HYPERBOLIC KAC-MOODY WEYL GROUPS, BILLIARD TABLES AND ACTIONS OF LATTICES ON BUILDINGS

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Abstract. Let G be a locally compact Kac-Moody group associated to a symmetrizable rank 3 Kac-Moody algebra of noncompact hyperbolic type. It is known that the fundamental chambers of Weyl groups of certain algebras in this class serve as billiard tables for a physical phenomenon known as cosmological billiards. We discuss the classification of Weyl groups in this class, we determine their group presentations and exhibit their tessellations on the Poincaré disk. We show that each Weyl group W is an amalgam of finite Coxeter groups by constructing an action of W on a tree $\mathcal{Y}$. It follows that among all Kac-Moody Weyl groups of affine or hyperbolic type, those of rank 2 and of rank 3 noncompact hyperbolic type are the only ones that have nontrivial amalgam decompositions. We show that the degrees of $\mathcal{Y}$ can be determined from the Dynkin diagram of G. We construct a bihomogeneous bipartite tree $\mathcal{X}$ embedded in the Tits building of G, a rank 3 locally finite hyperbolic building, on which the full Kac-Mooody group G acts and we determine the degrees of $\mathcal{X}$. We show that there is a nonuniform lattice subgroup of G which acts properly on this tree, that is, with finite vertex stabilizers.

1. Introduction

The topic of our study is a class of hyperbolic Kac-Moody groups and the action of their Weyl groups. In addition to their rich mathematical structure, this class of Kac-Moody symmetries has recently been discovered to occur in high energy physics in the context of supergravity, a theory incorporating general relativity and supersymmetry. It is known that the dynamics of certain supergravity theories near a space-like singularity gives rise to a phenomenon known as *cosmological billiards*, which may be identified with the motion of an abstract billiard ball moving freely on a hyperbolic billiard table and bouncing off its walls ([DHN]). Remarkably, the fundamental chamber of certain hyperbolic Kac-Moody Weyl groups serves as a 'billiard table' for the dynamics. The ambient Lorentz space for such hyperbolic Kac-Moody root systems is thus the setting for many interesting physical phenomena, including cosmological billiards. A concrete understanding of the geometry of the root space and its Weyl group action is therefore desirable.

In order to build a mathematical framework for studying these physical applications, we adopt a utilitarian approach to the subject. We work with explicit group theoretical constructions whose existence is theoretically known, but which may be nontrivial to obtain in practice. We consider the class of locally compact Kac-Moody groups associated to a symmetrizable rank 3 Kac-Moody algebra of noncompact hyperbolic type. The fundamental chambers of several of the associated Weyl groups in this class have been discovered to serve as billiard tables for cosmological billiards in $D = 3$ and $D = 4$ spacetime dimensions. In an appendix, we note that among the known gravity and supergravity Lagrangians giving rise to billiard motion in $D = 3$

Date: December 15, 2010.

The first author was supported in part by NSF grant #DMS-0701176. The second author was partially supported by NSF grant #PHY-0901237.

and $D = 4$, there are 5 possible billiard tables for cosmological billiards in this class of Weyl groups and their tessellations.

It is useful for physical applications to understand the classification of Weyl groups in this class, to determine their explicit group presentations and their tessellations on the Poincaré disk. We give a detailed answer to each of these questions. It turns out that these Weyl groups are amalgams of the form $H_1 *_K H_2$ where H_1, H_2 and K are finite Coxeter groups. In general, determining which groups are amalgams is nontrivial, but can be achieved by investigating actions of the groups on trees and applying the fundamental theory of Bass and Serre ([B], [Se]). This is the approach we adopt, which also leads naturally to the construction of a bihomogeneous bipartite tree embedded in the Tits building on which the full Kac-Mooody group G acts. We then show that there is a nonuniform lattice subgroup of G which acts properly on this tree, that is, with finite vertex stabilizers.

In order to discuss our result more precisely, we introduce the following notions. The data for constructing a Kac-Moody algebra includes a *generalized Cartan matrix* which is a generalization of the notion of a Cartan matrix of a finite dimensional Lie algebra, and which encodes the same information as a Dynkin diagram. A Dynkin diagram $\mathcal{D}$ is of *hyperbolic type* if it is neither of finite nor affine type, but every proper connected subdiagram is either of finite or of affine type. If $\mathcal{D}$ is of hyperbolic type, we say that $\mathcal{D}$ is of *compact hyperbolic type* if every proper, connected subdiagram is of finite type. 'Symmetrizability' is an important property of a generalized Cartan matrix, necessary for the existence of a well-defined symmetric invariant bilinear form $(\cdot \mid \cdot)$ on the Kac-Moody algebra which plays the role of 'squared length' of a root. It is necessary that a generalized Cartan matrix be symmetrizable in order for its root space to give rise to a cosmological billiard motion in the fundamental chamber of the Weyl group.

In order to associate a group to a Kac-Moody algebra, Tits associated a group functor $\mathcal{G}_A$ on the category of commutative rings, such that for any symmetrizable generalized Cartan matrix A and any ring R there exists a group $\mathcal{G}_A(R)$ ([Ti1], [Ti2]). He also showed that if R is a field, then $\mathcal{G}_A(R)$ is unique up to isomorphism.

Let G be a Kac-Moody group associated to a symmetrizable rank 3 Kac-Moody algebra of noncompact hyperbolic type. The classification of hyperbolic Dynkin diagrams shows that there are 33 possible such algebras up to isomorphism ([CCCMNNP], [Li], [S]). When completed over a finite field, G is locally compact and totally disconnected ([CG], [RR]). Moreover, G admits an action on a locally finite hyperbolic Tits building X ([CG], [RR]). The Tits building has a geometric realization which can be expressed as a union of subcomplexes (apartments) which are isomorphic Coxeter complexes satisfying certain axioms. For G of rank 3 noncompact hyperbolic type, apartments in X are hyperbolic planes tessellated by the action of a hyperbolic Weyl group W of noncompact type.

We recall that a discrete subgroup Γ of a locally compact group G is a *lattice* if the quotient $\Gamma\backslash G$ carries a finite invariant measure. If further, $\Gamma\backslash G$ is compact, we say that Γ is cocompact. Otherwise we say that Γ is nonuniform. It is known that the minimal parabolic subgroup B^- of the negative BN-pair for G is a nonuniform lattice subgroup of G ([CG], [Re]). In [C] the author deduced from the results of [DJ] that for all symmetrizable locally compact Kac-Moody groups G of either rank 2 (affine or hyperbolic) or of rank 3 noncompact hyperbolic type, the nonuniform lattice subgroup B^- has the Haagerup property, which is a strong negation of Kazhdan's property (T). It is also known that the Haagerup property predicts a continuous proper action by isometries on a 'median space' ([CDH]).

We explicitly construct a proper action of the lattice $B^- \leq G$ on a bihomogeneous tree $\mathcal{X}$, which is an example of a median space, for symmetrizable locally compact rank 3 Kac-Moody groups G of noncompact hyperbolic type (Section 8). We begin by associating a tree $\mathcal{Y}$ to the Weyl group W of G (Section 4). The tree $\mathcal{Y}$ is naturally embedded in the Poincaré disk and is naturally associated to the tessellation of the Poincaré disk by W. The existence of such a tree with a W-action is predicted by the fact that W is a discrete subgroup of $PSL_2(\mathbb{R})$ and hence is virtually free. We make use throughout the paper of an explicit construction of such an action.

Now let X be the Tits building of G. Let $\mathcal{A}_0$ denote the fundamental apartment of X. Then $\mathcal{A}_0$ is a copy of the Poincaré disk tessellated by W and $\mathcal{Y}$ is embedded in $\mathcal{A}_0$. We show that the full Tits building X contains a tree $\mathcal{X}$ that retracts onto the tree $\mathcal{Y}$ under the retraction of the building X onto $\mathcal{A}_0$ (Section 5.3). The tree $\mathcal{X}$ is bihomogeneous and bipartite and the vertices of $\mathcal{X}$ are barycenters of faces of X, barycenters of edges of X, or corners of triangles in X. We also show that the degrees of $\mathcal{Y}$ can be determined from the Dynkin diagram of G (Section 4).

Construction of a lattice acting properly on a tree cannot be made in a locally compact Kac-Moody group G of affine or hyperbolic type (compact or noncompact) if $rank(G) \geq 4$ since such groups have Kazhdan's Property (T) ([C] and [DJ]). By a theorem of de la Harpe and Valette ([HV]), every action of a Property (T) group on a tree fixes a vertex. In particular this applies to lattice subgroups of G (see Section 8.1).

On the other hand, if G is a rank 2 locally compact Kac-Moody group, then G is symmetrizable and is of affine or hyperbolic type, The Tits building of G is then itself a homogeneous tree X ([CG], [RR]). It is known that G has the Haagerup property and that the subgroup B^- acts properly on X ([CG], [Re]).

After preparation of this manuscript, we were notified that the existence of amalgam presentations of W in terms of finite Coxeter groups and that a continuous action of the full Kac-Moody group G on a locally finite tree can be deduced from the far reaching methods in the recent book of Mike Davis ([D]). However, we prefer explicit computation of the vertex stabilizers for the action of W on the tree $\mathcal{Y}$, the presentations they give and the subgroup relations among them, which we determined in Section 3.1. We also require an explicit embedding of a tree $\mathcal{X}$ in the full Tits building which retracts onto the tree $\mathcal{Y}$. We extended these constructions to give a proper action of the nonuniform lattice B^- in the Kac-Moody group G on a tree embedded in the rank 3 hyperbolic Tits building of G.

Using the classification of hyperbolic Dynkin diagrams ([CCCMNNP], [Li], [S]), in Section 3, we tabulate the 33 possible symmetrizable generalized Cartan matrices of rank 3 noncompact hyperbolic type, their Dynkin diagrams and their Weyl groups, from which the corresponding noncompact tessellations of the hyperbolic plane can be determined.

It turns out that among the 33 symmetrizable Dynkin diagrams of rank 3 noncompact hyperbolic type, there are only 9 distinct (nonisomorphic) Weyl groups. Therefore, among the 33 noncompact hyperbolic tessellations of the Poincaré disk that arise here, only 9 are distinct. We give detailed drawings of these 9 tessellations with the inscribed trees $\mathcal{Y}$ in Section 4.1. These intricate drawings were prepared by Sahar Waked Sati (using *AutoCAD 2009* from *Autodesk*), to whom we are deeply indebted for her patience and artistic talent.

Our amalgam presentations of W of the form $H_1 *_K H_2$ where H_1, H_2 and K are finite Coxeter groups come from the graph of groups presentations for the action of each W on $\mathcal{Y}$ (Section 4.2). These amalgam decompositions coincide with the JSJ-decompositions for Weyl groups given by Mihalik ([Mi]) and Ratcliffe and Tschantz ([RTs]). We deduce that among all

Kac-Moody Weyl groups of affine or hyperbolic type, those of rank 2 and of rank 3 noncompact hyperbolic type are the only ones that have nontrivial amalgam decompositions (Section 4).

We mention that there are a number of papers that use dynamical systems methods to study billiard motion on abstract hyperbolic manifolds, notably Bunimovich, Chernov and Sinai ([BCS1], [BCS2]), McMullen ([Mc]) , Sarnak ([Sa]), Schwartz ([Sc1], [Sc2], [Sc3]), Sinai ([Si]) and Tabachnikov ([T]). However, we are not aware of any mathematical studies of cosmological billiards, or of billiard motion in the fundamental chamber of a Weyl group of a Lie algebra.

We thank Mike Mihalik and John Ratcliffe for notifying us of their results on JSJ-decompositions for Coxeter groups and for very helpful discussions which led to the material in Section 4.3. We also thank Sophie de Buyl for clarifying some details about the appearance of Kac-Moody Weyl groups in cosmological billiards. We are very grateful to the anonymous referee whose careful reading and comments improved our exposition.

2. Locally compact Kac-Moody groups and Tits buildings

2.1. **Introduction to Kac-Moody algebras.** The data for constructing a Kac-Moody algebra includes a *generalized Cartan matrix*. This is a square matrix $A = (a_{ij})$, $i, j \in \{1, 2, \dots, \ell\}$ whose entries are integers such that:

(1) $a_{ii} = 2$,

(2) $a_{ij} \leq 0$, $i \neq j$,

(3) $a_{ij} = 0$ implies $a_{ji} = 0$.

A generalized Cartan matrix A is called *indecomposable* if there is no rearrangement of the indices so that A can be written in block diagonal form. A generalized Cartan matrix A is called *symmetrizable* if there exist nonzero rational numbers $d_1, \dots, d_\ell$, such that the matrix DA is symmetric, where $D = diag(d_1, \dots, d_\ell)$. We call DA a *symmetrization* of A.

The generalized Cartan matrix A is *affine* if A is positive semi-definite but not positive definite. If A is neither positive definite nor positive semi-definite, but every proper indecomposable submatrix is either positive definite or positive semi-definite, we say that A has *hyperbolic type.* If every proper indecomposable submatrix of A is positive definite, we say that A has *compact hyperbolic type.* Thus if A has a proper indecomposable affine submatrix, we say that G has *noncompact hyperbolic type.*

Given a generalized Cartan matrix and a finite dimensional vector space $\mathfrak{h}$ satisfying some natural conditions, one may construct a Kac-Moody algebra. An account of the construction and the subject, which also includes works of many others, is given in [K] (see also [Sel]).

2.2. **Examples.** Let $A = (a_{ij})$, $i, j \in \{1, 2, \dots, \ell\}$ be a generalized Cartan matrix.

Rank 3 compact hyperbolic type. If $rank(A) = 3$ then for any rank 2 proper indecomposable submatrix $\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$ we have $a_{ij}a_{ji} < 4$ for $i \neq j$. For example, $A = \begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -2 \\ -2 & -1 & 2 \end{pmatrix}$.

Rank 3 noncompact hyperbolic type. If $rank(A) = 3$ then there is a rank 2 proper indecomposable submatrix $\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$ with $a_{ij}a_{ji} = 4$ for $i \neq j$. For example, $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

2.3. **Introduction to Kac-Moody groups and their properties.** Subsequent to the discovery of infinite dimensional generalizations of Lie algebras by dropping the assumption that the matrix of Cartan integers is positive definite, the problem of associating groups to Kac-Moody algebras arose, the difficulty being that there is no obvious definition of a general 'Kac-Moody group'.

Several appropriate definitions of a Kac-Moody group have been discovered, many of them using a variety of techniques as well as additional external data, such as a $\mathbb{Z}$-form for the universal enveloping algebra. Many constructions use some version of the *Tits functor* ([Ti2]).

Though there is no obvious infinite dimensional generalization of finite dimensional Lie groups, Tits associated a group functor G_A on the category of commutative rings, such that for any symmetrizable generalized Cartan matrix A and any ring R there exists a group $G_A(R)$ ([Ti1], [Ti2]). Tits showed that if R is a field, then $G_A(R)$ is characterized uniquely up to isomorphism, apart from some degeneracy in the case of small fields. Tits defined not one group, but rather *minimal* and *maximal* groups. The value of the Tits functor G_A over a field k is called a *minimal Kac-Moody group*. The *maximal* or *complete Kac-Moody group* is defined relative to a completion of the Kac-Moody algebra and contains $G_A(k)$ as a dense subgroup.

Let A be an $\ell \times \ell$ symmetrizable generalized Cartan matrix. The existence of a completion $G = G_A(\mathbb{F}_q)$ of the Tits functor associated to A and the finite field $\mathbb{F}_q$ was noted by Tits ([Ti1]). Explicit completions have been constructed using distinct methods by Carbone and Garland ([CG]) and by Rémy and Ronan ([RR]). A complete Kac-Moody group G over a finite field is locally compact and totally disconnected.

While Kac-Moody groups may be described axiomatically, by generators and relations, by analytic, topological and geometric methods, for our purposes it will be convenient and sufficient for our purposes to characterize these groups by describing their BN-pairs.

2.4. **The BN-pair and Tits building of a complete Kac-Moody group.** A Kac-Moody group G may be described by certain group theoretic data, called a *Tits system* or *BN-pair*. This data carries a great deal of information about the group and its subgroups, and in particular determines a simplicial complex, a *Tits building* X on which the Kac-Moody group acts.

Let A be an $\ell \times \ell$ symmetrizable generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field $\mathbb{F}_q$. Such completions have been constructed by Carbone and Garland ([CG] and by Rémy and Ronan ([RR]). We shall use the Carbone-Garland completion. A complete Kac-Moody group G over a finite field $\mathbb{F}_q$ is locally compact and totally disconnected, and the Tits building X of G is locally finite. In this section we give a brief description of the Tits system for G and its corresponding Tits building.

The Carbone-Garland completion G of Tits' functor over the finite field $\mathbb{F}_q$ has subgroups $B^{\pm} \subseteq G$, $N \subseteq G$, and Weyl group $W = N/H$, where $H = N \cap B^{\pm}$ is a normal subgroup of N. We have $B^{\pm} = HU^{\pm}$ where U^{+} is generated by all positive real root groups , U^{-} is generated by all negative real root groups, B^{+} is compact, in fact a profinite neighborhood of the identity in G, and B^{-} is discrete. Then (G, B^{+}, N) and (G, B^{-}, N) are BN-pairs, and

$$G \quad = \quad B^{+}NB^{-} \quad = \quad B^{-}NB^{+}.$$

It follows that G has Bruhat decomposition

$$G \quad = \quad \sqcup_{w \in W} \; B^{\pm}wB^{\pm}.$$

There is a surjective homomorphism $\nu : N \longrightarrow W$. We identify W (non-canonically) with a subset (not a subgroup) of N which contains exactly one representative of every element of W.

