# LATTICES ON NON-UNIFORM TREES

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Abstract. Let $X$ be a locally finite tree, and let $G = Aut(X)$. Then $G$ is a locally compact group. We show that if $X$ has more than one end, and if $G$ contains a discrete subgroup $\Gamma$ such that the quotient graph of groups $\Gamma\backslash\backslash X$ is infinite but has finite covolume, then $G$ contains a non-uniform lattice, that is, a discrete subgroup $\Lambda$ such that $\Lambda\backslash G$ is not compact, yet has a finite $G$-invariant measure.

## 0. Notation, preliminaries and results

Let $X$ be a locally finite tree, and $G = Aut(X)$. Then $G$ is naturally a locally compact group with compact open vertex stabilizers $G_x$, $x \in VX$ ([BL], (3.1)). A subgroup $\Gamma \leq G$ is discrete if and only if $\Gamma_x$ is a finite group for some (hence for every) $x \in VX$.

Let $\mu$ be a (left) Haar measure on $G$. By a *$G$-lattice* we mean a discrete subgroup $\Gamma \leq G = Aut(X)$ such that $\Gamma\backslash G$ has finite measure $\mu(\Gamma\backslash G)$. We call $\Gamma$ a *uniform $G$-lattice* if $\Gamma\backslash G$ is compact, and a *non-uniform $G$-lattice* if $\Gamma\backslash G$ is not compact yet has finite invariant measure. Let $H \leq G$ be a closed subgroup. We may also refer to *$H$-lattices*, that is, discrete subgroups $\Gamma \leq H$ such that $\Gamma\backslash H$ has finite measure.

A discrete subgroup $\Gamma \leq G$ is called an *$X$-lattice* if

$$Vol(\Gamma\backslash\backslash X) := \sum_{x \in V(\Gamma\backslash X)} \frac{1}{|\Gamma_x|}$$

is finite, a *uniform $X$-lattice* if $\Gamma\backslash X$ is a finite graph, and a *non-uniform lattice* if $\Gamma\backslash X$ is infinite but $Vol(\Gamma\backslash\backslash X)$ is finite.

Bass and Kulkarni have shown ([BK]) that $G = Aut(X)$ contains a uniform $X$-lattice if and only if $X$ is the universal covering of a finite connected graph, or equivalently, that $G$ is unimodular, and $G\backslash X$ is finite. In this case, we call $X$ a *uniform* tree. In case $G\backslash X$ is infinite we call $X$ a *non-uniform tree.*

When $G$ is unimodular, $\mu(G_x)$ is constant on $G$-orbits, so we can define ([BL], (1.5)):

$$\mu(G\backslash\backslash X) := \sum_{x \in V(G\backslash X)} \frac{1}{\mu(G_x)}.$$

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**(0.1) Theorem ([BL], (1.6)).** *Let $H \leq G$ be a closed subgroup with compact open vertex stabilizers. For a discrete subgroup $\Gamma \leq H$, the following conditions are equivalent:*

(a) $\Gamma$ *is an $X$-lattice, that is, $Vol(\Gamma\backslash\backslash X) < \infty$.*
(b) $\Gamma$ *is an $H$-lattice (hence $H$ is unimodular), and $\mu(H\backslash\backslash X) < \infty$.*

*In this case*:
$$Vol(\Gamma\backslash\backslash X) = \mu(\Gamma\backslash H) \cdot \mu(H\backslash\backslash X).$$

In [BCR] we proved the '*Lattice existence theorem*', namely that $G$ contains an $X$-lattice $\Gamma$ if and only if $G$ is unimodular and $\mu(G\backslash\backslash X) < \infty$. In particular, it is shown in [BCR] that if $G$ is unimodular, $\mu(G\backslash\backslash X) < \infty$, and $G\backslash X$ is infinite, then $G$ contains a (necessarily non-uniform) $X$-lattice $\Gamma$. However, $\Gamma$ turns out to be a *uniform* $G$-lattice. Here our main result is the following:

**(0.2) Theorem.** *Let $X$ be a locally finite tree with more than one end, and let $G = Aut(X)$. The following conditions are equivalent:*

(a) *$G$ contains a non-uniform $X$-lattice.*
(b) *$G$ contains a non-uniform $G$-lattice and $\mu(G\backslash\backslash X) < \infty$.*

If $X$ is uniform, then (a) $\Longrightarrow$ (b) is automatic (in light of Lemma (0.3) below), and the question of the existence of a non-uniform ($X$- or $G$-) lattice is answered in [C1]. If $X$ has only one end, in [CC] we show that if $G$ contains a non-uniform $X$-lattice, then $G$ contains a non-uniform $G$-lattice if and only if any path directed towards the end of the edge-indexed quotient of $X$ has unbounded index. We are thus reduced to proving Theorem (0.2) in the case that $X$ is not a uniform tree (we call $X$ *non-uniform* in this case), and has more than one end.

Let $\Gamma$ be a non-uniform $X$-lattice. Let $H \leq G$ be a closed subgroup with compact open vertex stabilizers. Then the diagram of natural projections

$$\begin{array}{ccc} & X & \\ {\scriptstyle p_\Gamma}\swarrow & & \searrow{\scriptstyle p_H} \\ \Gamma\backslash X & \xrightarrow{p} & H\backslash X \end{array}$$

commutes. By Theorem (0.1), $\Gamma$ is an $H$-lattice. To determine if $\Gamma$ is uniform or non-uniform in $G$, we use the following:

**(0.3) Lemma ([BL], (1.5.8)).** *Let $x \in VX$. The following conditions are equivalent:*

(a) $\Gamma$ *is a uniform $H$-lattice.*
(b) *Some fiber $p^{-1}(p_H(x)) \cong \Gamma\backslash H/H_x$ is finite.*
(c) *Every fiber of $p$ is finite.*

It follows that if $G\backslash X$ is finite, then $\Gamma$ is a uniform (respectively non-uniform) $X$-lattice if and only if $\Gamma$ is a uniform (respectively non-uniform) $G$-lattice. However in this work, we assume that $X$ is not uniform, that is, $G\backslash X$ is infinite. To construct a non-uniform $G$-lattice in this case, our task is to construct a discrete group $\Gamma$ with $\Gamma\backslash X$ infinite, $Vol(\Gamma\backslash\backslash X) < \infty$, and some (hence every) fiber of the projection $p$ infinite.

We can now establish the implication (b) $\Longrightarrow$ (a) of Theorem (0.2). Let $\Gamma$ be a non-uniform $G$-lattice. From Theorem (0.1), $\Gamma$ is an $X$-lattice. From Lemma (0.3) some fiber of $\Gamma\backslash X \xrightarrow{p} G\backslash X$ is infinite. We claim that $\Gamma\backslash X$ is infinite. If $G\backslash X$ is infinite, then $\Gamma\backslash X$ must be infinite. Assume then that $G\backslash X$ is finite. Since some fiber of $\Gamma\backslash X \xrightarrow{p} G\backslash X$ is infinite, $\Gamma\backslash X$ must be infinite, and we are done.

In the case that $G\backslash X$ is infinite, the implication (a) $\Longrightarrow$ (b) of Theorem (0.2) will be deduced from the following results about 'edge-indexed graphs'(see Theorem (9.5)). Here we follow the notations and terminology of Sections 1 and 2. Further, we say that an edge-indexed graph $(A, i)$ is *parabolic*, if $X = \widetilde{(A, i)}$ is a parabolic tree, that is, $X$ has only one end.

