Preprint (2001).

[CR1] Carbone, L and Rosenberg, G, *Lattices on non-uniform trees*, Submitted (2001).

[CR2] Carbone, L and Rosenberg, G, *Infinite towers of non-uniform tree lattices*, (in preparation) (2001).

[KM] Kazhdan, D and Margulis, G, *A proof of Selberg's hypothesis. (Russian)*, Mat. Sb. (N.S.) 75 (117) (1968), 163–168.

[R] Rosenberg, G, *Towers and covolumes of tree lattices*, PhD. Thesis, Columbia University (2001).

[S] Serre, J.P, *Trees (Translated from the French by John Stilwell)* (1980), Springer-Verlag, Berlin Heidelberg.



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