By abuse of notation, this set of representatives will also be called W. Let S be the standard generating set for W consisting of simple root reflections. Let $U \subsetneq S$. The *standard parabolic subgroups* are

$$P_U \quad = \quad \sqcup_{w \in \langle U \rangle} \, B^{\pm} w B^{\pm}.$$

A *parabolic* subgroup is any subgroup containing a conjugate of $B^{\pm}$. The Tits building of G is a simplicial complex X of dimension $dim(X) = |S| - 1$. In fact we associate a building $X^{\pm}$ to each BN-pair (G, B^+, N) and (G, B^-, N). The buildings X^+ and X^- are isomorphic as chamber complexes and have constant thickness $q+1$ (see [DJ], Appendix KMT).

In the next sections, we will work only with the Tits building $X = X^+$ of the positive BN-pair (G, B^+, N) which we write as (G, B, N). The vertices of X are given by the cosets in G of the maximal parabolic subgroups of G. The incidence relation is described as follows. The $r+1$ vertices $Q_1, \dots, Q_{r+1}$ span an r-simplex if and only if the intersection $Q_1 \cap \cdots \cap Q_{r+1}$ is parabolic, that is, contains a conjugate of $B^{\pm}$. In our case, the Weyl group W is infinite, so by the Solomon-Tits theorem, X is contractible. The group G acts by left multiplication on cosets.

The building X satisfies the axioms of Tits:

Tits building axioms, (1965)

A building is a simplicial complex X that can be expressed as the union of subcomplexes Σ (called apartments) satisfying the following axioms:

(B0) Each apartment Σ is a Coxeter complex of the same dimension d which also equals $dim(X)$.

(B1) For any two simplices σ and ω there is an apartment Σ containing both of them.

(B2) If Σ and Σ' are two apartments containing σ and ω, then there is an isomorphism $\Sigma \longrightarrow \Sigma'$ fixing σ and ω pointwise.

It follows that all apartments are isomorphic.

Let X be a Tits building. Let Σ be an apartment, let $\mathcal{C} \in \Sigma$ be a chamber (maximal simplex) contained in Σ. There is a unique chamber map $\rho = \rho(\Sigma, \mathcal{C}) : X \longrightarrow \Sigma$ called a *retraction* which fixes $\mathcal{C}$ pointwise and maps every apartment containing $\mathcal{C}$ isomorphically onto Σ. The retraction map ρ preserves distances from $\mathcal{C}$ and preserves colors of vertices.

2.5. **Lattices in Kac-Moody groups.** Let G be a locally compact group and let μ be a (left) Haar measure on G. Let $\Gamma \leq G$ be a discrete subgroup with quotient $p : G \longrightarrow \Gamma\backslash G$. We call Γ a *lattice* in G if $\mu(\Gamma\backslash G) < \infty$, and a *cocompact* lattice if $\Gamma\backslash G$ is compact.

Symmetrizable locally compact Kac-Moody groups G over finite fields $\mathbb{F}_q$ are known to contain lattice subgroups. If q is sufficiently large then the minimal parabolic subgroup B^- of the negative BN-pair for G is a nonuniform lattice subgroup for G of rank $r \geq 3$ ([CG], [Re]), and also for $r = 2$ with no restriction on q ([CG]).

The action of G on its Tits building X permits combinatorial criteria for subgroups $\Gamma \leq G$ to be discrete, cocompact, or to have finite covolume, without reference to a Haar measure on G ([BL]). In particular, the discreteness of a lattice subgroup Γ in a locally compact Kac-Moody group G is equivalent to the property that Γ acts on the Tits building X of G with finite vertex stabilizers ([BL]). Thus locally compact Kac-Moody groups G contain lattices that act discretely and isometrically on the locally finite Tits building X.

We will show that if G has rank 3 noncompact hyperbolic type, then G also acts discretely and isometrically on a simplicial tree $\mathcal{X}$ embedded in the Tits building (Section 8).

3. Weyl groups of rank 3 noncompact hyperbolic type

Let G be a locally compact Kac-Moody group associated to a symmetrizable rank 3 Kac-Moody algebra of noncompact hyperbolic type. Then the Weyl group $W = N/H$ is given by generators and relations:

$$W = \langle w_1, w_2, w_3 \mid (w_i w_j)^{m_{ij}} = 1, \text{ if } \ m_{ij} \neq \infty \rangle,$$

where

$$m_{ii} := 1, \quad m_{ij} := 2, 3, 4, 6, \text{ or } \infty$$

when

$$a_{ij} a_{ji} = 0, 1, 2, 3, \text{ or } \ \geq 4, \quad i, j \in \{1, 2, 3\},$$

respectively, where a_{ij} are the entries of the generalized Cartan matrix A corresponding to G. The group W acts on the set of all roots, preserving the symmetric bilinear form $(\cdot \mid \cdot)$.

The Weyl groups W that arise in our setting are hyperbolic triangle groups $T(r, s, t)$ of noncompact type. That is, W is generated by reflections in the sides of a triangle in the hyperbolic plane with angles π/r, π/s, π/t, with

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} < 1$$

and one or more of the angles equal to 0. We sometimes denote W as $W(r, s, t)$. A fundamental region for W in the hyperbolic plane therefore has at least one boundary point corresponding to an angle of 0 in the fundamental region. We note that the isomorphism type of $W(r, s, t)$ is unchanged under a permutation of the indices (r, s, t).

We will not attempt to define the tessellation associated to the noncompact hyperbolic reflection group W, rather we refer the reader to the informative discussion in Chapter II of the book [M] of Magnus. However, we do provide diagrams of tessellations of the Poincaré disk by W in Section 4.1. The study of tessellations by hyperbolic triangle groups was initiated by Klein and Fricke (1890-1912). Carathéodory established the existence of hyperbolic tessellations by triangle groups where one or more angles in the fundamental region equals 0 ([Ca]).

Using the classification of hyperbolic Dynkin diagrams ([CCCMNNP], [Li], [S]), here we tabulate the 33 possible symmetrizable generalized Cartan matrices of rank 3 noncompact hyperbolic type. The first column of the tables given below gives an enumeration index for the rank 3 symmetrizable Dynkin diagrams of noncompact hyperbolic type. The second column displays the Dynkin diagram itself, and the third column gives the corresponding generalized Cartan matrix. The fourth column lists the triple (m_{12}, m_{23}, m_{31}), where the Weyl group for each Dynkin diagram is given by $W = \langle w_1, w_2, w_3 \mid (w_i w_j)^{m_{ij}} = 1, \text{ if } \ m_{ij} \neq \infty \rangle$.

Table 3.1a. Rank 3 symmetrizable Dynkin diagrams of noncompact hyperbolic type

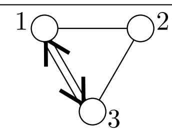
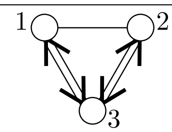
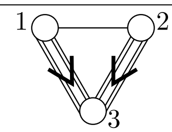
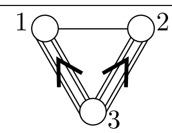
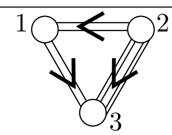
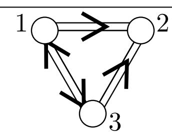
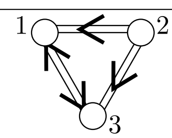
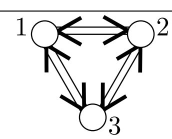
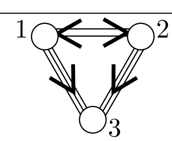
	Dynkin Diagram	GCM	(m_{12}, m_{23}, m_{31})
1.		$\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}$	$(3, 3, \infty)$
2.		$\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$	$(3, \infty, \infty)$
3.		$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -4 & -4 & 2 \end{pmatrix}$	$(3, \infty, \infty)$
4.		$\begin{pmatrix} 2 & -1 & -4 \\ -1 & 2 & -4 \\ -1 & -1 & 2 \end{pmatrix}$	$(3, \infty, \infty)$
5.		$\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & -1 \\ -2 & -4 & 2 \end{pmatrix}$	$(4, \infty, 4)$
6.		$\begin{pmatrix} 2 & -1 & -2 \\ -2 & 2 & -2 \\ -2 & -1 & 2 \end{pmatrix}$	$(4, 4, \infty)$
7.		$\begin{pmatrix} 2 & -2 & -2 \\ -1 & 2 & -1 \\ -2 & -2 & 2 \end{pmatrix}$	$(4, 4, \infty)$
8.		$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$	(∞, ∞, ∞)
9.		$\begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -3 & -3 & 2 \end{pmatrix}$	$(\infty, 6, 6)$

Table 3.1b. Rank 3 symmetrizable Dynkin diagrams of noncompact hyperbolic type

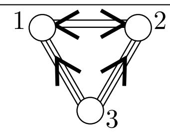
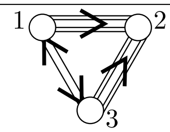
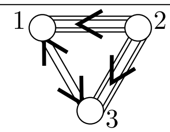
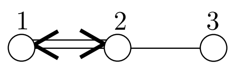
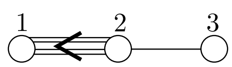
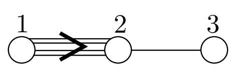
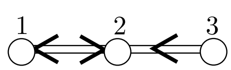
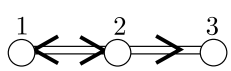
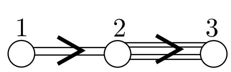
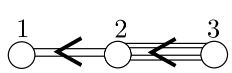
	Dynkin Diagram	GCM	(m_{12}, m_{23}, m_{31})
10.		$\begin{pmatrix} 2 & -2 & -3 \\ -2 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}$	$(\infty, 6, 6)$
11.		$\begin{pmatrix} 2 & -1 & -2 \\ -4 & 2 & -4 \\ -2 & -1 & 2 \end{pmatrix}$	(∞, ∞, ∞)
12.		$\begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ -2 & -4 & 2 \end{pmatrix}$	(∞, ∞, ∞)
13.		$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	$(\infty, 3, 2)$
14.		$\begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	$(\infty, 3, 2)$
15.		$\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	$(\infty, 3, 2)$
16.		$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$	$(\infty, 4, 2)$
17.		$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$	$(\infty, 4, 2)$
18.		$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$	$(4, \infty, 2)$
19.		$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -4 \\ 0 & -1 & 2 \end{pmatrix}$	$(4, \infty, 2)$

Table 3.1c. Rank 3 symmetrizable Dynkin diagrams of noncompact hyperbolic type

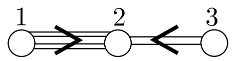
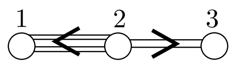
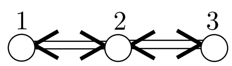
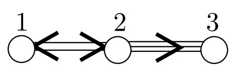
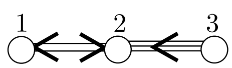
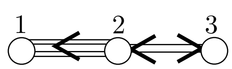
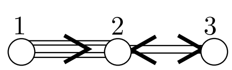
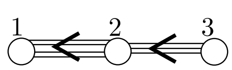
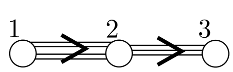
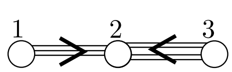
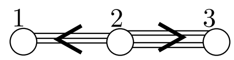
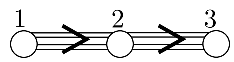
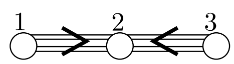
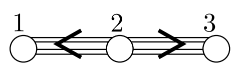
	Dynkin Diagram	GCM	(m_{12}, m_{23}, m_{31})
20.		$\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$	$(\infty, 4, 2)$
21.		$\begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$	$(\infty, 4, 2)$
22.		$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	$(\infty, \infty, 2)$
23.		$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$	$(\infty, 6, 2)$
24.		$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$	$(\infty, 6, 2)$
25.		$\begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	$(\infty, \infty, 2)$
26.		$\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	$(\infty, \infty, 2)$
27.		$\begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$	$(\infty, 6, 2)$
28.		$\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$	$(\infty, 6, 2)$
29.		$\begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -4 \\ 0 & -1 & 2 \end{pmatrix}$	$(6, \infty, 2)$

Table 3.1d. Rank 3 symmetrizable Dynkin diagrams of noncompact hyperbolic type

	Dynkin Diagram	GCM	(m_{12}, m_{23}, m_{31})
30.		$\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$	$(6, \infty, 2)$
31.		$\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$	$(\infty, \infty, 2)$
32.		$\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -4 \\ 0 & -1 & 2 \end{pmatrix}$	$(\infty, \infty, 2)$
33.		$\begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$	$(\infty, \infty, 2)$

3.1. **Subgroup structure.** Among the 33 symmetrizable Dynkin diagrams of rank 3 noncompact hyperbolic type in the above tables, there are only 9 isomorphism classes of Weyl groups. These are given by the 9 triples:

$$(\infty,\infty,\infty),\ (\infty,2,\infty),\ (\infty,3,\infty),\ (\infty,3,2),\ (\infty,4,2),\ (\infty,6,2),\ (\infty,3,3),\ (\infty,4,4),\ (\infty,6,6).$$

There are many examples of Weyl groups W^* associated to triples in this list that are subgroups of other Weyl groups W also in the list. These relationships are summarized in Figure 3.1 below. If we let w_1, w_2, w_3 be the generators of W, then Table 3.2 below also gives generators and coset representatives for W^* in terms of the generators of W, which can be used to obtain tessellations corresponding to subgroups.

FIGURE 3.1. Subgroup diagram

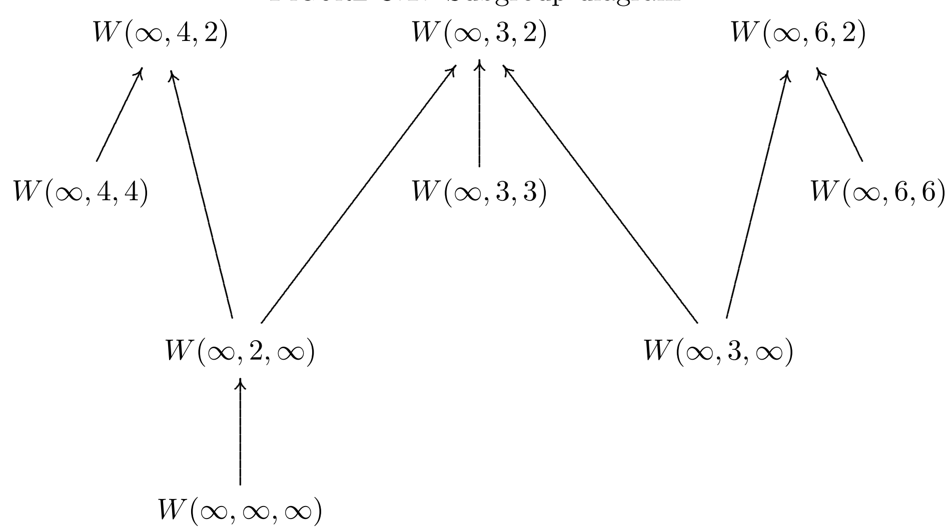


TABLE 3.2. Subgroup generators and cosets

W^*	W	Index($W^* \leq W$)	Generators for W^*	Coset Representatives
$W(\infty,\infty,\infty)$	$W(\infty,2,\infty)$	2	$w_1,\ w_2,\ w_3w_1w_3$	$1,\ w_3$
$W(\infty,\infty,\infty)$	$W(\infty,3,2)$	6	$w_1,\ w_2w_1w_2,$ $w_3w_2w_1w_2w_3$	$1,\ w_3,\ w_2w_3,$ $w_3w_2w_3,\ w_3w_2,\ w_2$
$W(\infty,\infty,\infty)$	$W(\infty,4,2)$	4	$w_1,\ w_2w_1w_2,\ w_3w_2w_3$	$1,\ w_3,\ w_3w_2,\ w_2$
$W(\infty,2,\infty)$	$W(\infty,3,2)$	3	$w_1,\ w_2w_1w_2,\ w_3$	$1,\ w_2,\ w_3w_2$
$W(\infty,2,\infty)$	$W(\infty,4,2)$	2	$w_1,\ w_2,\ w_3w_2w_3$	$1,\ w_3$
$W(\infty,3,\infty)$	$W(\infty,3,2)$	4	$w_1w_2w_1w_2w_1,\ w_2,\ w_3$	$1,\ w_1,\ w_2w_1,\ w_3w_2w_1$
$W(\infty,3,\infty)$	$W(\infty,6,2)$	2	$w_1,\ w_2,\ w_3w_2w_3$	$1,\ w_3$
$W(\infty,3,3)$	$W(\infty,3,2)$	2	$w_1w_2w_1,\ w_2,\ w_3$	$1,\ w_1$
$W(\infty,4,4)$	$W(\infty,4,2)$	2	$w_1w_2w_1,\ w_2,\ w_3$	$1,\ w_1$
$W(\infty,6,6)$	$W(\infty,6,2)$	2	$w_1w_2w_1,\ w_2,\ w_3$	$1,\ w_1$

In order to find a fundamental domain for a subgroup $W^* \leq W$, we fix a fundamental chamber $\mathcal{F}$ for W in the Poincaré disk and take the union of the images of $\mathcal{F}$ under the coset representatives for W^* in W. This gives the fundamental chamber for W^*. We give some examples below with reference to the diagrams in Section 4.1.

The Weyl group $W(\infty,\infty,\infty)$ is a subgroup of the Weyl group $W(\infty,3,2)$ of index 6. The 6 triangles in the tessellation of $W(\infty,3,2)$ meeting at the central vertex are coset representatives of the subgroup $W(\infty,\infty,\infty)$. As in [CV], we have

$$W(\infty,3,2) \cong W(\infty,\infty,\infty) \rtimes D_6,$$

where D_6 denotes the dihedral group of order 6, the symmetries of a triangle.

Both $W(\infty,r,r)$ and $W(\infty,r/2,\infty)$ are index 2 subgroups of $W(\infty,r,2)$, for $r=4,6$. For the fundamental domain of $W(\infty,r,r)$ in the tessellation of $W(\infty,r,2)$ we take a shaded triangle T with a corner at the origin together with an unshaded triangle sharing the edge of T opposite the center vertex. For the fundamental domain of $W(\infty,r/2,\infty)$ in the tessellation of $W(\infty,r,2)$ we take a shaded triangle T_1 centered at the origin together with an unshaded triangle T_2 that also has a corner at the origin and whose union $T_1 \cup T_2$ is also a triangle.