**(0.4) Theorem.** *Let $(A, i)$ be an infinite, unimodular edge-indexed graph with finite volume. If $(A, i)$ contains an arithmetic bridge $\beta$ with $n \geq 2$ edges, or if every ramified edge is separating and $(A, i)$ is not parabolic, then there is a covering $p : (B, j) \longrightarrow (A, i)$ of edge-indexed graphs with infinite fibers such that $(B, j)$ is infinite, unimodular, has finite volume and bounded denominators.*

Theorem (0.4) is proven in section 7. Theorem (8.2) and Theorem (8.3) obtain the conclusion of Theorem (0.4) in the case that there is a ramified non-separating edge. As a corollary of Theorem (0.4) we have the following:

**(0.5) Theorem.** *Let $(A, i)$ be an infinite, unimodular edge-indexed graph with finite volume. If $(A, i)$ contains an arithmetic bridge $\beta$ with $n \geq 2$ edges, or if every ramified edge is separating and $(A, i)$ is not parabolic, then there exists a (necessarily non-uniform) $X$-lattice $\Gamma \leq G_{(A,i)}$, which is a non-uniform $G_{(A,i)}$-lattice.*

We refer the reader to section 1 for the definition of the group $G_{(A,i)}$. (The terminology '$G_{(A,i)}$-lattice' is justified since $G_{(A,i)}$ is a closed subgroup of $G$ ([BL], (3.3)). Concerning existence of arithmetic bridges, we have the following:

**(0.6) Theorem.** *Let $(A, i)$ be a unimodular edge-indexed graph. Let $e \in EA$ be an edge with $\Delta(e) = c/d$, and $p$ a prime number such that $p|d$. If $e$ is not separating, then $(A, i)$ contains an arithmetic bridge $\beta$ of $n \geq 2$ edges with ramification factor $p$, such that $e \in \beta$.*

**(0.7) Corollary.** *Let $X$ be a locally finite tree, $G = Aut(X)$, $\mu$ a (left) Haar measure on $G$, $H \leq G$ a unimodular closed subgroup acting without inversions, $p_H : X \longrightarrow A = H\backslash X$, and $(A, i) = I(H\backslash\backslash X)$. Assume that $H = G_{(A,i)}$ and that $\mu(H\backslash\backslash X) < \infty$. If $X$ has more than one end, and $H\backslash X$ is infinite, then there exists a (necessarily non-uniform) $X$-lattice $\Gamma \leq H$, which is a non-uniform $H$-lattice.*

When $(A, i)$ is finite, Theorems (0.4), (0.5), (0.6) and (0.7) were proven in [C1] under some additional assumptions, natural to the setting there. Corollary (0.7) is a generalization of Corollary (0.9) in [BCR].

In section 1, we outline the basics of edge-indexed graphs and a method for constructing $X$-lattices. In section 2, we introduce the notion of an arithmetic bridge in an edge-indexed graph. In sections 3-6, we give some constructions with edge-indexed graphs containing arithmetic bridges. The material in sections 2-6 is taken from [C1] and adapted to infinite edge-indexed graphs, but is included here for reference.

In section 7, we prove Theorem (0.5) in the case that $(A, i)$ contains an arithmetic bridge $\beta$ with $n \geq 2$ edges, and in section 8, we prove Theorem (0.5) in the case that $(A, i)$ is has more than one end, considering separately the cases where every ramified edge is separating or else there is a ramified non-separating edge. In section 9, we prove Theorem (0.6), the existence theorem for arithmetic bridges and deduce Theorem (0.2) and Corollary (0.7). In section 10 we exhibit an infinite tower of coverings with infinite fibers and finite volume over an edge-indexed graph that admits an $X$-lattice and contains an arithmetic bridge with $n \geq 2$ edges.

## 1. Edge-indexed graphs and constructing $X$-lattices

An *edge-indexed graph* $(A, i)$ consists of an underlying graph $A$ , and an assignment of a positive integer $i(e) > 0$ to each oriented edge $e \in EA$. Our underlying graph $A$ will always be assumed to be locally finite. We assume that all indices $i(e)$ are *finite*. If $i(e) > 1$, we say that $e$ is a *ramified* edge. Otherwise, we say that $e$ is *unramified*.

Let $(A, i)$ be an edge-indexed graph. For $e \in EA$, we put

$$\Delta(e) = \frac{i(\overline{e})}{i(e)}.$$

For an edge path $\gamma = (e_1, \ldots, e_n)$ in $A$, we put $\Delta(\gamma) = \Delta(e_1) \ldots \Delta(e_n)$.

We say that $(A, i)$ is *unimodular* if $\Delta(\gamma) = 1$ for all closed paths $\gamma$ in $A$.

Now assume that $(A, i)$ is unimodular. Pick a base point $a_0 \in VA$, and define, for $a \in VA$,

$$N_{a_0}(a) = \frac{\Delta a}{\Delta a_0} (= \Delta(\gamma) \text{ for any path } \gamma \text{ from } a_0 \text{ to } a) \in \mathbb{Q}_{>0}.$$

For $e \in EA$, put

$$N_{a_0}(e) = \frac{N_{a_0}(\partial_0(e))}{i(e)}.$$

Following ([BL], (2.6)), we say that $(A, i)$ has *bounded denominators* if

$$\{N_{a_0}(e) \mid e \in EA\}$$

has bounded denominators, that is, if for some integer $D > 0$, $D \cdot N_{a_0}$ takes only integer values on edges. Since

$$N_{a_1} = \frac{\Delta a_0}{\Delta a_1} N_{a_0},$$

this condition is independent of $a_0 \in VA$.

Following ([BL], (2.6)), we define the *volume* of an edge-indexed graph $(A, i)$ at a base point $a_0 \in VA$:

$$Vol_{a_0}(A, i) := \sum_{a \in VA} \frac{1}{N_{a_0}(a)} = \sum_{a \in VA} (\frac{\Delta a_0}{\Delta a}).$$

We have ([BL], Ch. 2)

$$Vol_{a_1}(A,i) = \frac{\Delta a_1}{\Delta a_0} Vol_{a_0}(A,i),$$

so the condition

$$Vol(A,i) < \infty,$$

defined by $Vol_{a_0}(A,i) < \infty$, is independent of the choice of $a_0$.

Moreover, $(A,i,a_0)$ admits a covering tree $X = \widetilde{(A,i,a_0)}$ with a projection

$$p_{(A,i)} : X \longrightarrow A$$

([BL], (2.5)). We put $G = Aut(X)$ and $G_{(A,i)} = \{g \in G \mid p_{(A,i)} \circ g = p_{(A,i)}\}$. Then $G_{(A,i)}$ is a closed subgroup of $G$ with compact open vertex stabilizers ([BL], (3.3)).

We explain the notion of a *covering* of edge-indexed graphs ([BL], (2.5)),

$$p : (B,j) \longrightarrow (A,i).$$

Here $p : B \longrightarrow A$ is a graph morphism such that for all $e \in EA$, $\partial_0(e) = a$, and $b \in p^{-1}(a)$, we have

$$i(e) = \sum_{f \in p_{(b)}^{-1}(e)} j(f),$$

where $p_{(b)} : E_0^B(b) \longrightarrow E_0^A(a)$ is the local map on stars $E_0^B(b)$ and $E_0^A(a)$ of vertices $b \in VB$ and $a \in VA$ (*cf.* [BL], (2.5)). If $b \in VB$, $p(b) = a \in VA$, then we can identify

$$\widetilde{(A,i,a)} = X = \widetilde{(B,j,b)}$$

so that the diagram of natural projections

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle p_B}\swarrow & & \searrow{\scriptstyle p_A} & \\ B & & \xrightarrow{p} & & A \end{array}$$

commutes. Hence $G_{(B,j)} \leq G_{(A,i)}$.

Let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups, with underlying graph $A$, vertex groups $(\mathcal{A}_a)_{a \in VA}$, edge groups $(\mathcal{A}_e = \mathcal{A}_{\overline{e}})_{e \in EA}$ and monomorphisms $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}$. A graph of groups $\mathbb{A}$ naturally gives rise to an edge-indexed graph $I(\mathbb{A}) = (A,i)$ whose indices are the indices of the edge-groups as subgroups of the adjacent vertex groups: that is, $i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e \mathcal{A}_e]$, which we assume to be finite, for all $e \in EA$.

Given an edge-indexed graph $(A,i)$, a graph of groups $\mathbb{A}$ such that $I(\mathbb{A}) = (A,i)$, is called a *grouping* of $(A,i)$. We call $\mathbb{A}$ a *finite grouping* if the vertex groups $\mathcal{A}_a$ are finite and a *faithful grouping* if $\mathbb{A}$ is a faithful graph of groups, that is if $\pi_1(\mathbb{A}, a)$ acts faithfully on $X = \widetilde{(\mathbb{A}, a)}$.

We can now describe a method for constructing $X$-lattices. We begin with an edge-indexed graph $(A,i)$ which determines $X = \widetilde{(A,i,a_0)}$ up to isomorphism. We have the following:

**(1.7) Theorem ([BK], (2.4)).** *Let $(A,i)$ be an edge-indexed graph. Then $(A,i)$ admits a finite faithful grouping $\mathbb{A}=(A,\mathcal{A})$, if and only if $(A,i)$ is unimodular, and has bounded denominators.*

Assume that $(A,i)$ is unimodular and has bounded denominators (which is automatic if $A$ if finite). By Theorem (1.7), we can find a finite (faithful) grouping $\mathbb{A}$ of $(A,i)$ and a group $\Gamma=\pi_1(\mathbb{A},a_0)$ acting (faithfully) on $X$. Then we have

(i) $\Gamma$ is discrete, since $\mathbb{A}$ is a graph of *finite* groups.

(ii) $\Gamma$ is an $X$-*lattice* if and only if

$$Vol(\Gamma\backslash\backslash X)=Vol(\mathbb{A})(:=\sum_{a\in VA}\frac{1}{|\mathcal{A}_a|}=\frac{1}{|\mathcal{A}_a|}Vol_a(A,i))<\infty.$$

(iii) $\Gamma$ is a *uniform* $X$-lattice if and only if $A$ is finite.

(iv) $\Gamma$ is a *non-uniform* $X$-lattice if and only if $A$ is infinite.

(v) $\Gamma\leq G_{(A,i)}$.

We will say that a subgroup $H\leq G=Aut(X)$ is *saturated* if $H=G_{(A,i)}$ where $(A,i)=I(H\backslash\backslash X)$.

## 2. Geometric and arithmetic bridges in indexed graphs

In this section, following [C1], we recall the definition of an *arithmetic bridge* in an edge-indexed graph.

**(2.1) Definition ((p,q)-geometric bridge).** *Let $p,q\in\mathbb{Z}_{>0}\cup\{\infty\}$. Let $A$ be a connected locally finite graph (finite or infinite). We say that $\beta\subset EA$ is a* **(p,q)-geometric bridge** *for $A$ if:*

(i) *$\beta\neq\varnothing$, $\beta$ is oriented, $\beta\cap\overline{\beta}=\varnothing$,*

(ii) *$A\backslash(\beta\cup\overline{\beta})$ has $p+q$ connected components, $A_1,A_2,\dots A_p,B_1,B_2,\dots B_q$,*

(iii) *for every $e\in\beta$ we have $\partial_0e\in\mathcal{S}_\beta=A_1\cup A_2\cup\cdots\cup A_p$, the source of $\beta$ and $\partial_1e\in\mathcal{T}_\beta=B_1\cup B_2\cup\cdots\cup B_q$, the target of $\beta$,*

(iv) *the source $\mathcal{S}_\beta$ of $\beta$ does not contain any target vertex, and the target $\mathcal{T}_\beta$ of $\beta$ does not contain any source vertex; that is, for every $v\in A_1\cup A_2\cup\cdots\cup A_p$, $v\neq\partial_1e$ for any $e\in\beta$, for every $v\in B_1\cup B_2\cup\cdots\cup B_q$, $v\neq\partial_0e$ for any $e\in\beta$:*

A $(1,1)$-geometric bridge $\beta$ will be called a *geometric bridge*.

**(2.2) Definition ((p,q)-arithmetic bridge).** *A $(p,q)$-geometric bridge $\beta$ for $A$ is called a* **(p,q)-arithmetic bridge** *for $(A,i)$ if there exists a positive integer $d>1$ such that $d\mid i(e)$ for every $e\in\beta$, say $i(e)=di_0(e)$.*

We call $d$ the *ramification factor* of $\beta$. A $(1,1)$-arithmetic bridge will be called an *arithmetic bridge*: Our objective is to prove the following:

**(2.3) Theorem.** *Let $(A,i)$ be an infinite, unimodular edge-indexed graph with finite volume. If $(A,i)$ contains an arithmetic bridge $\beta$ with $n \geq 2$ edges, then there is a covering $p:(B,j) \longrightarrow (A,i)$ of edge-indexed graphs with infinite fibers such that $(B,j)$ is infinite, unimodular, has finite volume and bounded denominators.*

We prove Theorem (2.3) in Section 7 after describing some constructions with edge-indexed graphs in Sections 3-6.

## 3. Changing the ramification factor of an arithmetic bridge

In this section, we show how one may modify a unimodular edge-indexed graph containing an arithmetic bridge in such a way as to preserve unimodularity.

**(3.1) Construction ([C1], (4.2.1)) (Changing the ramification factor of an arithmetic bridge).**

If $(A,i)$ is an indexed graph with arithmetic bridge $\beta$ of ramification factor $d$, then we can make $\beta$ an arithmetic bridge of ramification factor $d'$, for any positive integer $d' > 0$, by replacing $di_0(e)$ by $d'i_0(e)$ for each positively oriented edge $e$ of $\beta$. We write $(\frac{d'}{d})\beta$ for the new arithmetic bridge.

**(3.2) Lemma ([C1], (4.2.2)) (Changing the ramification factor is unimodular).** *If $(A,i)$ is a unimodular edge-indexed graph with arithmetic bridge $\beta$, then the indexed graph $(A,i')$ obtained from $(A,i)$ by replacing $\beta$ by $(\frac{d'}{d})\beta$ is also unimodular.*

*Proof.* Let $E^+(\beta)$ be the set of positively oriented edges of $\beta$. For $e \in EA$ we define a new indexing $i'(e)$ as follows:

$$\begin{aligned} i'(e) \;&=\; i(e), \text{ if } e \notin E^+(\beta), \\ & d'i_0(e) = \frac{d'}{d} i(e), \text{ if } e \in E^+(\beta). \end{aligned}$$

For $e \in EA$, set $\Delta(e) = \dfrac{i(\overline{e})}{i(e)}$, and $\Delta'(e) = \dfrac{i'(\overline{e})}{i'(e)}$. Then for any $e \in EA$, we have:

$$\begin{aligned} \Delta'(e) = \quad & (\frac{d}{d'}) \cdot \Delta(e), \; e \in E^+(\beta), \\ & (\frac{d'}{d}) \cdot \Delta(e), \; \overline{e} \in E^+(\beta), \\ & \Delta(e), \; e \notin \beta. \end{aligned}$$

Let $\gamma$ be a (sufficiently long) closed path in $A$ with initial (and hence terminal) vertex in the connected component $A_0$. Then $\gamma$ crosses back and forth between $A_0$ and $A_1$, each time traversing an edge of $\beta$, returning finally to $A_0$.

It follows that $\gamma$ traverses $\beta$ and $\overline{\beta}$ an equal number of times, say $r$ times. Thus:

$$\begin{aligned}\Delta'_{(A,i')}(\gamma) &= (\frac{d}{d'})^r(\frac{d'}{d})^r\Delta(\gamma)\\ &= \Delta_{(A,i)}(\gamma)\\ &= 1,\ \text{since } (A,i) \text{ is unimodular.}\square\end{aligned}$$

## 4. Gluing unimodular subgraphs along connected intersections

In this section, we use a technique from [C1] for 'gluing' together unimodular edge-indexed graphs in such a way that unimodularity is preserved.

**(4.1) Lemma ([C1], (4.3.1)) (Gluing unimodular subgraphs along connected intersection).** *Let $(A,i)$, $(A_0,i)$, and $(A_1,i)$ be indexed graphs such that $(A,i) = (A_0,i) \cup (A_1,i)$, and $A_0$, $A_1$ and $A_0 \cap A_1$ are connected. If $(A_0,i)$ and $(A_1,i)$ are unimodular, then $(A,i)$ is unimodular.*

*Proof.* Observe that $(A_0 \cap A_1, i)$ is unimodular since it is a subgraph of a unimodular edge-indexed graph ($(A_0,i)$ or $(A_1,i)$).

Let $\gamma$ be a (sufficiently long) closed path in $A = A_0 \cup A_1$ with initial (and hence terminal) vertex $a_0$ in $A_0 \cap A_1$. Then $\gamma$ crosses back and forth between $A_0$ and $A_1$, say $n$ times, each time passing through $A_0 \cap A_1$, returning finally to $a_0$ in $A_0 \cap A_1$.