4. Tree associated to the Weyl group W

Let G be a symmetrizable locally compact Kac-Moody group of rank 3 noncompact hyperbolic type. In this section, we describe how to associate bihomogeneous bipartite tree $\mathcal{Y}$ to the Weyl group W of G. The tree $\mathcal{Y}$ is naturally related to the tessellation of the Poincaré disk by W. The existence of such a tree with a W-action is an expected consequence of the fact that W is a discrete subgroup of $PSL_2(\mathbb{R})$ and hence is virtually free. We will make use of the explicit construction of $\mathcal{Y}$ and the properties of the W-action on $\mathcal{Y}$ throughout the paper.

Theorem 4.1. *Let G be a symmetrizable locally compact Kac-Moody group of rank 3 noncompact hyperbolic type. Let W be the Weyl group of G. Then W determines a bihomogeneous bipartite tree $\mathcal{Y}$ whose vertices are barycenters of faces or edges in the tessellation of the Poincaré disk by W, or corners of triangles in the tessellation.*

Proof: We inscribe a tree in the tessellation of the Poincaré disk by W by first placing a vertex at each non-boundary corner of the triangles in the tessellation. If $W \neq W(\infty,\infty,\infty)$, then every fundamental chamber has such a corner. If each chamber has 2 such corners, then we connect pairs of vertices with an edge if there exists a chamber that contains both vertices.

Suppose that each chamber has only 1 non-boundary corner. In this case, there exist adjacent chambers (triangles) that share an edge with 2 boundary points. The remaining non-boundary corners have had vertices associated to them. We connect these vertices by an edge.

If $W = W(\infty,\infty,\infty)$, then the Poincaré disk is tessellated by ideal triangles, so there are no non-boundary corners. In this case, we place a vertex at the barycenter of face of each ideal triangle in the tessellation, and connect pairs of vertices at the barycenters of distinct adjacent chambers with an edge.

If e is an edge connecting vertices in distinct adjacent chambers (which occurs if and only if the Poincaré disk is tessellated by triangles with 2 or 3 boundary points), then e must cross the wall E common to both chambers. We now place a vertex at this intersection, which is naturally identified with the barycenter of E. This amounts to taking a barycentric subdivision of the originally inscribed tree. This also ensures that the Weyl group acts on the tree without inversions. In the cases where each chamber has 2 non-boundary corners and the edges of the inscribed tree connect vertices in the same chamber, we note that the Weyl group acts without inversions, so no further subdivision is necessary. In each case, by our construction, the graphs obtained here are trees. □

We use the notation $\mathcal{Y}_{a,b}$ to denote the bihomogeneous tree of degrees a and b, and we consider the following cases:

(i) W is of type (∞, ∞, ∞).

(ii) W is of type (∞, r, ∞), $r = 2, 3$.

(iii) W is of type $(\infty, r, 2)$, $r = 3, 4, 6$.

(iv) W is of type (∞, r, r), $r = 3, 4, 6$.

For case (i), the fundamental apartment of X is tessellated by ideal triangles. We place a vertex at the barycenter of each ideal triangle in the tessellation and a vertex at the barycenter of each edge of the ideal triangles. The simplicial complex spanned by these vertices is the tree $\mathcal{Y}_{3,2}$ which is naturally bipartite. Vertices at barycenters of the triangles have degree 3, and vertices at the barycenters of edges have degree 2.

For case (ii), the tessellation of the Poincaré disk by W suggests two possible types of vertices: those at (non-boundary) corners of triangles in the tessellation (which will have degree r) and those at barycenters of edges of the triangles (which will have degree 2). The simplicial complex spanned by these vertices is the tree $\mathcal{Y}_{r,2}$ which is naturally bipartite.

For case (iii), the tessellation of the Poincaré disk by W suggests two possible types of vertices for our tree: both are at (non-boundary) corners of triangles in the tessellation. Each corner in a tessellation of type $(\infty, r, 2)$ touches either 4 triangles or $2r$ triangles. A vertex at a corner that touches 4 triangles has degree 2, a vertex that touches 2r triangles has degree r. In this case, we associate the tree $\mathcal{Y}_{r,2}$ to W and this tree is naturally bipartite.

For case (iv), we make a tree by placing a vertex at every (non-boundary) corner of the triangles in the tessellation, and each vertex has degree r. In this case, we associate the tree $\mathcal{Y}_{r,r}$ to W and this tree is naturally bipartite, with the two different types of vertices corresponding to different orbits under the action of the Weyl group.

We may summarize these results in the following table. If $a = b$ then $\mathcal{Y}_{a,b}$ is homogeneous.

Table 4.1. Tree associated to the Weyl group

Type	$\mathcal{Y}_{a,b}$ in $\mathbb{H}$
(∞, ∞, ∞)	$\mathcal{Y}_{3,2}$
$(\infty, 2, \infty)$	$\mathcal{Y}_{4,2}$
$(\infty, 3, \infty)$	$\mathcal{Y}_{6,2}$
$(\infty, 3, 2)$	$\mathcal{Y}_{3,2}$
$(\infty, 4, 2)$	$\mathcal{Y}_{4,2}$
$(\infty, 6, 2)$	$\mathcal{Y}_{6,2}$
$(\infty, 3, 3)$	$\mathcal{Y}_{3,3}$
$(\infty, 4, 4)$	$\mathcal{Y}_{4,4}$
$(\infty, 6, 6)$	$\mathcal{Y}_{6,6}$

We remark that the trees associated with the Weyl groups $W(\infty, r, 2)$ for $r = 3, 4, 6$ are the same as the trees associated with their subgroups $W(\infty, s, \infty)$, for $s = \infty, 2, 3$, respectively. Note also that taking a barycentric subdivision of the tree associated to the Weyl group $W(\infty, r, r)$ would give the same tree as that associated to $W(\infty, r, 2)$ for $r = 3, 4, 6$.

We may determine the degrees of $\mathcal{Y}$ directly from the Dynkin diagram $\mathcal{D}$, using the rules listed below. In what follows, a *circuit diagram* refers to a Dynkin diagram in which every pair of vertices is connected by an edge or multi-edge, and a *linear diagram* refers to a Dynkin diagram in which there exists a pair of vertices with no edge between them.

For linear diagrams:

(a) If $\mathcal{D}$ has a single edge, then $\mathcal{Y} = \mathcal{Y}_{3,2}$.

(b) If $\mathcal{D}$ has a triple edge, then $\mathcal{Y} = \mathcal{Y}_{6,2}$.

(c) For all other cases, $\mathcal{Y} = \mathcal{Y}_{4,2}$.

For circuit diagrams:

(a) If $\mathcal{D}$ has only one single edge, then $\mathcal{Y} = \mathcal{Y}_{6,2}$.

(b) If $\mathcal{D}$ has two single edges, then $\mathcal{Y} = \mathcal{Y}_{3,3}$.

(c) If $\mathcal{D}$ has a double edge with a single arrow, then $\mathcal{Y} = \mathcal{Y}_{4,4}$.

(d) If $\mathcal{D}$ has a triple edge, then $\mathcal{Y} = \mathcal{Y}_{6,6}$.

(e) For all other cases $\mathcal{Y} = \mathcal{Y}_{3,2}$.

4.1. **Diagrams of Tessellations and Embedded Trees.** We recall that there are only 9 distinct noncompact hyperbolic tessellations of the Poincaré disk that arise here. Here we give detailed drawings of these 9 tessellations and the embedded tree $\mathcal{Y} = \mathcal{Y}_{a,b}$ in the Poincaré disk. These drawings were prepared by Sahar Waked Sati using *AutoCAD 2009*. Some of these pictures were reproduced from our hand drawn diagrams and therefore may not display perfect symmetry or have the exact angles we intend. The tessellations underlying Figures 4.1 and 4.4 appear in the book of Magnus ([M], Figures 18 and 17, respectively). We have copied and modified these diagrams.

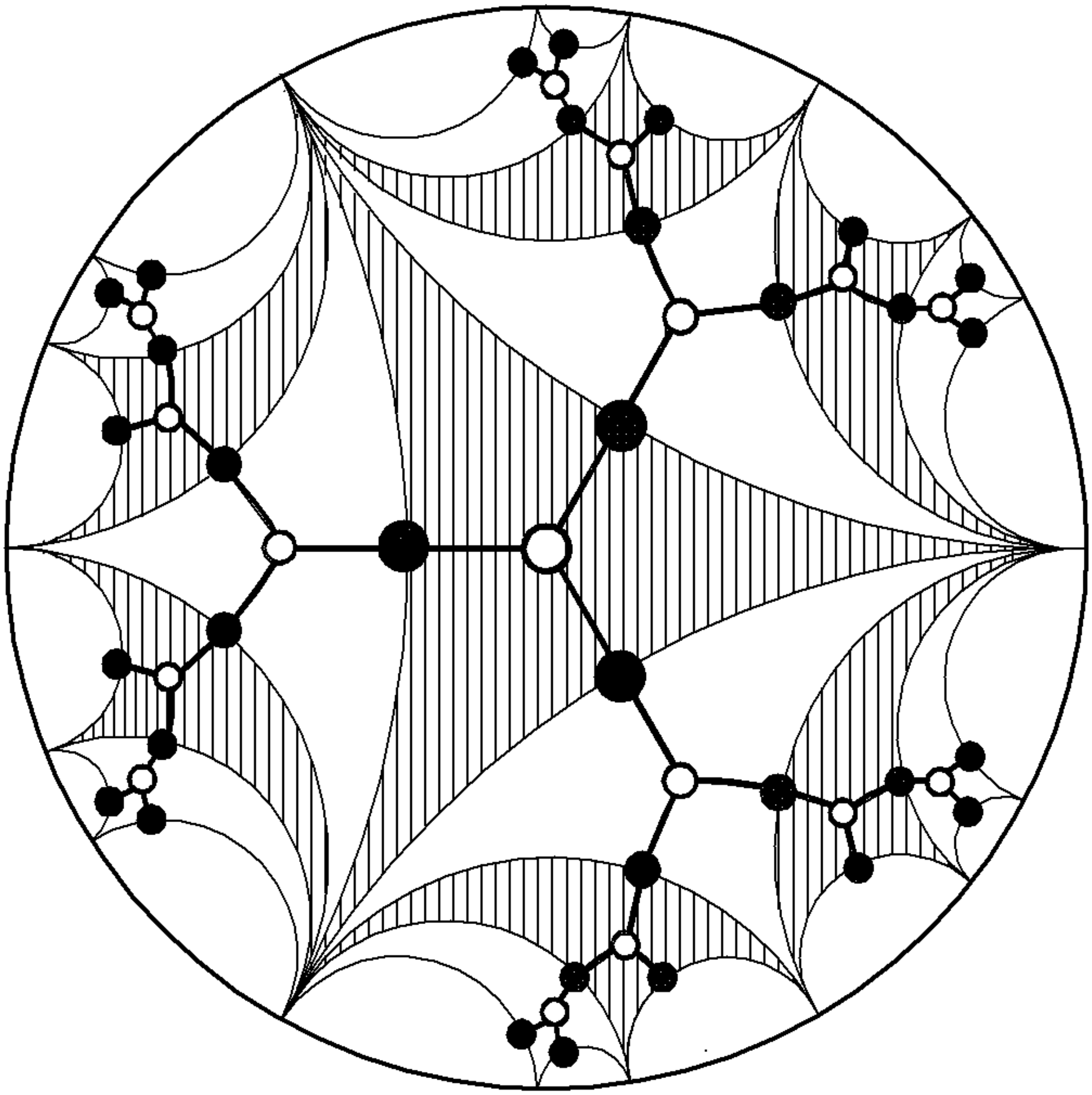


FIGURE 4.1. $\mathcal{Y}_{3,2}$ embedded in the tessellation induced by $W(\infty,\infty,\infty)$

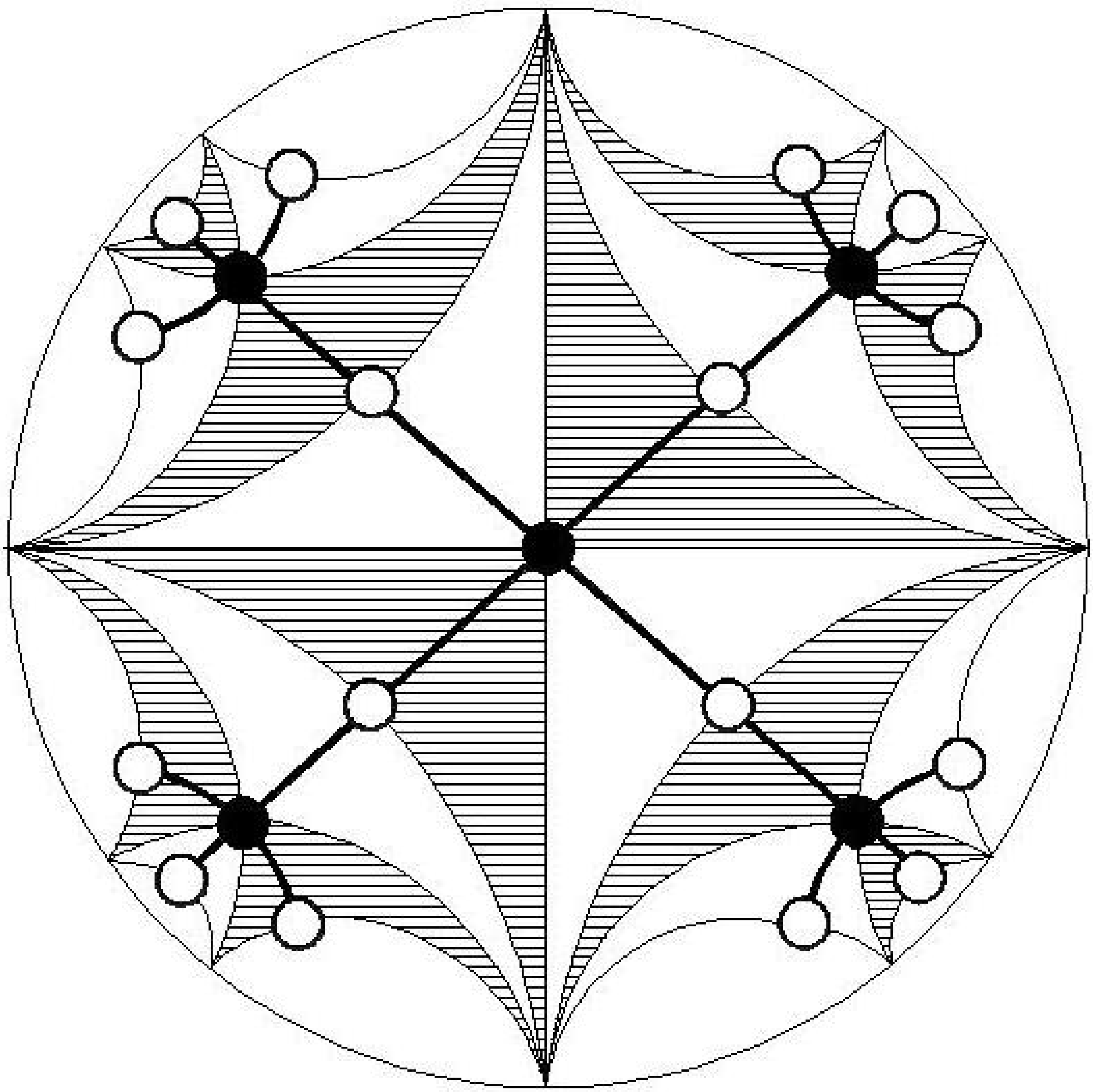


FIGURE 4.2. $\mathcal{Y}_{4,2}$ embedded in the tessellation induced by $W(\infty, 2, \infty)$

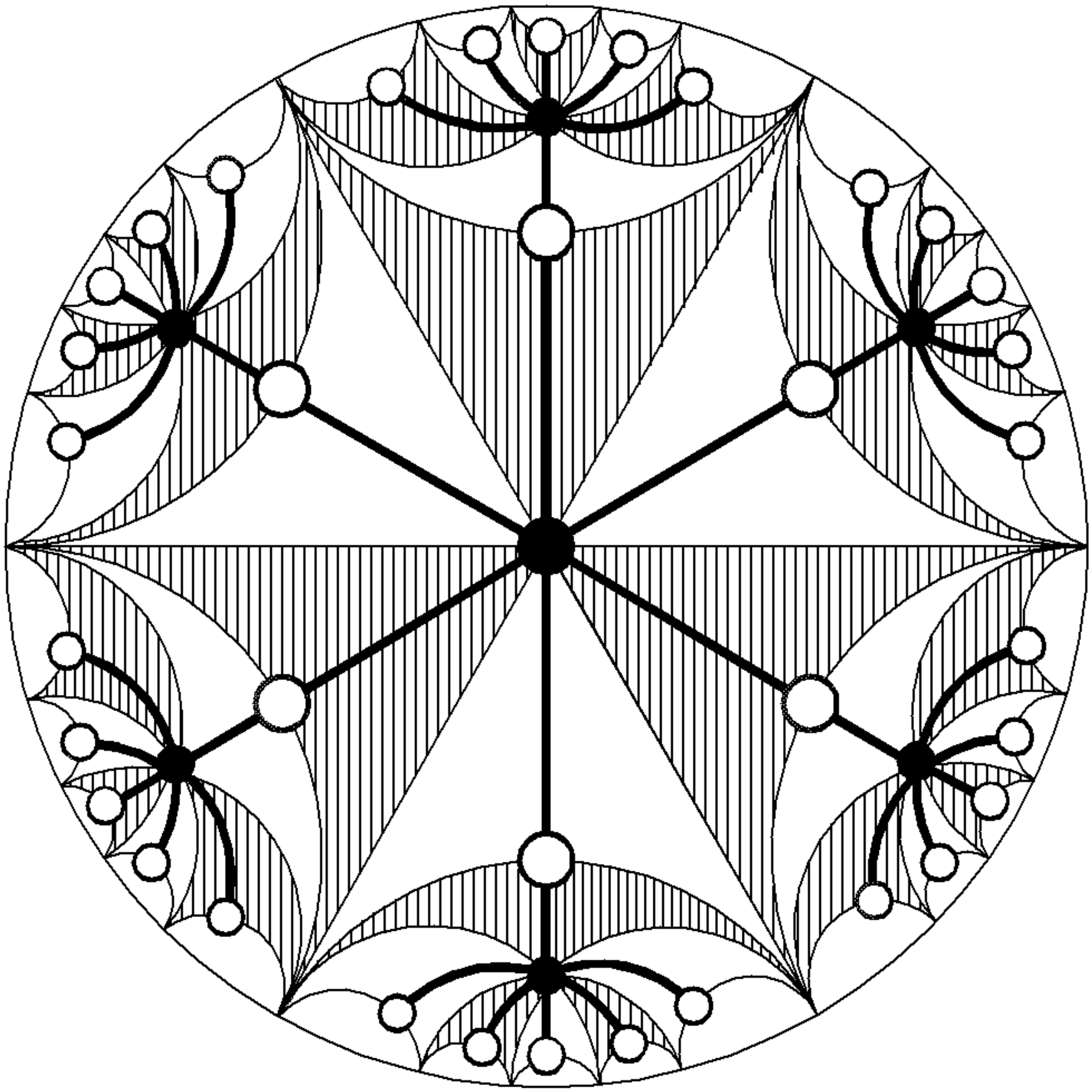


FIGURE 4.3. $\mathcal{Y}_{6,2}$ embedded in the tessellation induced by $W(\infty, 3, \infty)$