Suppose that the $j$-th time $\gamma$ passes through $A_0 \cap A_1$, $\gamma$ passes through a vertex $a_j \in A_0 \cap A_1$ (there are $n$ such vertices $a_1,\dots,a_n$). For each $j = 1,2,\dots,n$ , let $\gamma_j$ be a closed path initiating at $a_j \in A_0 \cap A_1$, passing through $a_{j-1} \in A_0 \cap A_1$, and remaining entirely in $A_0 \cap A_1$. Since $(A_0 \cap A_1, i)$ is unimodular, we have $\Delta_{(A_0 \cap A_1, i)}(\gamma_j) = 1$.

For $t = 1, 2$, and $s = 0, 1, \dots, n$, let $\gamma^{A_t}_{a_s a_{s+1}}$ denote the sub-path of $\gamma$ initiating at $a_s \in A_0 \cap A_1$, passing through $A_t$ and terminating at $a_{s+1} \in A_0 \cap A_1$. For $t = 1, 2$, and $s = 0, 1, \dots, n$, let $\gamma^j_{a_s a_{s+1}}$ denote the subpath of (the closed path) $\gamma_j$ initiating at $a_s \in A_0 \cap A_1$ and terminating at $a_{s+1} \in A_0 \cap A_1$.

Consider the closed path $\gamma'$ based at $a_0 \in A_0 \cap A_1$:

$$\gamma' = \gamma^{A_1}_{a_0 a_1} \cdot \gamma_1 \cdot \gamma^{A_0}_{a_1 a_2} \cdot \gamma_2 \cdot \gamma^{A_1}_{a_2 a_3} \cdot \gamma_3 \cdots \cdot \gamma^{A_1}_{a_{n-1} a_n} \cdot \gamma_n \cdot \gamma^{A_0}_{a_n q_0}.$$

Then $\Delta(\gamma') = \Delta(\gamma)$ since $\gamma'$ is obtained from $\gamma$ by inserting closed paths $\gamma_j$, $j = 1, 2, \dots, n$ between $a_s$ and $a_{s+1}$, $s = 0, 1, \dots, n$ (and we have $\Delta_{(A_0 \cap A_1, i)}(\gamma_j) = 1$, $j = 1, 2, \dots, n$).

Moreover, $\gamma'$ can be expressed as a product of paths $\gamma' = \sigma_1 \sigma_2 \dots \sigma_l$ such that $\sigma_j$ is contained entirely in either $A_0$ or $A_1$; namely

$$\begin{aligned}\sigma_1 &= \gamma^{A_1}_{a_0 a_1} \cdot \gamma^1_{a_1 a_0}\\ \sigma_2 &= \gamma^1_{a_0 a_1} \cdot \gamma^{A_0}_{a_1 a_2} \cdot \gamma^2_{a_2 a_1} \cdot \gamma^1_{a_1 a_0}\\ \sigma_3 &= \gamma^1_{a_0 a_1} \cdot \gamma^2_{a_1 a_2} \cdot \gamma^{A_1}_{a_2 a_3} \cdot \gamma^3_{a_3 a_2} \cdot \gamma^2_{a_2 a_1} \cdot \gamma^1_{a_1 a_0}\\ &\vdots\end{aligned}$$

Then each $\sigma_{2j}$ is a closed path through $A_0$ based at $a_0 \in A_0 \cap A_1$, each $\sigma_{2j+1}$ is a closed path through $A_1$ based at $a_0 \in A_0 \cap A_1$, and therefore:

$$\begin{aligned}\Delta_A(\gamma) &= \Delta_A(\gamma') \\ &= \Delta_{A_1}(\sigma_1)\Delta_{A_0}(\sigma_2)\dots\Delta_{A_0}(\sigma_l) \\ &= 1,\end{aligned}$$

since $(A_0, i)$ and $(A_1, i)$ are unimodular. □

## 5. Open fanning of arithmetic bridges

In this section, we use a technique from [C1] to modify a unimodular edge indexed graph $(A, i)$ with an arithmetic bridge $\beta$, in such a way that we preserve unimodularity, and obtain a new arithmetic bridge with a different ramification factor.

**(5.1) Construction ([C1], (4.4.1)) (Open fanning of arithmetic bridges I).**

Suppose that $(A, i)$ is an edge-indexed graph containing an arithmetic bridge $\beta$ with ramification factor d. The *open fanning of* $\beta$ *in* $(A, i)$ is the edge-indexed graph $(B, j)$ obtained by replacing $\beta$ by $d$ copies of $\frac{1}{d}\beta$; $\beta_1, \dots \beta_d$, such that each positively oriented edge $e$ of $\beta_l$ in $(B, j)$ has index $i_0(e)$, for $l = 1, \dots d$.

We observe that $p : (B, j) \longrightarrow (A, i)$ is a covering of indexed graphs. When $\beta$ consists of a single (ramified) edge, the open fanning of $\beta$ in $(A, i)$ coincides with the notion of 'open fanning of a separating edge' in ([BL], (7.2)). In this case, the edge $\beta$ with its ramification index $m$ is replaced by $m$ copies of $\frac{1}{m}\beta$, each with index 1.

**(5.2) Construction ([C1], (4.4.3)) (Open fanning of arithmetic bridges II).**

We shall also consider the following modification of open fanning:

Suppose that $(A, i)$ is an edge-indexed graph containing an arithmetic bridge $\beta$. Rather than fanning open the arithmetic bridge $\beta$ with its ramification factor d into $d$ copies of $\frac{1}{d}\beta$, we obtain an indexed graph $(B, j)$ by replacing $\beta$ with $\beta^+ = \frac{1}{d}\beta$ and $\beta^- = \frac{d-1}{d}\beta$. Thus each edge $e$ of $\beta^+$ has index $i_0(e)$, and each edge $e$ of $\beta^-$ has index $(d-1)i_0(e)$. We observe that $p : (B, j) \longrightarrow (A, i)$ is a covering of indexed graphs.

The following lemma indicates that the process of open fanning preserves unimodularity:

**(5.3) Lemma ([C1], (4.4.4)) (Open fanning is unimodular).** *Let* $(A, i)$ *be a unimodular edge-indexed graph containing an arithmetic bridge* $\beta$*. Then the open fanning (I and II) of* $\beta$ *in* $(A, i)$ *is unimodular.*

*Proof.* Let $(A, i')$ and $(A, i'')$ denote the indexed graphs obtained by changing the ramfication factor of $\beta$ from $d$ to 1, and from $d$ to $d - 1$ respectively. By Lemma (3.2) (Changing the ramification factor is unimodular), $(A, i')$ and $(A, i'')$ are unimodular.

The open fanning (I) of $\beta$ in $(A, i)$ (see (5.1)) is an indexed graph $(B, j)$ obtained by 'gluing' $d$ copies of $(A, i')$ together along the connected subgraph $A_0$. Similarly, in open fanning (II) (see (5.2)), we glue $(A, i')$ and $(A, i'')$ together along the connected subgraph $A_0$. By Lemma (4.1) (Gluing unimodular subgraphs along connected intersection), the result of open fanning (I) (5.1) or open fanning (II) (5.2) is unimodular.□

## 6. Indexed topological coverings

In this section, we use a technique from ([C1], Section 4) for constructing coverings of edge-indexed graphs by taking topological coverings of the underlying graph, and lifting the indexing in such a way that the projection is index preserving. The resulting *indexed topological covering* will automatically be unimodular, and an edge-indexed covering of the original indexed graph.

**(6.1) Definition (Indexed topological coverings).**

Let $p : B \longrightarrow A$ be a graph morphism. If $i : EA \longrightarrow \mathbb{Z}$ is an indexing on $A$, we can lift it to an indexing $j = i \circ p$ on $B$ so that $p : (B, j) \longrightarrow (A, i)$ is index preserving: $i(p(f)) = j(f)$ for each $f \in B$. If $(A, i)$ is unimodular, then so also is $(B, j)$. In fact, if $\gamma = (e_1, \ldots, e_n)$ be a closed path in $(B, j)$, then

$$\Delta_{(B,j)}(\gamma) = \Delta_{(A,i)}(p(\gamma)) \text{ since } p \text{ is index preserving.}$$

Since $p(\gamma)$ is closed and $(A, i)$ is unimodular, we have $\Delta_{(A,i)}(p(\gamma)) = 1$, and so $\Delta_{(B,j)}(\gamma) = 1$.