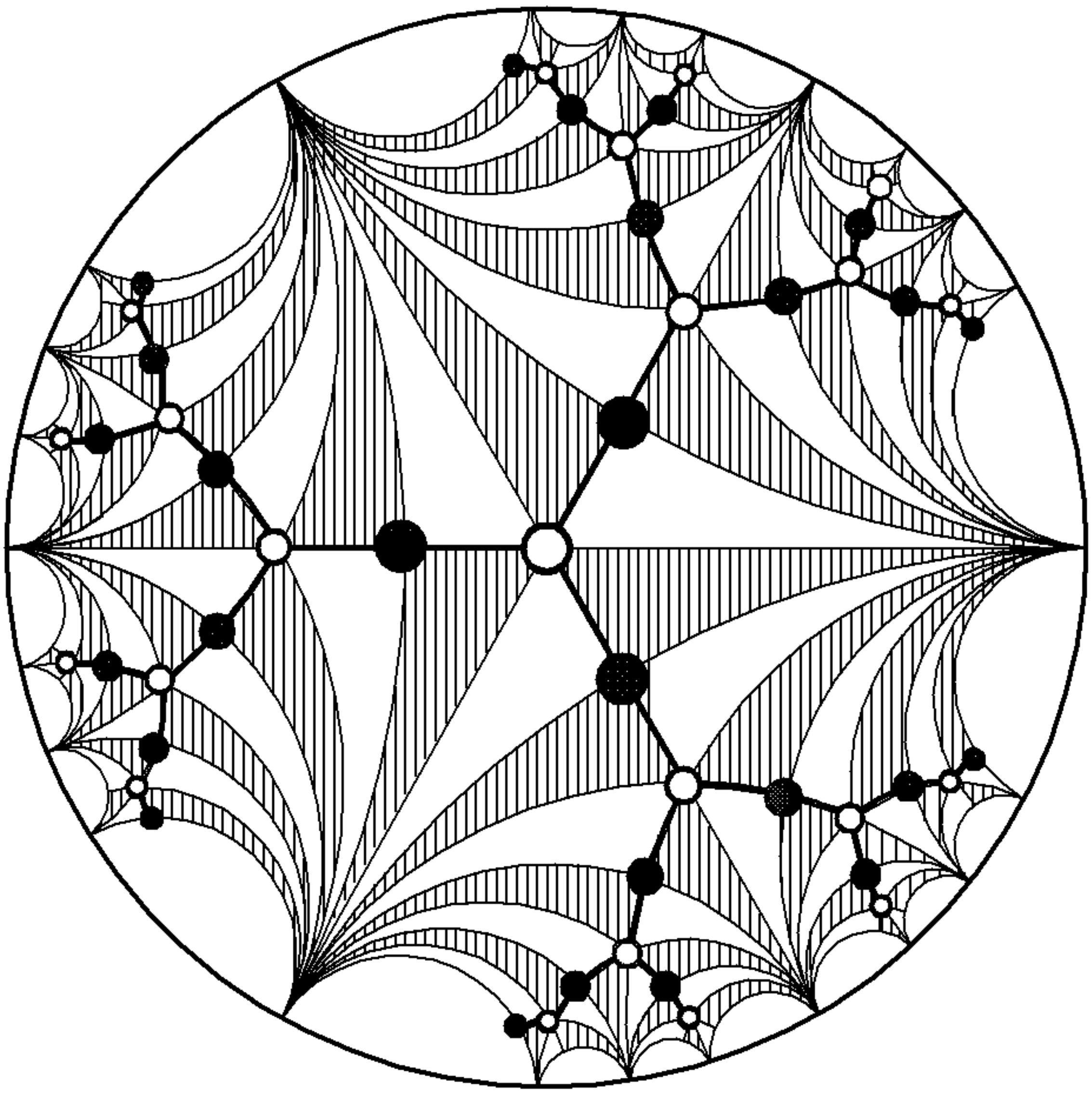


FIGURE 4.4. $\mathcal{Y}_{3,2}$ embedded in the tessellation induced by $W(\infty, 3, 2)$

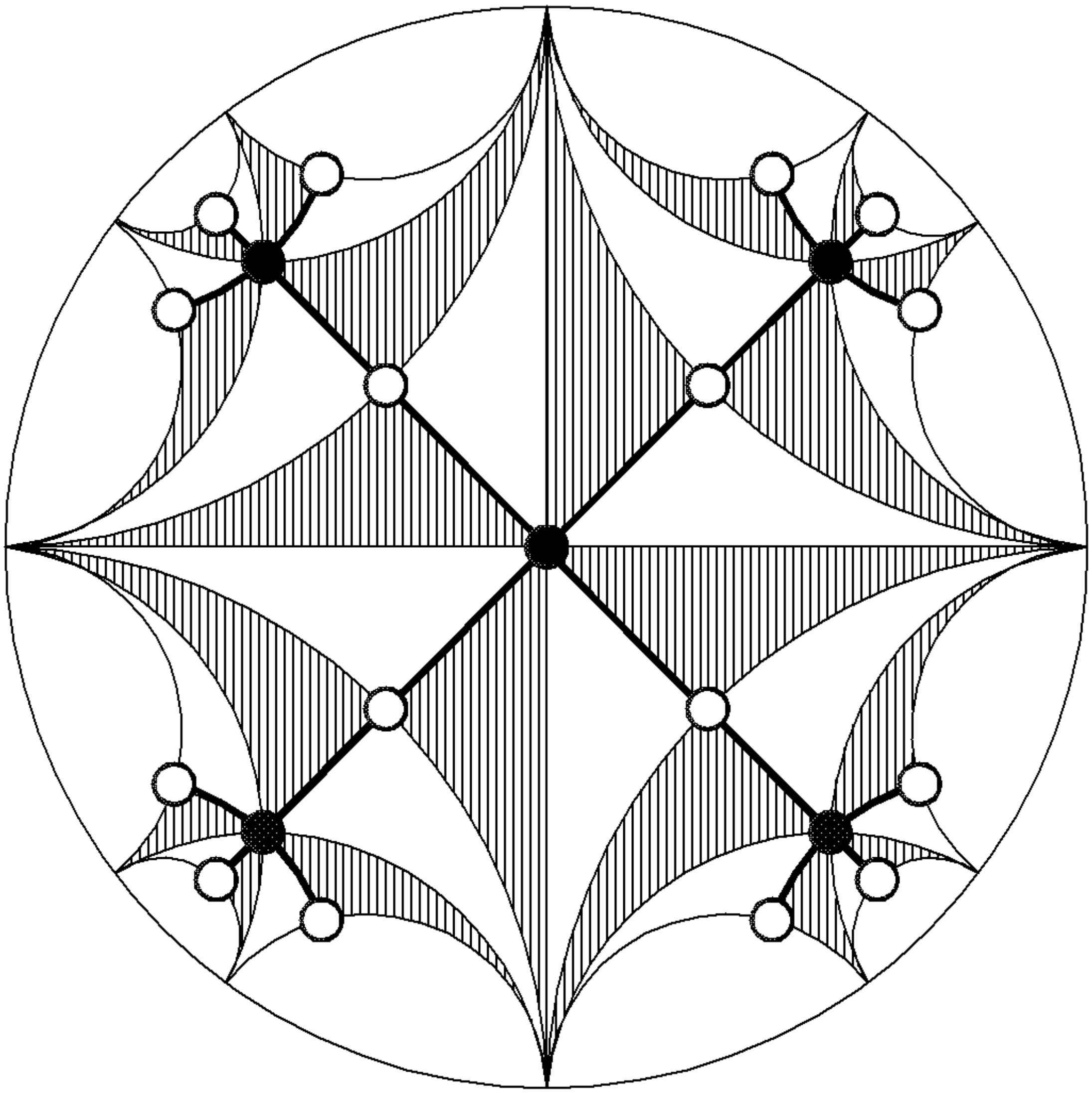


FIGURE 4.5. $\mathcal{Y}_{4,2}$ embedded in the tessellation induced by $W(\infty, 4, 2)$

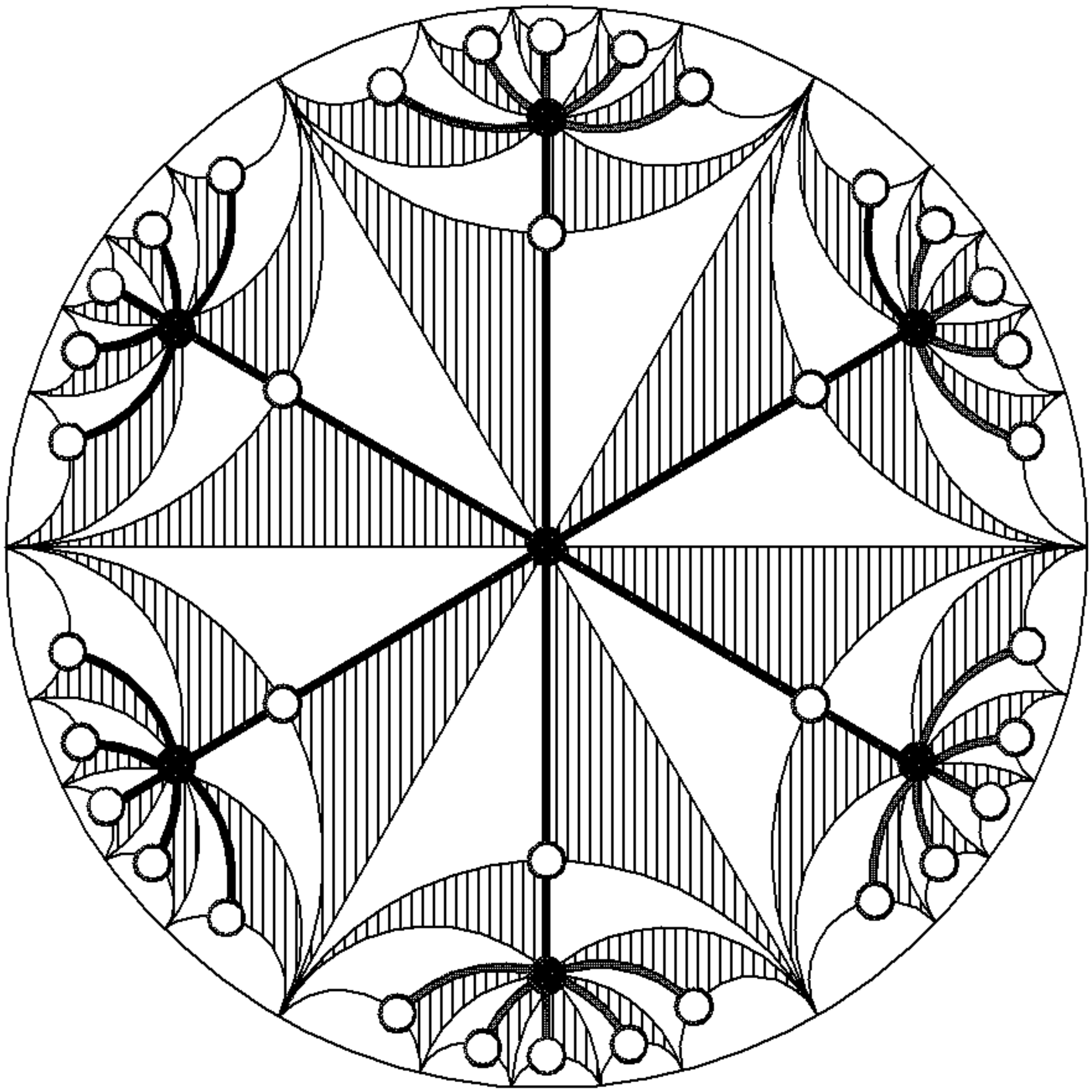


FIGURE 4.6. $\mathcal{Y}_{6,2}$ embedded in the tessellation induced by $W(\infty, 6, 2)$

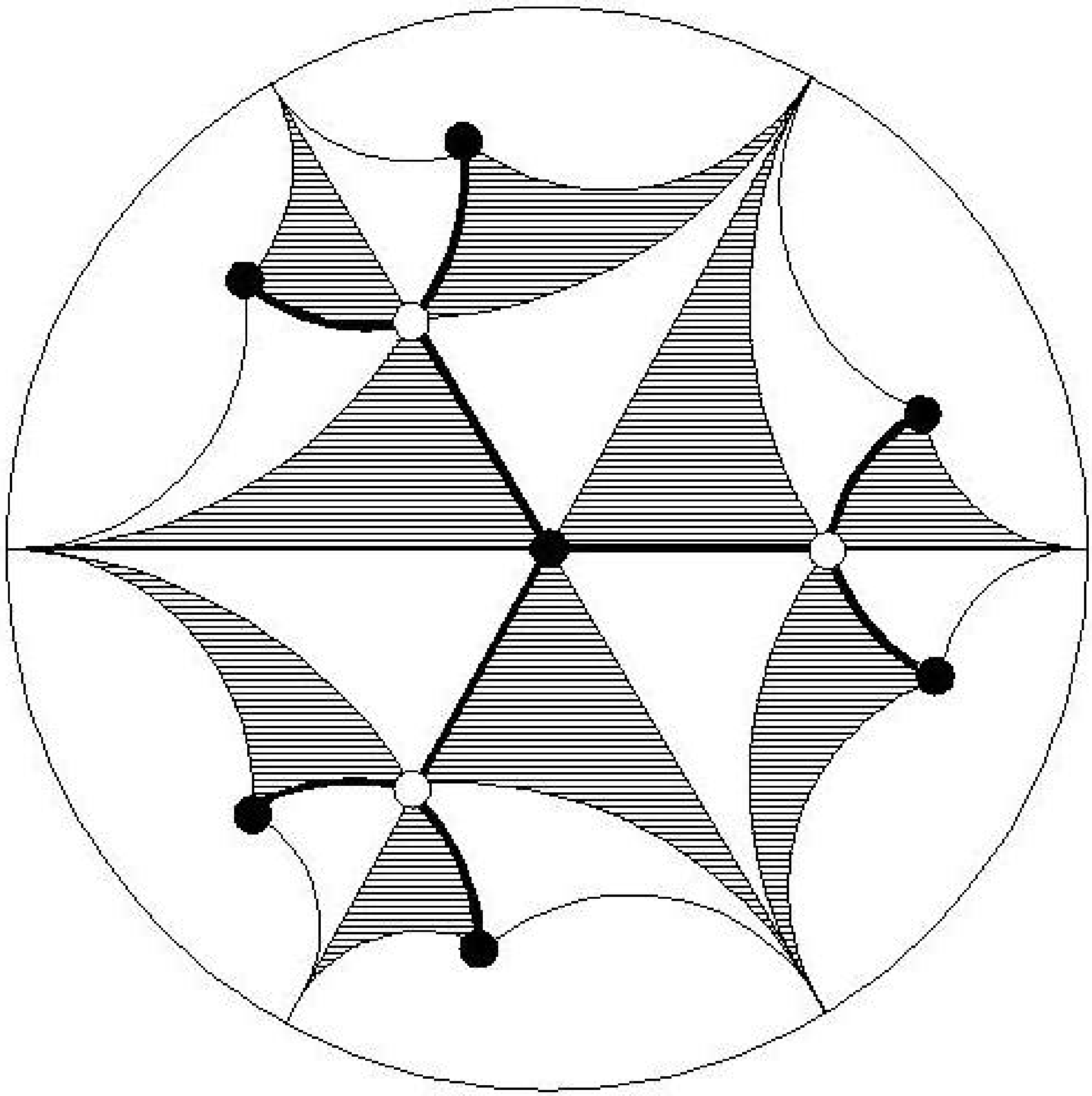


FIGURE 4.7. $\mathcal{Y}_{3,3}$ embedded in the tessellation induced by $W(\infty, 3, 3)$