We call a graph morphism $p : B \longrightarrow A$ a *topological covering* if the local map:

$$p_{(b)} : E_0(b) \longrightarrow E_0(p(b))$$

is bijective, for every $b \in VB$.

If $(A, i)$ and $(B, j)$ are indexed graphs, and $p : B \longrightarrow A$ is an index preserving topological covering, then $p : (B, j) \longrightarrow (A, i)$ is a covering of edge-indexed graphs.

## 7. A covering with infinite fibers for an edge-indexed graph with an arithmetic bridge

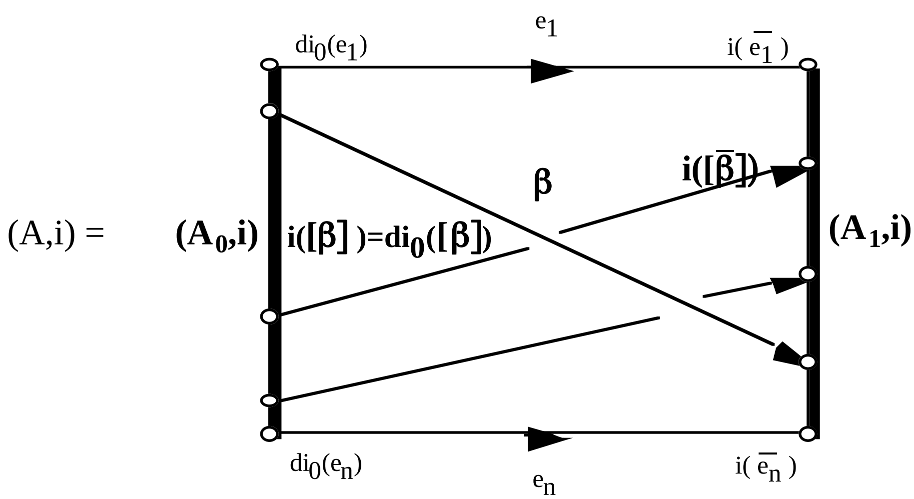
We are now able to prove:

**(7.1) Theorem.** *Let $(A, i)$ be an infinite, unimodular edge-indexed graph with finite volume. If $(A, i)$ contains an arithmetic bridge $\beta$ with $n \geq 2$ edges, then there exists a (necessarily non-uniform) $X$-lattice $\Gamma \leq G_{(A,i)}$, which is a non-uniform $G_{(A,i)}$-lattice.*

The terminology '$G_{(A,i)}$-lattice' is justified since $G_{(A,i)}$ is a closed subgroup of $G$ ([BL], (3.3). Theorem (7.1) follows immediately from the following:

**(7.2) Theorem.** *Let $(A, i)$ be an infinite, unimodular edge-indexed graph with finite volume. If $(A, i)$ contains an arithmetic bridge $\beta$ with $n \geq 2$ edges, then there is a covering $p : (B, j) \longrightarrow (A, i)$ of edge-indexed graphs with infinite fibers such that $(B, j)$ is infinite, unimodular, has finite volume and bounded denominators.*

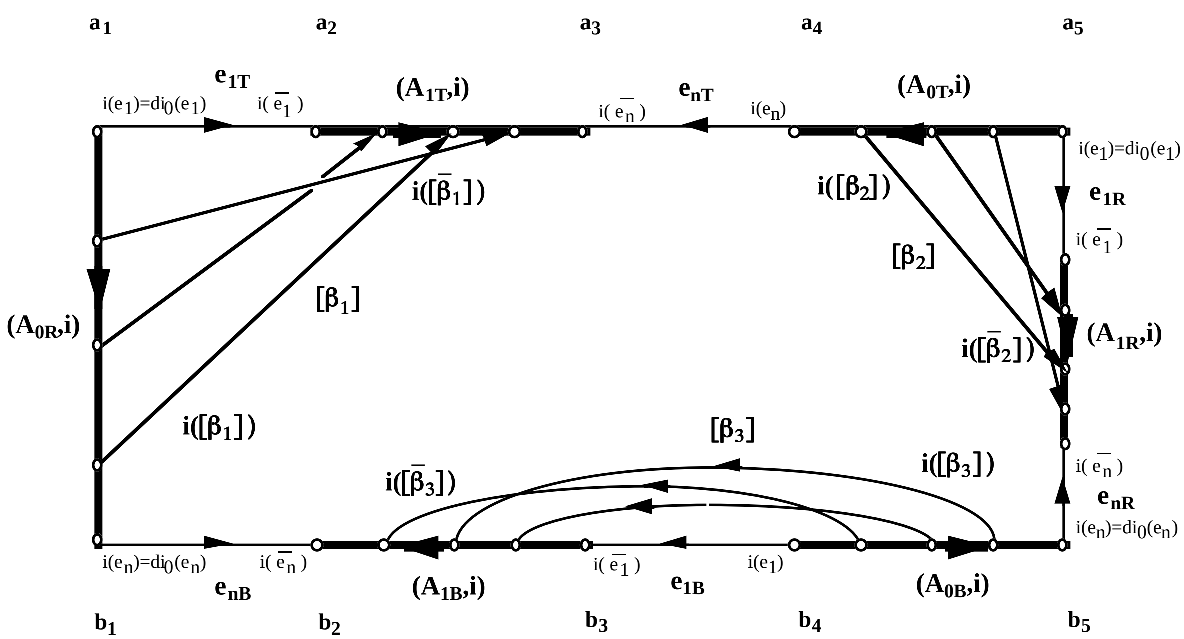
*Proof of Theorem (7.2).* The proof follows the proof of Theorem (4.1.6) in [C1], adapted to infinite edge-indexed graphs. For convenience, we represent $(A, i)$ schematically as follows:




(7.3) We assume that $\beta$ has $n \geq 2$ edges. In fact the number of edges of $\beta$ may be infinite. We choose two edges $e_1$ and $e_n$ of $\beta$, and we let $[\beta] = \beta - \{e_1, e_n\}$. We have schematically denoted the indexing of $[\beta]$ as '$i([\beta]) = di_0([\beta])$'; more precisely, $i(e) = di_0(e)$ for every $e \in [\beta]$.

(7.4) We form a 3-fold topological covering $p: A_3 \to A$ and lift the indexing $i_A$ to an indexing $(i \circ p)$ on $A_3$ such that $p$ is index preserving. We also denote this indexing on $A_3$ by $i$. Then by (6.1), $(A_3, i)$ is unimodular and $p\colon (A_3, i) \to (A, i)$ is a covering of edge-indexed graphs.

(A3,i) =




We have denoted the three copies of $\beta$, as $\beta_1,\ \beta_2,\ \beta_3$ and the corresponding copies of $A_0$ and $A_1$ as $A_{0R},\ A_{OT},\ A_{OB},\ A_{1T},\ A_{1L},\ A_{1B}$ where the lower labels 'T', 'B', 'R', 'L'; 'top', 'bottom', 'right', 'left', respectively, signify the position in the schematic diagram of $(A_3, i)$. The vertex and edge labels are suggested by the notation. We observe that $\beta_1,\ \beta_2$, and $\beta_3$ are each arithmetic bridges for $(A_3, i)$.

Observe that the paths from $a_1$ to $a_5$ and $b_1$ to $b_5$, are both liftings of closed paths in $(A, i)$, and since $p$ is index-preserving:

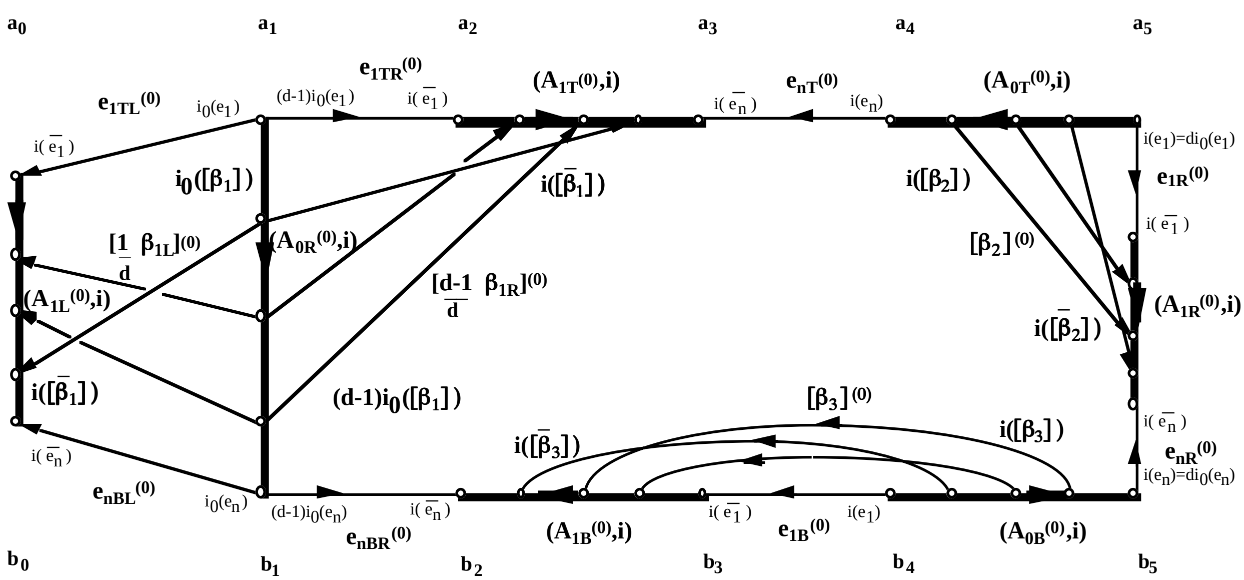
$$\frac{\Delta a_5}{\Delta a_1} = \frac{\Delta b_5}{\Delta b_1} = 1$$

since $(A, i)$ is unimodular.

(7.5) We form a new indexed graph $(R_0^{(0)}, i)$ from $(A_3, i)$ as follows: we open fan $\beta_1$ to '$\frac{d-1}{d}\beta_1$' and '$\frac{1}{d}\beta_1$' (that is, we apply 'open fanning II' (5.2)) to $\beta_1$ which is an arithmetic bridge in

$(A_3, i)$ from $A_{OR}$ to $A_3 - \{\beta_1 \cup A_{1T}\})$ :

$(R_0^{(0)}, i) =$




By Lemma (5.3) (Open fanning is unimodular), $(R_0^{(0)}, i)$ is unimodular. We observe that:

$$\frac{\Delta a_5}{\Delta a_1} = \frac{i(\overline{e_1})}{(d-1)i_0(e_1)} \cdot \frac{\Delta_{(A_1,i)} a_3}{\Delta_{(A_1,i)} a_2} \cdot \frac{d i_0(e_n)}{i(\overline{e_n})} \cdot \frac{\Delta_{(A_0,i)} a_5}{\Delta_{(A_0,i)} a_4} = \frac{d}{d-1}$$

by unimodularity of $(A, i)$. Similarly,

$$\frac{\Delta b_5}{\Delta b_1} = \frac{d}{d-1}.$$

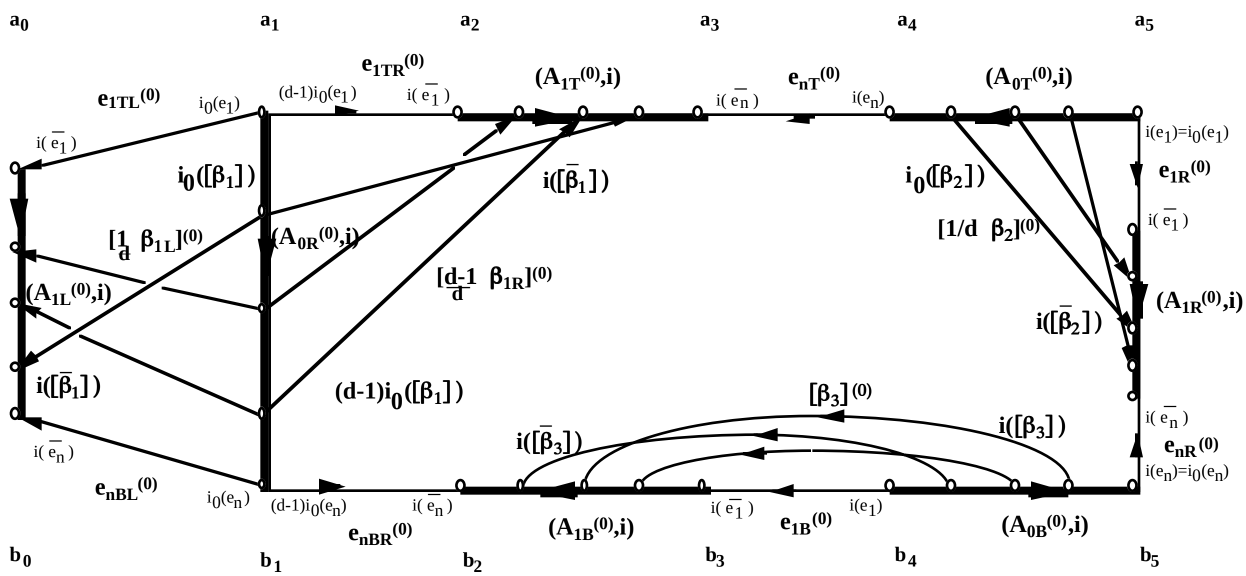
Moreover, the projection $p: (R_0^{(0)}, i) \to (A, i)$ is a covering of edge-indexed graphs.

We observe that $[\beta_2] \cup e_{1R}^{(0)} \cup e_{nR}^{(0)}$ is an arithmetic bridge in $(R_0^{(0)}, i)$ from

$$R_0^{(0)} - \{[\beta_2] \cup e_{1R}^{(0)} \cup e_{nR}^{(0)} \cup A_{1R}^{(0)}\}$$

to $A_{1R}^{(0)}$. Next, we form a new indexed graph $(R^{(0)}, i)$, from $(R_0^{(0)}, i)$, by changing the ramification factor of the arithmetic bridge $[\beta_2] \cup e_{1R}^{(0)} \cup e_{nR}^{(0)}$ in $(R_0^{(0)}, i)$ from $d$ to 1 (using (3.1)):

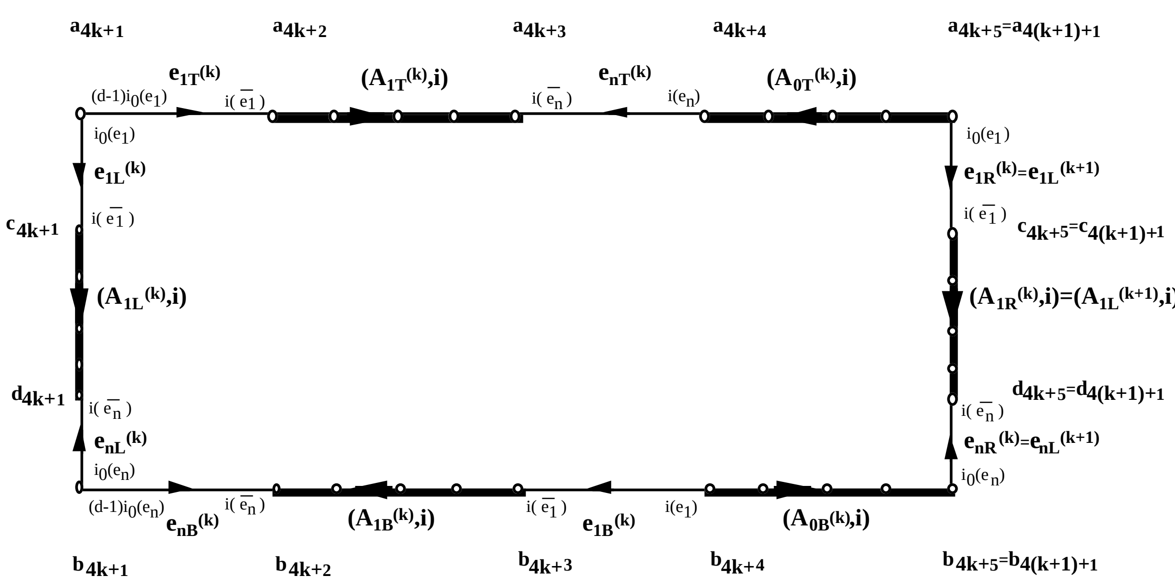
(R$^{(0)}$,i) =




and by Lemma (3.2) (Changing the ramification factor is unimodular), $(R^{(0)}, i)$ is unimodular. The notation $[\frac{1}{d}\beta_{1L}]^{(0)}$ denotes $[\beta_1]$ with its ramification factor changed from d to 1, $\left[\frac{d-1}{d}\beta_{1R}\right]^{(0)}$ denotes $[\beta_1]$ with its ramification-factor changed from d to d-1. The upper label (0) signifies that $[\frac{1}{d}\beta_{1L}]$ and $[\frac{d-1}{d}\beta_{1R}]$ belong to $R^{(0)}$; the lower labels 'L' and 'R' denote 'left' and 'right' respectively.