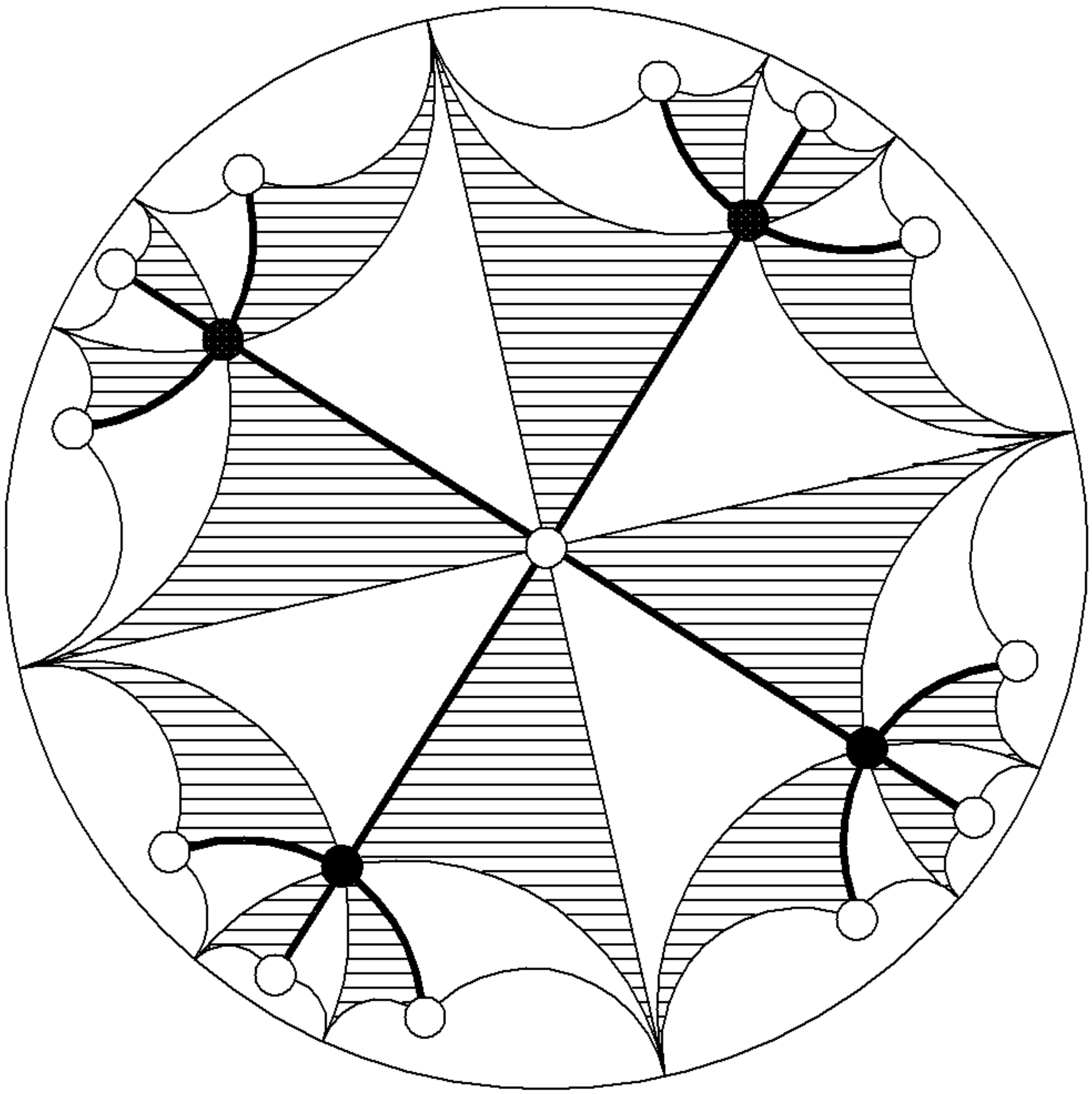


FIGURE 4.8. $\mathcal{Y}_{4,4}$ embedded in the tessellation induced by $W(\infty, 4, 4)$

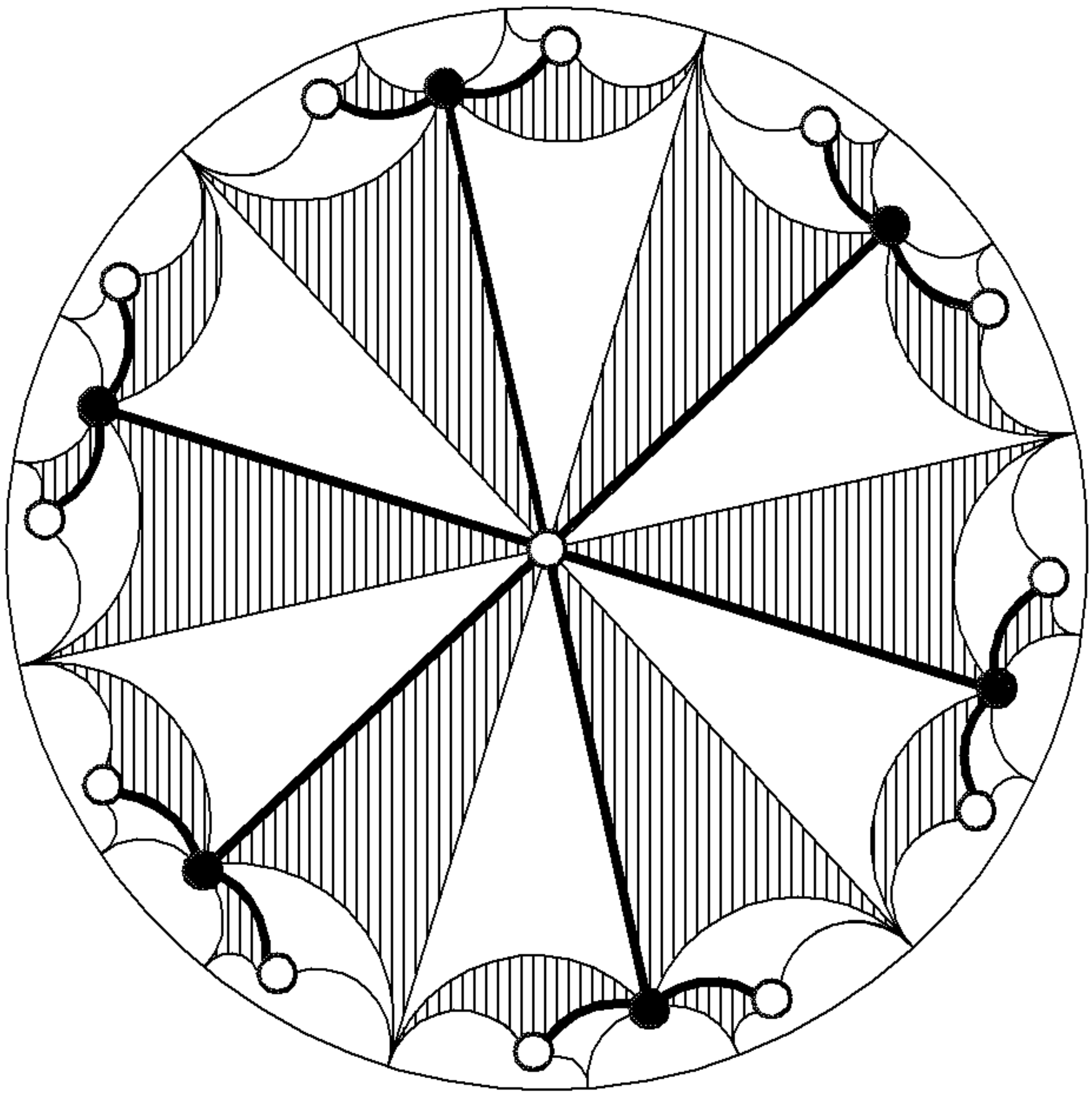


FIGURE 4.9. $\mathcal{Y}_{6,6}$ embedded in the tessellation induced by $W(\infty, 6, 6)$

4.2. **Quotient graph of groups for W on $\mathcal{Y}$.** For W as in Section 3, we have associated a tree $\mathcal{Y}$ to W that is naturally associated to the tessellation of W on the Poincaré disk. In fact, the action of W the Poincaré disk induces an action of W on $\mathcal{Y}$. If $\mathcal{F}$ is a fundamental triangle for the action of W on the Poincaré disk, then the intersection of the closure of $\mathcal{F}$ with $\mathcal{Y}$ will be a fundamental domain for the action of W on $\mathcal{Y}$.

Thus if $W = W(\infty,\infty,\infty)$, then the quotient graph for the action of W on $\mathcal{Y} = \mathcal{Y}_{3,2}$ consists of a tripod; namely a vertex of degree 3 (at the barycenter of $\mathcal{F}$) together with 3 terminal vertices (at the barycenters of edges of $\mathcal{F}$).

If $W \neq W(\infty,\infty,\infty)$ then the intersection of the closure of $\mathcal{F}$ with $\mathcal{Y}$ consists of a single edge with its initial and terminal vertices.

We can then form the quotient graph of groups for the action of W on $\mathcal{Y}$ by associating to the vertices and edges of the quotient graph $W\backslash\mathcal{Y}$ the stabilizers of liftings to $\mathcal{Y}$ of these vertices and edges for the action of W on the Poincaré disk. By applying the fundamental Bass-Serre theory for reconstructing group actions on trees, this gives a graph of groups presentation for W.

In this subsection, we explicitly compute the quotient graph of groups for the action of W on $\mathcal{Y}$ for all W that occur, except for $W = W(\infty,\infty,\infty)$ (which we consider in Section 6). For each tessellation of the Poincaré disk by W with inscribed tree $\mathcal{Y}$, let a denote the vertex at the center of the Poincaré disk, and let b denote the other vertex in the fundamental chamber of the tessellation. Then the stabilizers of vertices a and b and the edge (a,b) under the action of W are given in the following table:

Table 4.2. Stabilizers of vertices and edges

W	Stab(a)	Stab(b)	Stab$((a,b))$
$W(\infty,r,2)$	$\langle w_2,w_3 \mid w_2^2,w_3^2,(w_2w_3)^r\rangle$	$\langle w_1,w_3 \mid w_1^2,w_3^2,(w_1w_3)^2\rangle$	$\langle w_3 \mid w_3^2\rangle$
$W(\infty,r,r)$	$\langle w_2,w_3 \mid w_2^2,w_3^2,(w_2w_3)^r\rangle$	$\langle w_1,w_3 \mid w_1^2,w_3^2,(w_1w_3)^r\rangle$	$\langle w_3 \mid w_3^2\rangle$
$W(\infty,r,\infty)$	$\langle w_2,w_3 \mid w_2^2,w_3^2,(w_2w_3)^r\rangle$	$\langle w_1 \mid w_1^2\rangle$	$\{1\}$

Since the edge (a,b) is a fundamental domain for the action of W on $\mathcal{Y}$, the corresponding graph of groups consists of a single edge which we also denote (a,b) together with vertex and edge stabilizers as in the above table. The fundamental group of such a quotient graph of groups is a free product of the vertex groups amalgamated over their intersection. Thus we obtain the following graph of groups presentations for W:

$$W(\infty,r,2) \cong \langle w_2,w_3 \mid w_2^2 = w_3^2 = (w_2w_3)^r = 1\rangle *_{\langle w_3|w_3^2=1\rangle} \langle w_1,w_3 \mid w_1^2 = w_3^2 = (w_1w_3)^2 = 1\rangle\,,$$

$$W(\infty,r,r) \cong \langle w_2,w_3 \mid w_2^2 = w_3^2 = (w_2w_3)^r = 1\rangle *_{\langle w_3|w_3^2=1\rangle} \langle w_1,w_3 \mid w_1^2 = w_3^2 = (w_1w_3)^r = 1\rangle\,,$$

$$W(\infty,r,\infty) \cong \langle w_2,w_3 \mid w_2^2 = w_3^2 = (w_2w_3)^r = 1\rangle *_{\{1\}} \langle w_1 \mid w_1^2 = 1\rangle\,,$$

These presentations coincide with the ‘JSJ decompositions’ of Mihalik ([Mi]) over virtually abelian subgroups and of Ratcliffe and Tschantz ([RTs]) over FA subgroups. Our amalgamated subgroups are either trivial or cyclic of order 2, which fit both JSJ-decompositions. In particular, our work gives the amalgam presentation $W(\infty,3,2) \cong D_6 *_{\mathbb{Z}/2\mathbb{Z}} D_4$ for $W(\infty,3,2) \cong PGL_2(\mathbb{Z})$ (see Section 6.2 for more detail).

In Section 6 we also determine the quotient graph of groups for $W(\infty,\infty,\infty)$ and for the index 2 subgroup $PSL_2(\mathbb{Z})$ of $W(\infty,3,2)$. The graph of groups presentations for $W(\infty,\infty,\infty)$ and $PSL_2(\mathbb{Z})$ coincide with the well known presentations $W(\infty,\infty,\infty) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

4.3. **Amalgam decompositions of higher rank hyperbolic Weyl groups.** A group G is said to have Property (FA) if every action of G on a tree has a global fixed point. Serre showed in [Se] that if a group has Property (FA), then it cannot split as a free product with amalgamation or HNN extension. In particular, a finitely generated group with Property (FA) has finite abelianization. It is also known that Kazhdan's Property (T) implies Property (FA) of Serre ([Wa]).

From [Se] (Exercise 3, p 66), we see that if the Coxeter matrix of a finite rank Coxeter group conatins no infinities, then the group has Property (FA). We thank John Ratcliffe and Mike Mihalik for communicating the following:

Theorem 4.2. *([MT]) If the Coxeter matrix of a finite rank Coxeter group contains an infinity, then the group has a nontrivial visual amalgamated product decomposition, and so it does not have Property (FA).*

Thus a finite rank Coxeter group has Property (FA) if and only if its Coxeter matrix contains no infinities. In ([C]), the author deduced the following using the results of [DJ]:

Theorem 4.3. *Let A be a symmetrizable affine or hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a locally compact Kac-Moody group associated to A and the finite field $\mathbb{F}_q$, with q sufficiently large. If $r = rank(G) = 3$ and G has compact hyperbolic type, or if $rank(G) \geq 3$ and G has affine type, or if $4 \leq r \leq 10$ and G has hyperbolic type then*

(i) G has Property (T).

(ii) All entries in the Coxeter matrix for G are finite.

We deduce that for G as in Theorem 4.3, the associated Weyl group $W = W(A)$ has Serre's Property (FA) and hence is *not* an amalgam. In particular, the Weyl group of any Dynkin diagram containing only single bonds has Property (FA). Therefore the Weyl groups of the exceptional Lie algebras E_n, $n = 6,7,8,9,10$ are *not* amalgams.

We combine these results to obtain the following correspondence between the representation theoretic properties of the Kac-Moody group G and the properties of the Weyl group G. We refer the reader to Section 7 for a discussion of the Haagerup property and Kazhdan's Property (T).

Theorem 4.4. *Let A be a symmetrizable affine or hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a locally compact Kac-Moody group associated to A and the finite field $\mathbb{F}_q$, with q sufficiently large.*

(i) If $r = rank(G) = 2$ or if $rank(G) = 3$ and G has noncompact hyperbolic type, then G has the Haagerup property and $W = W(A)$ has a nontrivial amalgamated product decomposition. Thus W does not have Property (FA).

(ii) If $r = rank(G) = 3$ and G has compact hyperbolic type, or if $rank(G) \geq 3$ and G has affine type, or if $4 \leq r \leq 10$ and G has hyperbolic type, then G has Property (T) and $W = W(A)$ has Property (FA).

5. The Tits building X and an associated tree

Let G be a symmetrizable locally compact Kac-Moody group of rank 3 noncompact hyperbolic type. Let X be the Tits building of G. In this section, we show that there is a bihomogeneous bipartite tree $\mathcal{X}$ associated to the Tits building X of G.

5.1. **Hyperbolic buildings of rank 3 Kac-Moody groups.** Let G be a symmetrizable locally compact Kac-Moody group of rank 3 noncompact hyperbolic type over $\mathbb{F}_q$. Let X be the Tits building of G. Apartments in X are hyperbolic planes tessellated by the action of the hyperbolic Weyl group W. We construct X using the following labelling.

Let B be the minimal parabolic subgroup of the positive BN-pair (G, B, N). The maximal simplices (chambers) of X correspond to cosets G/B. Note that these chambers are 2-simplices in this case, and hence are the triangular faces in the building.

The faces of the chambers, which are the edges in X, correspond to cosets G/P_i where P_i, $i = 1, 2, 3$ are the non-maximal standard parabolic subgroups

$$\begin{aligned} P_1 &:= B \sqcup Bw_1B, \\ P_2 &:= B \sqcup Bw_2B, \\ P_3 &:= B \sqcup Bw_3B. \end{aligned}$$

The vertices of the Tits building X are given by the cosets G/P_{ij} where P_{ij} $(i, j = 1, 2, 3, i \neq j)$ are the maximal standard parabolic subgroups

$$\begin{aligned} P_{12} &:= \bigsqcup_{w \in \langle w_1, w_2 \rangle} BwB, \\ P_{23} &:= \bigsqcup_{w \in \langle w_2, w_3 \rangle} BwB, \\ P_{13} &:= \bigsqcup_{w \in \langle w_1, w_3 \rangle} BwB. \end{aligned}$$

(Note that $P_{ij} = P_{ji}$.)

The incidence relation is described as follows. The $r + 1$ vertices $v_1, \ldots, v_{r+1}$ span an r-simplex if and only if the intersection of the labels of $v_1 \cap \cdots \cap v_{r+1}$ contains a contains a coset of B. In particular, $gB \sim_i hB$ (i.e. gB and hB are i-adjacent) if and only if $gP_i = hP_i$. Note that this implies that if A and B are simplices in X, then A is a subsimplex of B if and only if the label of B is contained in the label of A. The group G acts on vertices, edges and chambers of X by left multiplication on cosets.

Every coset representative g for a coset of B in G can be written in the form

$$\chi_{i_1}(s_1) w_{i_1} \chi_{i_2}(s_2) w_{i_2} \ldots \chi_{i_n}(s_n) w_{i_n},$$

where $i_j = 1, 2, 3$, w_k is a generator of W, and $\chi_k(t)$ denotes $\chi_{\alpha_k}(t) = exp(te_k)$, $t \in \mathbb{F}_q$ and e_k are the Lie algebra generators. Moreover, by Lemma 7.4 in [R], an element in this form can also be written as

$$\chi_{i_1}(t_1) \chi_{i_2}(t_2) \ldots \chi_{i_n}(t_n) w_{i_1} w_{i_2} \ldots w_{i_n}$$

where $t_k = \pm s_k$, with sign determined by choice of Lie algebra generators. Therefore, if we require that $w_{i_1} \ldots w_{i_n}$ be a reduced word in W, this representation is unique up to choice of Lie algebra generators, and so we have a natural labelling of each chamber in X given by the Lie algbera generators. Such decompositions for coset representatives were used by Iwahori and

Matsumoto in [IM] for $\mathrm{SL}_2(K)$ where K is a p-adic field, and by Carbone and Garland in [CG] for rank 2 Kac-Moody groups over $\mathbb{F}_q$. However, the result is a local one and holds for Moufang buildings in general, and hence our case in particular, as shown in [RTi].

We can now prove the following lemma.

Lemma 5.1. *Let G be a symmetrizable locally compact Kac-Moody group over $\mathbb{F}_q$, and let X be the Tits building associated to a BN-pair for G. Every face of a chamber in X is shared by $q+1$ chambers.*

Proof: We observe that every face of a chamber in X, given by a coset of P_i in G, is shared by n chambers, where n is the number of cosets of B contained in the given coset of P_i. In particular gP_i contains the coset $g\chi_i(s)w_iB$ for all $s \in \mathbb{F}_q$, as well as the coset gB. It follows that $n = q+1$. $\square$

We also observe that if a vertex v_i is labelled by a coset of P_{ij}, then its link is the building corresponding to the Lie group over $\mathbb{F}_q$ with Weyl group W' generated by $\{w_i, w_j\}$ (see [Br], Chapters III and IV). Thus, the link of each vertex is the rank 2 building of type $A_1 \times A_1, A_2, B_2, G_2$ or $A_1^{(1)}$. In particular, every nonboundary vertex has a link corresponding to a rank 2 building of finite (spherical) type. These buildings are discussed in Subsection 5.2.

We will also make use of the retraction of a building onto an apartment. Let $\mathcal{A}$ be an apartment of X, let $\mathcal{C} \in \mathcal{A}$ be a chamber contained in $\mathcal{A}$. We recall that a retraction of X onto $\mathcal{A}$ is a unique chamber map $\rho = \rho(\mathcal{A}, \mathcal{C}) : X \longrightarrow \mathcal{A}$ which fixes $\mathcal{C}$ pointwise and maps every apartment containing $\mathcal{C}$ isomorphically onto $\mathcal{A}$. The retraction ρ preserves distances from $\mathcal{C}$ and preserves colors of vertices in X.

5.2. **Links of vertices in the building.** Let G be a rank 2 Lie group of finite type over $\mathbb{F}_q$. Then the Weyl group W of G has the presentation $\langle w_1, w_2 \mid w_1^2 = w_2^2 = (w_1w_2)^m = 1\rangle$, where $m = 2, 3, 4, 6$ for $G = A_1 \times A_1, A_2, B_2, G_2$, respectively. This implies that the standard apartment is of G is a polygon with $2m$ sides, and the full building for G is a generalized m-gon.

To determine the number of vertices and edges and the degrees of the vertices in the buildings of G, we use a labelling that is analogous to the one used in the Rank 3 case. Let B be the minimal parabolic subgroup of the positive BN-pair (G, B, N). We label the maximal simplices (which are edges in this case) by the cosets G/B. We label the vertices with the cosets G/P_i where P_i, $i = 1, 2, 3$ are the maximal standard parabolic subgroups

$$\begin{aligned} P_1 &:= B \sqcup Bw_1B, \\ P_2 &:= B \sqcup Bw_2B. \end{aligned}$$

Note that the two standard parabolic subgroups give us a natural bipartion of the vertices of the m-gon.

As before, all cosets of B can be written in the form

$$\chi_{i_1}(t_1)\chi_{i_2}(t_2)\dots\chi_{i_n}(t_n)w_{i_1}w_{i_2}\dots w_{i_n}B,$$

where $i_j = 1, 2$ in this case, and $i_{j+1} = i_j + 1 \bmod 2$ for $1 \le j \le (n-1)$. Since we require that $w = w_{i_1}w_{i_2}\dots w_{i_n}$ be a reduced word in W, we note that $0 \le n \le m$. If $0 < n < m$, then i_1 can equal either 1 or 2, and each choice will give us a different element of W. If $n = 0$, $w = 1$, and if $n = m$, then w is the unique word in W of longest length. We also have q choices for each t_k. Therefore, a simple counting argument shows that we have $(1 + 2\sum_{i=1}^{m-1} q^i + q^m)$ different cosets of B in G, and hence the generalized m-gon has $(1 + 2\sum_{i=1}^{m-1} q^i + q^m)$ edges.

We can apply Lemma 5.1 to these rank 2 buildings as well, and so each vertex has degree $q+1$. We also note that the cosets of P_i $(i = 1, 2)$ all have the form

$$\chi_{i_1}(t_1)\chi_{i_2}(t_2)\ldots\chi_{i_n}(t_n)w_{i_1}w_{i_2}\ldots w_{i_n}P_i,$$

with the additional requirement that $i_n \neq i$. This implies that $0 \leq n < m$, since a word in W of length m can always be taken to end in w_i. This also implies that $i_1 = i + n \bmod 2$, since we have the condition that $i_{j+1} = i_j + 1 \bmod 2$ for $1 \leq j \leq (n-1)$. Therefore, given a fixed parabolic subgroup P_i, the word $w = w_{i_1}w_{i_2}\ldots w_{i_n}$ is determined entirely by n, but we still have q choices for each t_k. Therefore there are $1 + q + ... + q^{(m-1)}$ cosets of P_i in G for each i, and so the generalized m-gon has $1 + q + ... + q^{(m-1)}$ vertices of each color.

5.3. **Tree associated to the Tits building X.**

Theorem 5.2. *Let G be a symmetrizable locally compact Kac-Moody group of rank 3 noncompact hyperbolic type. Let X be the Tits building of G. For every apartment $\mathcal{A}$ of X, let $\mathcal{Y}_{\mathcal{A}}$ be the tree associated to the Weyl group W of G and embedded in $\mathcal{A}$. Define the simplicial complex*

$$\mathcal{X} = \bigcup_{\textit{apartments } \mathcal{A} \textit{ of } X} \mathcal{Y}_{\mathcal{A}}.$$

Then $\mathcal{X}$ has the following properties:

(1) The vertices of $\mathcal{X}$ are barycenters of faces of X, barycenters of edges of X, or corners of triangles in X. The retraction ρ of X onto an apartment $\mathcal{A}$ induces a retraction of $\mathcal{X}$ onto $\mathcal{Y}_{\mathcal{A}}$. This retraction takes barycenters of faces and edges to barycenters of faces and edges respectively, and takes corners of triangles to corners of triangles, preserving types of vertices.

(2) $\mathcal{X}$ is connected and a tree.

(3) The tree $\mathcal{X}$ is bihomogeneous and bipartite. If we let $f(q,r) = q^r + 2\sum_{i=1}^{r-1} q^i + 1$ and $g(q,r) = \sum_{i=0}^{r-1} q^i$, then the degrees of the vertices of $\mathcal{X}$ are as given in Table 5.1 below.

TABLE 5.1. Tree embedded in the Tits building

Type	$\mathcal{Y}_{a,b}$ *in* $\mathcal{A}$	$\mathcal{X}_{a,b}$ *in* X
(∞,∞,∞)	$\mathcal{Y}_{3,2}$	$\mathcal{X}_{3,\, q+1}$
(∞,r,∞)	$\mathcal{Y}_{2r,2}$	$\mathcal{X}_{f(q,r),\, q+1}$
$(\infty,r,2)$	$\mathcal{Y}_{r,2}$	$\mathcal{X}_{g(q,r),\, q+1}$
(∞,r,r)	$\mathcal{Y}_{r,r}$	$\mathcal{X}_{g(q,r),\, g(q,r)}$

Proof: (1) Each vertex of $\mathcal{X}$ is a vertex in some $\mathcal{Y}_{\mathcal{A}}$, and hence the barycenter of a face or edge of $\mathcal{A}$, or a corner of a triangle in $\mathcal{A}$. We can thus identify each vertex of $\mathcal{X}$ with a simplex of $\mathcal{A}$, and hence of X. If $x \in V\mathcal{X}$ is a vertex at a barycenter of a face in $\mathcal{Y}_{\mathcal{A}}$ in the apartment $\mathcal{A}$, we identify x with a chamber)σ_x of X. If $x \in V\mathcal{X}$ is a vertex at a barycenter of an edge in $\mathcal{Y}_{\mathcal{A}}$, we identify x with an edge σ_x of X. If $x \in V\mathcal{X}$ is a vertex at a corner of a triangle in an apartment $\mathcal{A}$, we identify x with a vertex σ_x at a corner of a triangle in X.

Now fix an apartment $\mathcal{A}_0$ of X. Then the retraction ρ of X onto $\mathcal{A}_0$ induces an isomorphism of subtrees $\mathcal{Y}_{\mathcal{A}} \longrightarrow \mathcal{Y}_{\mathcal{A}_0}$ for all apartments $\mathcal{A}$ of X.

If σ_x is a chamber of X, then $\rho(\sigma_x)$ is at the barycenter of a face in $\mathcal{A}$. If σ_x is an edge of X, then $\rho(\sigma_x)$ is at the barycenter of an edge in $\mathcal{A}$. If σ_x is a vertex at a corner of a triangle in X, then $\rho(\sigma_x)$ is at the corner of a triangle in $\mathcal{A}$.

(2) Pick any two vertices x and y in $\mathcal{X}$. Then by building axiom (B1), there exists an apartment $\mathcal{A}$ in X such that $x, y \in \mathcal{A}$, and in particular, $x, y \in \mathcal{Y}_{\mathcal{A}}$. Thus, any two vertices in $\mathcal{X}$ lie in the same connected component, and hence it follows that $\mathcal{X}$ is connected.

To show that $\mathcal{X}$ is a tree, we have to make sure that $\mathcal{X}$ does not contain closed circuits. To continue the proof of the theorem, we make use of the following lemma which we prove here.

Lemma 5.3. *If x and y are adjacent vertices of $\mathcal{X}$, then there is a chamber of the Tits building X containing x and y.*

Proof (of Lemma 5.3): We prove the lemma in cases. Since $\mathcal{X}$ is a union of copies of $\mathcal{Y}$, we have the following 3 possibilities for pairs of adjacent vertices of $\mathcal{X}$:

Case (a): If x is a vertex at a barycenter of a face in X and y is adjacent to x in $\mathcal{X}$, then y is at a barycenter of an edge. In this case, there is a chamber (ideal triangle) of X containing x and y.

Case (b): If x is a vertex at a barycenter of an edge in X, y is adjacent to x in $\mathcal{X}$ and we are not in Case (a), then y is at the corner of a triangle and is opposite x. In this case, the triangle is a chamber of X containing both x and y.

Case (c): If x is a vertex at a corner of a triangle in X, y is adjacent to x in $\mathcal{X}$ and we are not in case (b) then x is a vertex at a corner of the same triangle. In this case, the triangle is a chamber of X containing both x and y. $\square$

We now continue the proof of Theorem 5.2. We claim that $\mathcal{X}$ has no closed paths. Suppose there is a closed path $v_1, v_2, \dots v_n = v_1$ in $\mathcal{X}$ containing n vertices, for $n \geq 3$. Pick any two adjacent vertices v_k and v_{k+1} in this closed path, and another vertex on the closed path, say v_m such that $v_m \neq v_k$ and $v_m \neq v_{k+1}$. Let γ_k be the path $(v_k, v_{k-1}, \dots, v_m)$ in the loop (with indices taken cyclically, and let γ_{k+1} be the path $(v_{k+1}, v_{k+2}, \dots, v_m)$ in the loop.

By Lemma 5.3, there is a chamber $\mathcal{C}$ containing v_k, v_{k+1} and the edge e between them. By building axiom (B1), there is an apartment $\mathcal{A}$ containing $\mathcal{C}$ and v_m. Let ρ be the retraction of X onto $\mathcal{A}$ that fixes $\mathcal{C}$ pointwise and maps every apartment containing $\mathcal{C}$ isomorphically onto $\mathcal{A}$. Thus ρ fixes v_m. Since ρ is a simplicial map, it takes the path γ_k from v_k to v_m to a path σ_k in $\mathcal{Y}_{\mathcal{A}}$ from v_k to v_m, and the path γ_{k+1} from v_{k+1} to v_m to a path σ_{k+1} in $\mathcal{Y}_{\mathcal{A}}$ from v_{k+1} to v_m. Since $v_k \neq v_{k+1}$, σ_k and σ_{k+1} are distinct paths connecting v_k to v_m and v_{k+1} to v_m respectively in $\mathcal{A}$. However, ρ also fixes the edge e in $\mathcal{C}$ that connects v_k to v_{k+1}. Thus, the image under ρ of the closed path $v_1, v_2, \dots v_n = v_1$ in $\mathcal{X}$ is a closed path in $\mathcal{A}$. This is a contradiction, since $\rho(\mathcal{X}) \subseteq \mathcal{Y}_{\mathcal{A}}$, and $\mathcal{Y}_{\mathcal{A}}$ is a tree. Thus, there are no closed paths in $\mathcal{X}$ containing 3 or more vertices.

It is easy to see that there are no closed paths in $\mathcal{X}$ containing only 2 vertices v_1 and v_2. Any 2 such vertices are adjacent in $\mathcal{X}$ and are contained in a single apartment $\mathcal{A}$ along with the 2 edges e_1 and e_2 forming a closed path between them. The retraction of X onto $\mathcal{A}$ fixes v_1, v_2, and e_1, e_2 so a closed path in $\mathcal{X}$ containing 2 vertices would be a closed path in $\mathcal{A}$, which is a contradiction.

(3) There is a natural bipartition of the vertices of $\mathcal{X}$ inherited from $\mathcal{Y}$ and preserved by ρ. Since the local structure of the Tits building is the same for each chamber, $\mathcal{X}$ is bihomogeneous. In particular, we have the following 4 cases:

(i) W is of type (∞, ∞, ∞).

(ii) W is of type (∞, r, ∞), $r = 2, 3$.

(iii) W is of type $(\infty, r, 2)$, $r = 3, 4, 6$.

(iv) W is of type (∞, r, r), $r = 3, 4, 6$.

In case (i), we have $\mathcal{Y} = \mathcal{Y}_{3,2}$. In each apartment of X, $\mathcal{Y}$ has a vertex of degree 3 at the barycenter of the face of each ideal triangle and a vertex of degree 2 at the barycenter of each edge of each ideal triangle. In $\mathcal{X}$, each vertex at the barycenter of a face of a triangle is connected to the 3 edges of the triangle, and hence still has degree three. Each vertex at the barycenter of an edge is connected to vertices at the barycenter of the chambers that share that edge. By Lemma 5.1, each edge lies on $q + 1$ chambers, and so the degree of vertices at the barycenter of edges has degree $q + 1$.

In case (ii), we have $\mathcal{Y} = \mathcal{Y}_{r,2}$. In each apartment of X, $\mathcal{Y}$ has vertices at (non-boundary) corners of triangles in the tessellation which have degree r and vertices at barycenters of edges of the triangles which have degree 2. In the building X, the link of each vertex at a non-bounday corner is a generalized r-gon, and the vertex connects to each edge in the r-gon. As discussed in Subsection 5.2, the number of edges in the r-gon is $(q^r + 2\sum_{i=1}^{r-1} q^i + 1)$. Each vertex at the barycenter of an edge is connected to vertices at the barycenter of the chambers that share that edge. By Lemma 5.1, each edge lies on $q + 1$ chambers, and so the degree of vertices at the barycenter of edges has degree $q + 1$.

In case (iii), we have $\mathcal{Y} = \mathcal{Y}_{r,2}$. In each apartment of X, $\mathcal{Y}$ has vertices at (non-boundary) corners of triangles in the tessellation which have degree r and degree 2. The link of each vertex of degree n is a generalized n-gon, and the vertex is connected to each of the vertices of a single color in the n-gon. By the results of Subsection 5.2, we see that the number of vertices of a single color in the n-gon is $\sum_{i=0}^{n-1} q^i$. (Note that if $n = 2$, $\sum_{i=0}^{n-1} q^i = q + 1$.)

In case (iv), we have $\mathcal{Y} = \mathcal{Y}_{r,r}$. In each apartment of X, $\mathcal{Y}$ has a vertex at every (non-boundary) corner of triangles in the tessellation, and each vertex has degree r. As in the previous case, the link of each vertex of degree n is a generalized n-gon, and the vertex is connected to each of the vertices of a single color in the n-gon. The number of vertices of a single color in the n-gon is $\sum_{i=0}^{n-1} q^i$. $\square$

6. Examples

Let G be a symmetrizable locally compact Kac-Moody group of rank 3 noncompact hyperbolic type. Let X be the Tits building of G. In this section we describe X and its inscribed tree $\mathcal{X}$ in more detail in certain cases.

6.1. $W(\infty, \infty, \infty)$**.** When $W = W(\infty, \infty, \infty)$ we can describe the tree $\mathcal{X}$ explicitly. The tree $\mathcal{Y}$ associated to W is $\mathcal{Y} = \mathcal{Y}_{3,2}$. In each apartment of X, $\mathcal{Y}$ has a vertex of degree 3 at the barycenter of the face of each ideal triangle and a vertex of degree 2 at the barycenter of each edge of each ideal triangle.

Vertices that had degree 3 in $\mathcal{Y}$ have degree 3 in $\mathcal{X}$. Vertices that had degree 2 in $\mathcal{Y}$ have degree $q+1$ in $\mathcal{X}$. Thus if $W = W(\infty,\infty,\infty)$, the tree $\mathcal{X} = \mathcal{X}_{3,q+1}$ is naturally inscribed in X and G acts on $\mathcal{X}$ without inversions.