(7.6) For $k = 1, 2, 3, \ldots$ let $(R^{(k)}, i)$ be the following edge-indexed graph:

(R$^{(k)}$,i)=



where the lower labels 'T', 'B', 'L', 'R' on edges $e_1^{(k)}$, $e_n^{(k)}$ and graphs $A_1^{(k)}$, $A_0^{(k)}$ indicate 'top', 'bottom', 'left' and 'right' respectively and signify the position within $R^{(k)}$.

For each $k = 1, 2, 3, \ldots$ the 'rectangle' $(R^{(k)}, i)$ has as its 'top' a path from $a_{4k+1}$ to $a_{4k+5}$ which coincides with the path from $a_1$ to $a_5$ in $(R^{(0)}, i)$, a path from $b_{4k+1}$ to $b_{4k+5}$ which coincides with the path from $b_1$ to $b_5$ in $(R^{(0)}, i)$, and paths from $a_{4k+1}$ to $b_{4k+1}$ and $a_{4k+5}$ to $b_{4k+5}$ each of which coincide with the path from $a_5$ to $b_5$ in $(R^{(0)}, i)$. Therefore,

$$\frac{\Delta a_{4k+5}}{\Delta a_{4k+1}} = \frac{\Delta b_{4k+5}}{\Delta b_{4k+1}} = \frac{d}{d-1}$$
$$\frac{\Delta b_{4k+1}}{\Delta a_{4k+1}} = \frac{\Delta b_{4k+5}}{\Delta a_{4k+5}},$$

and it follows easily that $(R^{(k)}, i)$ is unimodular.

(7.7) We construct an infinite indexed graph from $(R^{(0)}, i)$ and $(R^{(k)}, i)$, $k = 1, 2, 3, \ldots$ by an infinite sequence of gluings: we identify the edges $e_{1R}^{(0)}$ and $e_{nR}^{(0)}$ and the subgraph $A_{1R}^{(0)}$ of $(R^{(0)}, i)$ with $e_{1L}^{(1)}$ and $e_{nL}^{(1)}$ and $A_{1L}^{(1)}$ of $(R^{(1)}, i)$, respectively.

For $k = 1, 2, 3, \ldots$ we identify $e_{1R}^{(k)}$ and $e_{nR}^{(k)}$ and $A_{1R}^{(k)}$ of $(R^{(k)}, i)$ with $e_{1L}^{(k+1)}, e_{nL}^{(k+1)}$ and $A_{1L}^{(k+1)}$ of $(R^{(k+1)}, i)$, respectively.

We denote the resulting indexed graph by $(R^{(\infty)}, i)$. We refer the reader to fig 4.10.2 in [C1] for a detailed schematic diagram. By (7.3), the indexed graph $(R^{(0)}, i)$ is unimodular, and by (7.4), $(R^{(k)}, i)$ is unimodular, for $k = 1, 2, 3, \ldots$. Moreover, we have glued $(R^{(0)}, i)$ to $(R^{(1)}, i)$ and $(R^{(k)}, i)$ to $(R^{(k+1)}, i)$ respectively, for $k = 1, 2, 3, \ldots$ along connected subgraphs:

$$\{e_{1R}^{(k)} = e_{1L}^{(k+1)}\} \cup \{A_{1R}^{(k)} = A_{1L}^{(k+1)}\} \cup \{e_{nR}^{(k)} = e_{nL}^{(k+1)}\}.$$

We apply Lemma (4.1) (Gluing unimodular subgraphs along connected intersection) to verify that the indexed graph $(R^{(\infty)}, i)$ is unimodular.

We compute in $(R^{(\infty)}, i)$ for each $s = 1, 2, 3, \ldots$

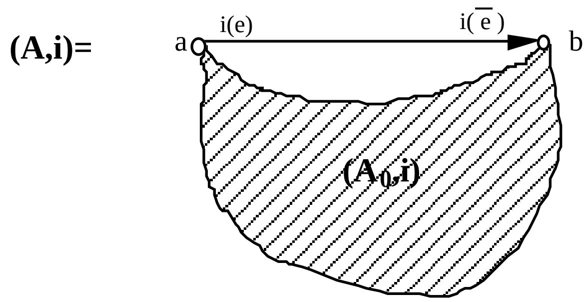
$$\frac{\Delta a_{4s+1}}{\Delta a_1} = (\frac{d}{d-1})^s = \frac{\Delta b_{4s+1}}{\Delta b_1},$$
$$\frac{\Delta c_{4s+1}}{\Delta a_1} = (\frac{d}{d-1})^s \frac{i(\overline{e_1})}{i_0(e_1)},$$
$$\frac{\Delta d_{4s+1}}{\Delta b_1} = (\frac{d}{d-1})^s \frac{i(\overline{e_n})}{i_0(e_n)}.$$

Since $d > 1$, it follows easily that $(R^{(\infty)}, i)$ has finite volume. We observe, however, that $(R^{(\infty)}, i)$ does *not* have bounded denominators.□

**(7.8) Lemma (Adjoining an edge).** *let $(A, i)$ and $(A_0, i)$ be indexed graphs such that $(A, i)$ is obtained from $(A_0, i)$ by attaching an edge $e$ to vertices $a, b \in VA_0$. If $(A_0, i)$ is unimodular and*

$$\frac{\Delta_{A_0} b}{\Delta_{A_0} a} = \frac{i(\overline{e})}{i(e)}$$

*then $(A, i)$ is unimodular.*




*Proof.* Obvious.□

The lemma extends easily to adjoining a set of edges satisfying the hypothesis of the lemma. We fix the following notation: for $k = 1, 2, 3, \ldots$

$$\begin{aligned}
[\frac{d-1}{d}\beta_{2T}]^{(k)} &\text{ denotes } [\beta_2] \text{ with its ramification} \\
&\text{ factor changed from d to d-1,} \\
[\frac{1}{d}\beta_{2R}]^{(k)} &\text{ denotes } [\beta_2] \text{ with its ramification} \\
&\text{ factor changed from d to 1,} \\
[\beta_3]^{(k)} &\text{ denotes } [\beta_3].
\end{aligned}$$

The upper labels $(k)$ indicate the $k$-th rectangle. The lower labels 'T', 'R' denote 'top' and 'right'.

(7.9) Next we construct an indexed graph $(B^-, j^-)$ from $(R^{(\infty)}, i)$ as follows: for $k = 1, 2, 3, \ldots$ we adjoin:

$$\begin{aligned}
[\frac{1}{d}\beta_{2R}]^{(k)} &\text{ to } (R^{(\infty)}, i) \text{ from } A^{(k)}_{OT} \text{ to} A^{(k)}_{1R}, \\
[\frac{d-1}{d}\beta_{2T}]^{(k)} &\text{ to } (R^{(\infty)}, i) \text{ from } A^{(k)}_{OT} \text{ to} A^{(k+1)}_{1T}, \\
[\beta_3]^{(k)} &\text{ to } (R^{(\infty)}, i) \text{ from } A^{(k)}_{OB} \text{ to } A^{(k)}_{1B}.
\end{aligned}$$

We refer the reader to Fig (4.11.2) in [C1] for a detailed schematic diagram of $(B^-, j^-)$.