For the action of $W = W(\infty,\infty,\infty)$ on the Poincaré disk, we take a fundamental triangle $\mathcal{F}$ to be the central triangle in the tessellation. The quotient graph for the action of W on $\mathcal{Y} = \mathcal{Y}_{3,2}$ then consists of a tripod; namely a vertex of degree 3 (at the barycenter of $\mathcal{F}$) together with 3 terminal vertices (at the barycenters of edges of $\mathcal{F}$).

To determine the stabilizers of vertices and edges of $\mathcal{Y} = \mathcal{Y}_{3,2}$ under the action of $W(\infty,\infty,\infty)$, we identify $W(\infty,\infty,\infty)$ with the subgroup of $W(\infty,3,2)$ generated by w_1, $w_2w_1w_2$ and $w_3w_2w_1w_2w_3$, where w_1, w_2, and w_3 are the generators of $W(\infty,3,2)$. The isomorphism $\pi : W \longrightarrow PGL_2(\mathbb{Z})$ is given in Section 6.2. For this choice of generators, it is convenient to identify the center of the Poincaré disk with the point $e^{i\pi/3} \in \mathbb{H}$.

It is easy to check that no element of $W(\infty,\infty,\infty)$ fixes the point $e^{i\pi/3}$ at the center of the Poincaré disk. We associate the trivial group $\{1\}$ to the vertex of degree 3 in the quotient graph $W/\mathcal{Y}$. The stabilizers of the barycenters of edges of $\mathcal{F}$ are all conjugate to the stabilizer of i under the action of $W(\infty,\infty,\infty)$: this is the subgroup $\mathbb{Z}/2\mathbb{Z}$. We associate the vertex groups $\mathbb{Z}/2\mathbb{Z}$ to the remaining (terminal) vertices of $W/\mathcal{Y}$. The fundamental group of this graph of groups is $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

6.2. $W(\infty,3,2)$**.** We consider the following 3 generalized Cartan matrices A_{13}, A_{14} and A_{15}, corresponding to items 13, 14 and 15 in Table 3.1b in Section 3.

$$A_{13} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \ A_{14} = \begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \ A_{15} = \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

In each of these cases A has non-compact hyperbolic type and the Weyl group W is

$$W = \langle w_1, w_2, w_3 \mid w_i^2 = (w_1w_2)^\infty = (w_2w_3)^3 = (w_3w_1)^2 = 1\rangle,$$

which is isomorphic to $PGL_2(\mathbb{Z})$ ([FF]). The isomorphism $\pi : W \longrightarrow PGL_2(\mathbb{Z})$ is given by the map ([K])

$$w_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ w_2 \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \ w_3 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let G be a symmetrizable locally compact Kac-Moody group of noncompact hyperbolic type over the finite field $\mathbb{F}_q$ corresponding to a generalized Cartan matrix A_{13}, A_{14} or A_{15}. In the Tits building X of G, each apartment is a copy of the hyperbolic plane, tessellated by the action of $PGL_2(\mathbb{Z})$. The action of W on the upper-half plane is given by:
If $ad - bc = 1$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \ = \ \frac{az+b}{cz+d},$$

and if $ad - bc = -1$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \ = \ \frac{a\overline{z}+b}{c\overline{z}+d}.$$

The group $PSL_2(\mathbb{Z})$ is a subgroup of index 2 in $PGL_2(\mathbb{Z})$ consisting of matrices of determinant 1. Let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $S \simeq w_3w_1$ and $TS \simeq w_2w_3$ generate $PSL_2(\mathbb{Z})$. Let $\mathcal{F}$ be the fundamental domain for $PSL_2(\mathbb{Z})$ on $\mathbb{H}$. Then S rotates $\mathcal{F}$ about the point i, fixing i, and TS fixes $e^{i\pi/3}$.

Let $\mathcal{Y} = \mathcal{Y}_{3,2}$ be the tree of $PSL_2(\mathbb{Z})$ in $\mathbb{H}$ as in [Se]. The vertices of $\mathcal{Y}$ are the $PSL_2(\mathbb{Z})$-translates of the endpoints of the arc from i to $e^{i\pi/3}$ along the geodesic $x^2 + y^2 = 1$ bounding the fundamental chamber in $\mathbb{H}$ for the action of the extended modular group $PGL_2(\mathbb{Z})$. The $PSL_2(\mathbb{Z})$ translates of i have degree 2 in $\mathcal{Y}$ and the $PSL_2(\mathbb{Z})$ translates of $e^{i\pi/3}$ have degree 3. The subgroup $\langle S\rangle$ of $PSL_2(\mathbb{Z})$ stabilizes vertices of degree 2 in $\mathcal{Y}$, and the subgroup $\langle TS\rangle$ stabilizes vertices of degree 3.

Figure 6.1 shows the upper half plane tessellated by $PGL_2(\mathbb{Z})$ and with the inscribed tree $\mathcal{Y} = \mathcal{Y}_{3,2}$. This is a modification of diagram 15 in [M].

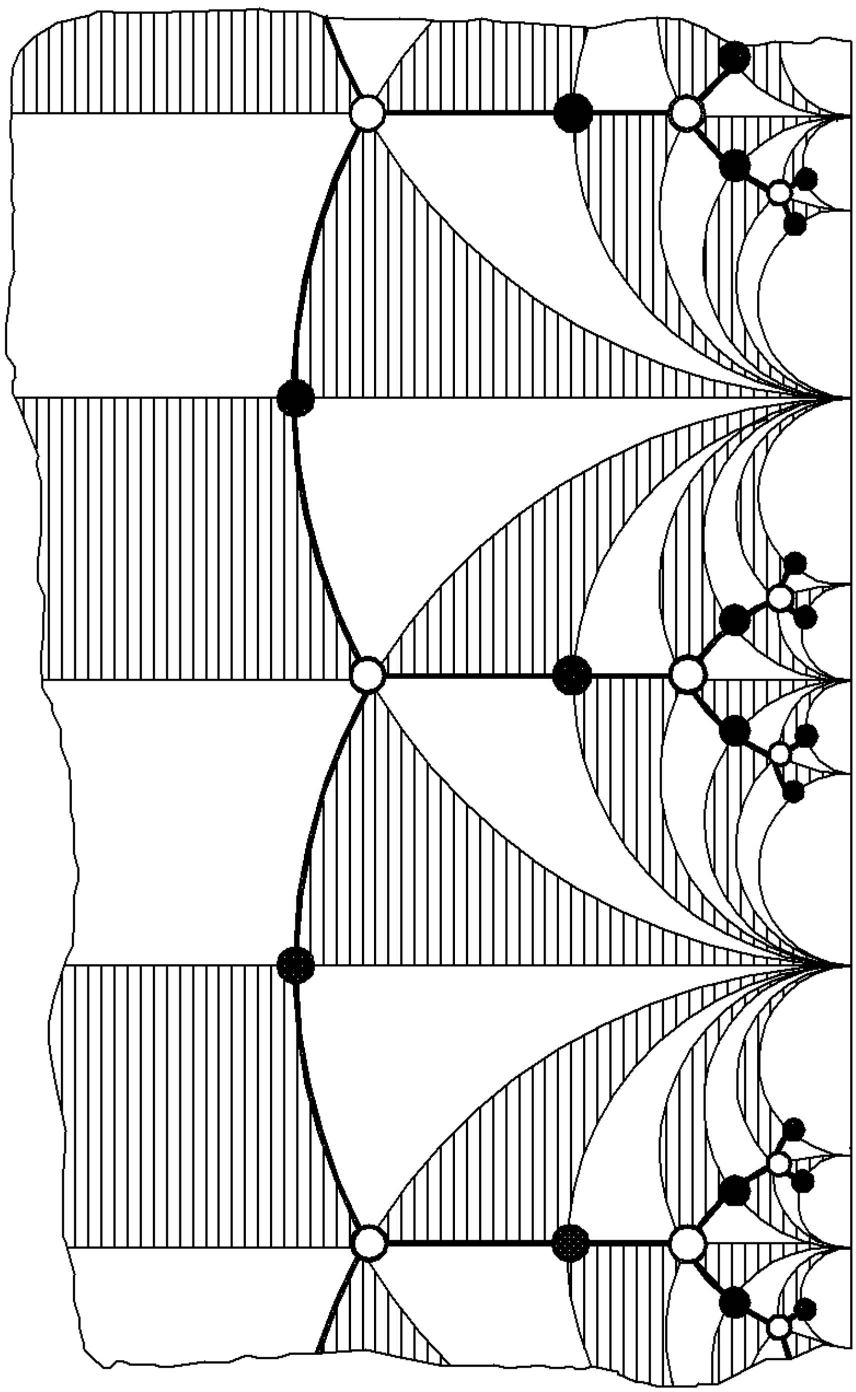


Figure 6.1. Upper halfplane model of Figure 4.4

The quotient graph for $PSL_2(\mathbb{Z})$ on $\mathcal{Y}$ therefore consists of 2 vertices: one corresponding to the orbit of vertices of degree 2 (referring to Figure 6.1, such a vertex is colored black), the other corresponding to the orbit of vertices of degree 3 (referring to Figure 6.1, such a vertex is colored white). The stabilizer of a lifting to $\mathcal{Y}$ of a black vertex is the group $\langle S\rangle \cong \mathbb{Z}/2\mathbb{Z}$. The stabilizer of a lifting to $\mathcal{Y}$ of a white vertex is the group $\langle TS\rangle \cong \mathbb{Z}/3\mathbb{Z}$. The fundamental group of this quotient graph of groups is the amalgam $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. Applying the Bass-Serre correspondence between group actions on trees and quotient graphs of groups, we get the familiar isomorphism $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

The building X has 3 types of vertices, denoted by cosets in G of P_{12}, P_{23} and P_{13}, which are the maximal parabolic subgroups of G. We have $P_{12} = B\langle w_1, w_2\rangle B$, $P_{23} = B\langle w_2, w_3\rangle B$, $P_{13} = B\langle w_1, w_3\rangle B$. For a vertex gP_{ij} of type ij, the intersection of the link of gP_{ij} with any apartment $\mathcal{A}$ containing this vertex is a Coxeter complex corresponding to $W_{ij} = \langle w_1, w_2\rangle$. Thus, this intersection is a closed circuit of length 4 for $ij = 13$ and of length 6 for $ij = 23$. For $ij = 13$, the intersection of $\mathcal{A}$ with the link of the vertex P_{12} 'at ∞' is a bi-infinite line isomorphic to $\mathbb{Z}$. The full link of P_{ij} in X is a generalized m_{ij}-gon (where a generalized ∞-gon is a tree with no end points).

The tree $\mathcal{Y}$ has vertices corresponding to the non-boundary vertices of the fundamental apartment. Every apartment $\mathcal{A}$ of X contains an isomorphic copy of $\mathcal{Y}$ denoted $\mathcal{Y}_{\mathcal{A}}$. Then

$$\mathcal{X} = \bigcup_{\text{apartments } \mathcal{A} \text{ of } X} \mathcal{Y}_{\mathcal{A}}$$

is connected, and is a tree on which G acts without inversions. The tree $\mathcal{X}$ is locally finite over $\mathbb{F}_q$ and has two types of vertices - those inherited from vertices of degree 2 and of degree 3 in $\mathcal{Y}$. Vertices of degree 2 in $\mathcal{Y}$ correspond to cosets of P_{13} in G and have degree $(q+1)$ in $\mathcal{X}$, and vertices of degree 3 in $\mathcal{Y}$ correspond to cosets of P_{23} in G and have degree (q^2+q+1) in $\mathcal{X}$.

6.3. **Lorentzian but not hyperbolic.** Consider the infinite family of generalized Cartan matrices

$$A(m) = \begin{pmatrix} 2 & -m & 0 \\ -m & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

where $m \geq 3$. Each generalized Cartan matrix in this infinite family is Lorentzian but not hyperbolic since each $A(m)$ contains the rank 2 hyperbolic generalized Cartan matrix

$$H(m) = \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$$

as a proper submatrix. The Weyl group of $A(m)$ is the group $W(3, 2, \infty)$ for each $m \geq 3$. However the fundamental domain for $W(A(m))$ on the Poincaré disk no longer has finite volume. Thus under the action of $W(A(m))$, there is no tessellation of the Poincaré disk in the usual sense.[1]

[1]We thank Axel Kleinschmidt for drawing this class of examples to our attention.

7. The Haagerup property and actions on trees

Let G be a locally compact group. We say that there is a continuous, isometric action of G on some affine Hilbert space H if there is a a continuous map $G \longrightarrow Isom(H)$. We say that the action of G on H is metrically proper if for any bounded subset B in H the set

$$K(G,B) := \{g \in G \ s.t. \ gB \cap B \neq \emptyset\}$$

has compact closure in G. The locally compact group G satisfies the *Haagerup property*, (or is *a-T-menable*) if it admits a continuous, isometric, proper action on an affine Hilbert space. The Haagerup property is a strong negation of Property (T) which states that every continuous action of G by isometries on a Hilbert space has a fixed point.

Concerning the Haagerup property, a theorem of Haagerup ([H]) states that a free group Γ has the Haagerup property. It follows that if Γ is a discrete group acting freely on the vertices of a Bruhat-Tits building X of rank 2, that is, a homogeneous or bi-homogeneous tree, then Γ is a free group and hence has the Haagerup property. If Γ is a lattice in G, then G has the Haagerup property ([CCJJV]).

We may also ask: If a locally compact group G acts on a tree, when does G have the Haagerup property? This question has been answered in [CMV] where the authors introduce spaces with 'measured walls', generalizing Haglund and Paulin's spaces with 'walls' ([HP]). They show that if a locally compact group G acts on a space with measured walls, then G has the Haagerup property. They also conjecture that the converse is true, proving the converse in a number of cases, including the class of discrete groups with the Haagerup property.

This conjecture has been proven in [CDH] where the authors prove that a group G has the Haagerup property if and only if it admits a proper continuous action by isometries on a 'median space'. A median space of [CDH] is a metric space for which, given any triple of points, there exists a unique median point, that is a point which is simultaneously between any two points in that triple. Simplicial trees are examples of median spaces. Spaces with measured walls are naturally endowed with a (pseudo)metric. In [CDH] the authors showed that a (pseudo)metric on a space is induced by a structure of measured walls if and only if it is induced by an embedding of the space into a median space.

Let G be a symmetrizable locally compact Kac-Moody group of rank 3 noncompact hyperbolic type. It is known that the minimal parabolic subgroup B^- of the negative BN-pair for G is a nonuniform lattice subgroup of G ([CG], [Re]). We recall that for all symmetrizable locally compact Kac-Moody groups G of either rank 2 (affine or hyperbolic type) or of rank 3 noncompact hyperbolic type, the nonuniform lattice subgroup B^- has the Haagerup property.

In the next section, we explicitly construct a proper action of the lattice $B^- \leq G$ on the homogeneous or bihomogeneous bipartite tree $\mathcal{X}$, which is an example of a median space, for each of the possible 33 symmetrizable locally compact rank 3 Kac-Moody groups of noncompact hyperbolic type that occur.

We mention also Proposition 2.1 of [DJ] where the authors showed that subgroups of finite covolume in groups without Property (T) admit actions on trees without fixed points.

Proposition 2.1 of [DJ]. *Let X be a building corresponding to a BN-pair for a group G. Suppose that one of the entries of the Coxeter matrix is ∞. Let H be a subgroup of finite covolume in $G = Aut_+(X)$. Then H acts without a fixed point on an infinite tree T.*

However the tree T in Proposition 2.1 of [DJ] may not be locally finite and the action of H on T is not proper in general.

8. Proper actions on inscribed trees

In this section, we prove our main theorem, that the lattice subgroup B^- of a locally compact symmetrizable rank 3 Kac-Moody group of noncompact hyperbolic type acts properly on the tree $\mathcal{X}$.

Theorem 8.1. *Let G be a symmetrizable locally compact Kac-Moody group of rank 3 noncompact hyperbolic type. Let X be the Tits building of G. Let $\mathcal{X}$ be the bihomogeneous tree inscribed in X as in Theorem 5.2. Let B^- be the minimal parabolic subgroup of the negative BN-pair for G. Then B^- acts on $\mathcal{X}$ with finite vertex stabilizers.*

Proof: As in previous sections, we consider the following cases:

(i) W is of type (∞, ∞, ∞).

(ii) W is of type (∞, r, ∞), $r = 2, 3$.

(iii) W is of type $(\infty, r, 2)$, $r = 3, 4, 6$.

(iv) W is of type (∞, r, r), $r = 3, 4, 6$.

We recall that for case (i), there are two possible types of vertices of $\mathcal{X}$: those at barycenters of ideal triangles in the tessellation and those at barycenters of edges of the ideal triangles.

For case (ii), there are two possible types of vertices of $\mathcal{X}$: those at (non-boundary) corners of triangles in the tessellation and those at barycenters of edges of the triangles.

For case (iii), there are two possible types of vertices of $\mathcal{X}$, but both are at (non-boundary) corners of triangles in the tessellation.

For case (iv), vertices of $\mathcal{X}$ occur at every (non-boundary) corner of the triangles in the tessellation.

We recall that we can identify each vertex of $\mathcal{X}$ with a simplex of X: if $x \in V\mathcal{X}$ is a vertex at a barycenter of a face in some copy of $\mathcal{Y} = \mathcal{Y}_{\mathcal{A}}$ in an apartment $\mathcal{A}$, we identify x with a maximal simplex (chamber) of X. If $x \in V\mathcal{X}$ is a vertex at a barycenter of an edge in some copy of $\mathcal{Y} = \mathcal{Y}_{\mathcal{A}}$, we identify x with an edge of X. If $x \in V\mathcal{X}$ is a vertex at a corner of a triangle in an apartment $\mathcal{A}$, we identify x with a vertex at a corner of a triangle in X.

We therefore identify vertices at corners of triangles not on the boundary of X with cosets G/P_{ij} where P_{ij} are the maximal standard parabolic subgroups. We identify vertices at barycenters of edges of triangles with cosets G/P_i where P_i, $i = 1, 2, 3$ are the non-maximal standard parabolic subgroups. We identify vertices at barycenters of faces of triangles with cosets G/B for the minimal standard parabolic subgroup B.

In case (i), $W = W(\infty, \infty, \infty)$, all vertices at corners of triangles are on the boundary of X: these vertices are also on the boundary of the tree $\mathcal{X}$. We associate only non-boundary vertices of $\mathcal{X}$ with cosets in the Tits building. There are 2 types of such vertices here, those at barycenters of edges of triangles which we identify with cosets G/P_i, $i = 1, 2, 3$ and those at barycenters of faces of triangles which we identify with cosets G/B.

We claim that the isotropy groups of the (non-boundary) vertices for the action of B^- on $\mathcal{X}$ are finite. It suffices to do this for the vertices on the fundamental apartment $\mathcal{A}_0$ since all other vertices (cosets gP_i and gP_{ij}) are conjugate to these. Each vertex on the interior of the fundamental apartment is a coset of the form wP_i or wP_{ij}, $w \in W$. We let Γ denote the group B^-, and let Γ_{wB}, $w \in W$, denote the isotropy group of the chamber wB:

$$\Gamma_{wB} \quad = \quad \{\gamma \in \Gamma \mid \gamma wB = wB\}.$$

From [CG], we have that if G is constructed over the finite field $\mathbb{F}_q$, then

$$\mid \Gamma_{wB} \mid \quad = \quad q^{l(w)}(q-1)^3,$$

where $l(\cdot)$ is the length function of the Weyl group W. It remains to show that for a vertex wP_i or wP_{ij}, $w \in W$, the groups Γ_{wP_i} and $\Gamma_{wP_{ij}}$ are finite. But each coset wP_i or wP_{ij} is its own stabilizer, so $\Gamma_{wP_i} = \Gamma \cap wP_i$ and $\Gamma_{wP_{ij}} = \Gamma \cap wP_{ij}$. Then $\Gamma_{wB} \leq \Gamma_{wP_i} \leq \Gamma_{wP_{ij}}$ and Γ_{wB} is a subgroup of finite index in both Γ_{wP_i} and $\Gamma_{wP_{ij}}$. Since $\mid \Gamma_{wB} \mid = q^{l(w)}(q-1)^3 < \infty$, Γ_{wP_i} and $\Gamma_{wP_{ij}}$ are finite groups. $\square$

8.1. **Kac-Moody groups of ranks 2 and ≥ 4.** If G is a rank 2 locally compact Kac-Moody group (of affine or hyperbolic type), then the Tits building of G is itself a homogeneous tree X ([CG], [RR]). It is known that G has the Haagerup property and that the subgroup B^- acts properly on X ([CG], [Re]).

Let G be a rank r symmetrizable locally compact Kac-Moody group of affine or hyperbolic type (compact or noncompact). The conclusion of Theorem 8.1, that the minimal parabolic subgroup B^- of the negative BN-pair for G acts on the tree $\mathcal{X}$ inscribed in the Tits building with finite vertex stabilizers, is available in rank $r = 3$ (for noncompact hyperbolic type) but not ranks $r \geq 4$. This follows from the fact that minimal parabolic subgroup B^- of the negative BN-pair for G of rank $r \geq 4$ has Kazhdan's Property (T) ([C] and [DJ]). We now apply a well known theorem of de la Harpe and Valette that states that if a Property (T) group H acts on a tree, then the group H must fix a vertex ([HV]). Thus if G has rank $r \geq 4$, a lattice subgroup Γ of G fixes a vertex for *any* action of Γ on a tree. In particular, this applies to $\Gamma = B^-$.

8.2. **Quotient graph of groups.** Let G be a symmetrizable locally compact Kac-Moody group of rank 3 noncompact hyperbolic type. Let X be the Tits building of G. Let B^- be the minimal parabolic subgroup of the negative BN-pair for G. As we have discussed, B^- acts on X with finite covolume. As proven in Theorem 8.1, B^- acts on the inscribed bihomogeneous tree $\mathcal{X}$ with finite vertex stabilizers. We may thus ask the following:

Question. What is the structure of the quotient graph of groups of for the action of B^- on $\mathcal{X}$?

We know that the quotient $B^-\backslash\mathcal{X}$ is infinite and has finite vertex stabilizers. It follows that $B^-\backslash\mathcal{X}$ also has finite covolume. Here the covolume of $B^-\backslash\mathcal{X}$ is defined as the sum over all vertices of the reciprocals of orders of vertex stabilizers. Thus B^- is also an '$\mathcal{X}$-lattice', that is, a lattice in the automorphism group of the tree $\mathcal{X}$. We can then apply the full theory of tree lattices ([BL]) to investigate the structure of the group B^-. We also claim that a graph of groups presentation for B^- in rank 3 noncompact hyperbolic type would give a solution to the Kac-Peterson conjecture on the structure of U^-, where $B^- = HU^-$ ([KP]). We hope to take this up elsewhere.

Appendix A. Billiard tables for cosmological billiards

In 2002, Damour, Henneaux and Nicolai investigated how theories of gravity, and in particular supergravity, close to a spacelike cosmological singularity decouple in the sense that nearby spacetime points become causally disconnected ([DHN]). This leads to a local dynamics in the presence of effective 'potential walls which can be modeled by a billiard moving in a region of hyperbolic space. This phenomenon, known as 'cosmological billiards' was first discovered in relation to pure gravity in $D = 4$ spacetime dimensions by Belinskii, Khalatnikov and Lifschitz ([BKL]) and was reformulated by Chitre ([Ch]) and Misner ([Mis]) in terms of billiard motion in hyperbolic space.

Under some natural assumptions on the Lagrangian, the billiard table for this dynamics is a Coxeter polyhedron ([DH]). Remarkably, the hyperbolic billiard table can be identified with the fundamental chamber of the Weyl group of a hyperbolic Kac-Moody Lie algebra determined by the inner products of the 'wall forms'.

In [dBS], the authors identify the hyperbolic Kac-Moody algebras for which there exists a Lagrangian of gravity, dilatons and p-forms which gives rise to a billiard motion whose billiard table can be identified with the fundamental chamber of the corresponding Weyl group. In [dBS] it is not assumed that the corresponding Kac-Moody groups or algebras are symmetries of the gravity or supergravity theories. In Table A.1 there are 2 Lagrangians of [dBS] that come from supergravity theories in $D = 4$ spacetime dimensions. The remaining Lagrangians of [dBS] in $D = 3$ spacetime dimensions may also correspond to higher dimensional gravity theories but these would not be supersymmetric.

In [HJ], Henneaux and Julia compute the billiards that emerge in the Belinskii-Khalatnikov-Lifschitz limit for all pure supergravities in $D = 4$ spacetime dimensions, as well as for $D = 4$, $N = 4$ supergravities coupled to an arbitrary number k of Maxwell supermultiplets. They show that the billiards tables for all these models are the Weyl chambers of hyperbolic Kac-Moody algebras. Their explicit computations for $D = 4$, $N = 2$, $N = 3$ supergravities reveal rank 3 hyperbolic noncompact Weyl chambers arising from a twisted form of the Kac-Moody algebra.

In Table A.1 we summarize the results of [BKL], [DHN], [dBS] and [HJ] for gravity and supergravity in $D = 3$ and $D = 4$ spacetime dimensions, where N denotes the number of supersymmetries in the supergravity theory. These results give rise to 5 distinct billiard tables which can be identified with fundamental chambers of Weyl groups of rank 3 noncompact hyperbolic type.

Table A.1. Lagrangians of gravity and supergravity that admit a billiard table which can be identified with the fundamental chamber of a rank 3 Kac-Moody Weyl group of noncompact hyperbolic type.

Gen. Cartan matrix	Gravity / billiard Lagrangian	Billiard table
$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	$D = 4$, $N = 1$ supergravity ([DHN]) Corresponds to a Lagrangian of pure gravity in $D = 3$ ([dBS])	$(\infty, 3, 2)$
$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$	$D = 4$, $N = 2, 3$ supergravity ([HJ]) Corresponds to a Lagrangian of gravity coupled to a one form (without dilaton) in $D = 3$ ([dBS])	$(\infty, 3, 2)$
$A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	Corresponds to a Lagrangian of gravity with 1 dilaton in $D = 3$ ([dBS])	$(\infty, 4, 2)$
$A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$	Corresponds to a Lagrangian of gravity with 1 dilaton in $D = 3$ ([dBS])	$(\infty, 4, 2)$
$A = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	Corresponds to a Lagrangian of gravity with 1 dilaton in $D = 3$ ([dBS])	$(\infty, 6, 2)$
$A = \begin{pmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$	Corresponds to a Lagrangian of gravity with 1 dilaton in $D = 3$ ([dBS])	$(\infty, 6, 2)$
$A = \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$	Corresponds to a Lagrangian of gravity with 1 dilaton in $D = 3$ ([dBS])	$(\infty, \infty, 2)$
$A = \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$	Corresponds to a Lagrangian of gravity with 1 dilaton in $D = 3$ ([dBS])	$(\infty, \infty, 2)$
$A = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$	$D = 4$ pure gravity ([BKL], [DHN]) without the presence of symmetry walls	(∞, ∞, ∞)

References

[B] Bass, H. *Covering theory for graphs of groups* J. Pure Appl. Algebra 89 (1993), no. 1-2, 3–47

[BL] Bass, H. and Lubotzky, A. *Tree Lattices*, with appendices by H. Bass, L. Carbone, A. Lubotzky, G. Rosenberg and J. Tits. Progress in Mathematics, **176**. Birkhauser Boston, Inc., Boston, MA, (2001)

[Br] Brown, Kenneth S. *Buildings*, Springer-Verlag, New York, (1989)

[BCS1] Bunimovich, L, Chernov, N and Sinai, Y. G. *Markov partitions for two-dimensional hyperbolic billiards*, Russ. Math. Surv. 45 (1990), 105–152

[BCS1] Bunimovich, L, Chernov, N and Sinai, Y. G. *Statistical properties of two-dimensional hyperbolic billiards*, Russ. Math. Surv. 46 (1991), 47–106

[BKL] V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, *Oscillatory approach to a singular point in the relativistic cosmology*, Adv. Phys.19 (1970) 525

[Ca] Carathéodory, C. *Theory of functions of a complex variable*. Vol. 2. Translated by F. Steinhardt. Chelsea Publishing Company, New York, (1954) 220 pp.

[C] Carbone, L. *The Haagerup property, Property (T) and the Baum-Connes conjecture for locally compact Kac-Moody groups*, To appear in Advances in Algebra (2010)

[CG] Carbone, L., and Garland, H. *Existence of Lattices in Kac-Moody Groups over Finite Fields*, Communications in Contemporary Math, Vol 5, No.5, (2003), 813–867

[CCCMNNP] Carbone, L., Chung, S., Cobbs, L., McRae, R., Nandi, D., Naqvi Y. and Penta, D. *Classification of hyperbolic Dynkin diagrams, root lengths and Weyl group orbits*, J. Phys. A: Math. Theor. 43 155209, (2010) arXiv:1003.0564v1

[CDH] Chatterji, I., Drutu, C., and Haglund, F. *Kazhdan and Haagerup properties from the median viewpoint*, arXiv:0704.3749v2 [math.GR]

[CV] Cheng, M. C. N., Verlinde, E. P. *Wall crossing, discrete attractor flow and Borcherds algebra*, SIGMA Symmetry Integrability Geom. Methods Appl. 4 (2008), Paper 068, 33 pp

[CCJJV] Cherix, P.-A., Cowling, M., Jolissaint, P., Julg, P. and Valette, A. *Groups With the Haagerup Property: Gromov's A-T-Menability*, Progress in Mathematics, Birkhauser, (2001)

[CMV] Cherix, P.-A., Martin, F. and Valette, A. *Spaces with measured walls, the Haagerup property and Property (T)*, Ergodic theory and dynamical systems, **24**, (2004), 1895–1908

[Ch] Chitre, D. M. *Investigations of vanishing of a horizon for Bianchy type X (the Mixmaster) Universe* PhD Thesis, Univ of Maryland, (1972)

[DH] Damour, T, Henneaux, M, *E(10), BE(10) and Arithmetical Chaos in Superstring Cosmology*, hep-th/0012172, Phys.Rev.Lett. 86 (2001) 4749-4752

[DHN] Damour, T., Henneaux, M., Nicolai, H. *Cosmological billiards*, Classical Quantum Gravity 20 (2003), no. 9, R145–R200

[D] Davis, M. *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, 32. Princeton University Press, Princeton, NJ, 2008. xvi+584 pp. ISBN: 978-0-691-13138-2

[dBS] de Buyl, S, and Schomblond, C *Hyperbolic Kac Moody Algebras and Einstein Billiards* J.Math.Phys. 45 (2004) 4464–4492

[DJ] Dymara, J., and Januszkiewicz, T. *Cohomology of buildings and of their automorphism groups*, Inventiones Math 150, **3**, (2002), 579–627

[FF] Feingold, A., and Frenkel, I. *A hyperbolic Kac- Moody algebra and the theory of Siegel modular forms of genus 2*, Math. Ann. 263, no. 1, 87–144, (1983)

[H] Haagerup, U. *An example of a non-nuclear C^*-algebra which has the metric approximation property*, Inventiones Math. 50 (1979), 279–293

[HJ] Henneaux, M and Julia, B *Hyperbolic billiards of pure D=4 supergravities*, hep-th/0304233, JHEP 0305 (2003) 047

[HP] Haglund, F., Paulin, F. *Simplicite de groupes d'automorphismes d'espaces a courbure negative*, Geom. Topol. Monogr. 1 (1998), 181–248

[HV] de la Harpe, P. and Valette, A. *La propriété (T) de Kazhdan pour les groupes localement compacts*, Asterisque, **175**, Soc. Math. France, (1989)

[IM] Iwahori, N., Matsumoto, H. *On some Bruhat decomposition and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups*, Inst. Hautes Études Sci. Publ. Math. No. 25 (1965), 5–48.

[K] Kac, V. *Infinite dimensional Lie algebras*, Cambridge University Press, (1990)

[KP] Kac, V. and Peterson, D. *Defining relations of certain infinite-dimensional groups*, Astérisque Numéro Hors Série (1985), 165–208

[Li] Li, W. *Generalized Cartan matrices of hyperbolic type*, Chinese Annals of Mathematics, 9B, (1988), 68–77

[M] Magnus, W. *Non-Euclidean tessellations and their groups*, Academic. Press, New York and London (1974)

[Mc] McMullen, C. T., *Billiards and Teichmller curves on Hilbert modular surfaces*, Journal: J. Amer. Math. Soc. 16 (2003), 857-885

[Mi] Mihalik, M. *JSJ decompositions of Coxeter groups over virtually abelian splittings* , Preprint (2009)

[MT] Mihalik, M., and Tschantz, S. *Visual decompositions of Coxeter groups*, Groups Geom. Dyn. 3 (2009), no. 1, 173–198

[Mis] Misner, C. W. *Mixmaster Universe*, Phys. Rev. Lett. 2 (20), (1969) 1071–1074

[RTs] Ratcliffe, J., and Tschantz, S. *JSJ decompositions of Coxeter groups over FA subgroups*, arXiv:0910.5732v1 [math.GR]
[Re] Rémy, B. *Groupes de Kac-Moody déployés et presque déployés. (French) [Split and almost split Kac-Moody groups]*, Astérisque No. 277 (2002), viii+348 pp
[RR] Rémy, B., and Ronan, M. *Topological groups of Kac-Moody type, right-angled twinnings and their lattices*, Commentarii Mathematici Helvetici 81 (2006) 191–219
[R] Ronan, M. *Lectures on buildings*, Perspectives in Mathematics, 7. Academic Press, Inc., Boston, MA, (1989)
[RTi] Ronan, M., and Tits, J. *Building buildings*, Math. Annalen **278** (1987), 291–306
[S] Saçlioğlu, C. *Dynkin diagrams for hyperbolic Kac-Moody algebras*, Journal of Physics A: Mathematics and General **22** 3753–3769, (1989)
[Sa] Sarnak, P *Arithmetic Quantum Chaos*, Israel Math. Conference Proceedings, 8, (1995)
[Sc1] Schwartz, R *Research announcement: unbounded orbits for outer billiards*, Electron. Res. Announc. Math. Sci. 14 (2007), 1–6 (electronic)
[Sc2] Schwartz, R *Unbounded orbits for outer billiards. I*, J. Mod. Dyn. 1 (2007), no. 3, 371–424
[Sc3] Schwartz, R. *Obtuse triangular billiards. II. One hundred degrees worth of periodic trajectories*, Experiment. Math. 18 (2009), no. 2, 137–171
[Sel] Seligman, George B. *Book Review: Infinite dimensional Lie algebras*, Bull. Amer. Math. Soc. (N.S.) 16 (1987), no. 1, 144–149
[Se] Serre, J. P. *Trees*, Translated from the French by John Stillwell. Springer-Verlag, Berlin-New York, (1980)
[Si] Sinai, Y. G *Hyperbollic billiards*, Proceedings of the International Congress of Mathematicians (Kyoto, 1990), 249-260, Math. Soc. Japan, Tokyo, (1991)
[T] Tabachnikov, S. *Geometry and billiards*, Student Mathematical Library, 30. American Mathematical Society, Providence, RI; Mathematics Advanced Study Semesters, University Park, PA, 2005. xii+176 pp. ISBN: 0-8218-3919-5
[Ti1] Tits, J. *Resume de Cours - Theorie des Groupes*, Annuaire du College de France, (1980–1981), 75 –87
[Ti2] Tits, J. *Uniqueness and presentation of Kac-Moody groups over fields*, Journal of Algebra, **105** (1987), 542–573
[Ti3] Tits, J. *Ensembles ordonnés, immeubles et sommes amalgamées.* (French) [Ordered sets, buildings and amalgamated sums]. Bull. Soc. Math. Belg. A 38 (1986), 367–387 (1987)
[Wa] Watatani, Y. *Property (T) of Kazhdan implies Property (FA) of Serre*, Math. Japon., 27 (1982), no. 1, 97–103

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