**(7.10) Proposition.** *The indexed graph $(B^-, j^-)$ is unimodular and has finite volume.*

*Proof.* By (7.7), the indexed graph $(R^{(\infty)}, i)$ is unimodular. For $k = 1, 2, 3, \ldots,$ Let $e \in [\frac{1}{d}\beta_{2R}]^{(k)}$ we attach

$$\begin{aligned}
\partial_0 e &\text{ to } (R^{(\infty)}, i) \text{ at } a \in A^{(k)}_{OT}, \\
\partial_1 e &\text{ to } (R^{(\infty)}, i) \text{ at } b \in A^{(k)}_{1R}.
\end{aligned}$$

Let '$\frac{1}{d}(\frac{\Delta_A b}{\Delta_A a})$' denote $\frac{\Delta_A b}{\Delta_A a}$ with all occurences of d changed to 1. Then

$$\begin{aligned}\frac{\Delta_{R^{(\infty)}} b}{\Delta_{R^{(\infty)}} a} &= \frac{1}{d}(\frac{\Delta_A b}{\Delta_A a}) \\ &=_A \frac{d}{1}(\frac{i(\overline{e})}{d_{i_0}(e)}) \\ &=_A \frac{i(\overline{e})}{i_0(e)} \\ &=_{R^{(\infty)}} \frac{j^-(\overline{e})}{j^-(e)},\end{aligned}$$

and thus by Lemma (7.8) (Adjoining an edge), the result of adjoining $[\frac{1}{d}\beta_{2R}]^{(k)}$ is unimodular.

Let $e \in [\frac{d-1}{d}\beta_{2T}]^{(k)}$ we attach

$$\partial_0 e \text{ to } (R^{(\infty)}, i) \text{ at } a \in A_{OT}^{(k)},$$
$$\partial_1 e \text{ to } (R^{(\infty)}, i) \text{ at } b \in A_{1T}^{(k)}.$$

Let '$\frac{d-1}{d}(\frac{\Delta_A b}{\Delta_A a})$' denote $\frac{\Delta_A b}{\Delta_A a}$ with all occurences of d changed to d-1. Then

$$\begin{aligned}\frac{\Delta_{R^{(\infty)}} b}{\Delta_{R^{(\infty)}} a} &= \frac{d-1}{d}(\frac{\Delta_A b}{\Delta_A a}) \\ &=_A \frac{d}{d-1}(\frac{i(\overline{e})}{d_{i_0}(e)}) \\ &=_A \frac{i(\overline{e})}{(d-1)i_0(e)} \\ &=_{R^{(\infty)}} \frac{j^-(\overline{e})}{j^-(e)}.\end{aligned}$$

By Lemma (7.8) (Adjoining an edge), the result of adjoining $[\frac{d-1}{d}\beta_{2T}]^{(k)}$ is unimodular.

Let $e \in [\beta_3]^{(k)}$ we attach

$$\partial_0 e \text{ to } a \in A_{0B}^{(k)},$$
$$\partial_1 e \text{ to } b \in A_{1B}^{(k)}.$$

Then

$$\frac{\Delta_{R^{(\infty)}} b}{\Delta_{R^{(\infty)}} a} = \frac{\Delta_A b}{\Delta_A a} =_A \frac{i(\overline{e})}{i(e)} =_{R^{(\infty)}} \frac{j^-(\overline{e})}{j^-(e)}$$

and by Lemma (7.8) (Adjoining an edge), the result of adjoining $[\beta_3]^{(k)}$ is unimodular.

Since the result of adjoining $[\frac{1}{d}\beta_{2R}]^{(k)}$, $[\frac{d-1}{d}\beta_{2T}]^{(k)}$, $[\beta_3]^{(k)}$ is unimodular for $k = 1, 2, 3, \ldots$ it follows that the resulting indexed graph $(B^-, j^-)$ inherits finite volume from $(R^{(\infty)}, i)$.□

**(7.11) Remark.** *The indexed graph $(B^-, j^-)$ does not have bounded denominators.*

**(7.12) Proposition.** *There is a morphism $q\colon (B^-, j^-) \to (A, i)$ which is a covering of indexed graphs.*

*Proof.* Recall that for the arithmetic bridge $\beta \subset EA$, $[\beta]$ denotes $\beta - \{e_1, e_n\} \subset EA$. For $j = 1, 2, 3$ and $k = 1, 2, \dots [\beta_j]^{(k)} \subset EB^-$ denotes the j-th copy of $[\beta]$ in $R^{(k)}$.

We define a morphism $q\colon B^- \to A$:

$$q\mid_{A_{1R}^{(0)}}\colon A_{1R}^{(0)} \xrightarrow{\cong} A_1$$
$$q(e^{(0)}_{1_T L}) = e_1$$
$$q\mid_{[\beta_{1L}]^{(0)}}\colon [\beta_{1L}]^{(0)} \xrightarrow{\cong} [\beta]$$
$$q(e^{(0)}_{n_B L}) = e_n$$
$$q(e^{(0)}_{1_T R}) = e_1$$
$$q\mid_{[\beta_{1R}]^{(0)}}\colon [\beta_{1R}]^{(0)} \xrightarrow{\cong} [\beta]$$
$$q(e^{(0)}_{n_b R}) = e_n$$

for $k = 1, 2, 3, \dots$

$$q\mid_{A_{1T}^{(k)}}\colon A_{1T}^{(k)} \xrightarrow{\cong} A_1$$
$$q\mid_{A_{0T}^{(k)}}\colon A_{0T}^{(k)} \xrightarrow{\cong} A_0$$
$$q\mid_{A_{0B}^{(k)}}\colon A_{0B}^{(k)} \xrightarrow{\cong} A_1$$
$$q\mid_{A_{1B}^{(k)}}\colon A_{1B}^{(k)} \xrightarrow{\cong} A_1$$
$$q\mid_{A_{1L}^{(k)}}\colon A_{1L}^{(k)} \xrightarrow{\cong} A_1$$

$$q(e^{(k)}_{nT}) = e_n$$
$$q(e^{(k)}_{1B}) = e_1$$
$$q(e^{(k+1)}_{1T}) = e_1$$
$$q(e^{(k+1)}_{nB}) = e_n$$
$$q(e^{(k+1)}_{1L}) = e_1$$
$$q(e^{(k+1)}_{nL}) = e_n$$

$$q\,|_{\frac{1}{d}[\beta_{2R}]^{(k)}}:[\frac{1}{d}\beta_{2R}]^{(k)}\overset{\cong}{\longrightarrow}[\beta]$$

$$q\,|_{\frac{(d-1)}{d}[\beta_{2T}]^{(k)}}:[\frac{(d-1)}{d}\beta_{2T}]^{(k)}\overset{\cong}{\longrightarrow}[\beta]$$

$$q\,|_{[\beta_3]^{(k)}}:[\beta_3]^{(k)}\overset{\cong}{\longrightarrow}[\beta]$$

The projection $q:(B^-,j^-)\to(A,i)$ 'erases upper and lower indices'.

Let $e\in EA_0$ with $q^{-1}(\partial_0 e)=b\in VB^-$. Then $q^{-1}_{(b)}(e)$ is isomorphic to $e$, so

$$\sum_{f\in q^{-1}_{(b)}(e)} j^-(f)=i(e).$$

For $e\in EA_1$, with $q^{-1}(\partial_0 e)=b\in VB^-$, $q^{-1}_{(b)}(e)$ is isomorphic to $e$, so

$$\sum_{f\in q^{-1}_{(b)}(e)} j^-(f)=i(e).$$

Now let $e\in\beta$, with $q^{-1}(\partial_0\overline{e})=b\in VB^-$, then $q^{-1}_{(b)}(\overline{e})$ is isomorphic to $\overline{e}$ so

$$\sum_{f\in q^{-1}_{(b)}(\overline{e})} j^-(f)=i(\overline{e}).$$

It is a routine check that the morphism $q:(B^-,j^-)\to(A,i)$ is a covering of edge-indexed graphs by computing the local fibers over each $e\in\beta$. The reader is referred to ([C1], (4.11.5)) for a detailed computation. □

(7.13) By the 'Bounding denominators theorem' ([BCR], (0.6)) we can construct a covering $p:(B,j)\longrightarrow(B^-,j^-)$ such that $(B,j)$ is unimodular with finite volume and bounded denominators. The composition $q\cdot p:(B,j)\longrightarrow(A,i)$ is the desired covering of Theorem (7.2). This completes the proof of Theorem (7.2).□

The reader is referred to Fig (7.14) for a schematic diagram of the covering $q\cdot p:(B,j)\longrightarrow$

$(A, i)$ of Theorem (7.2), and to [C1] for more detailed diagrams.

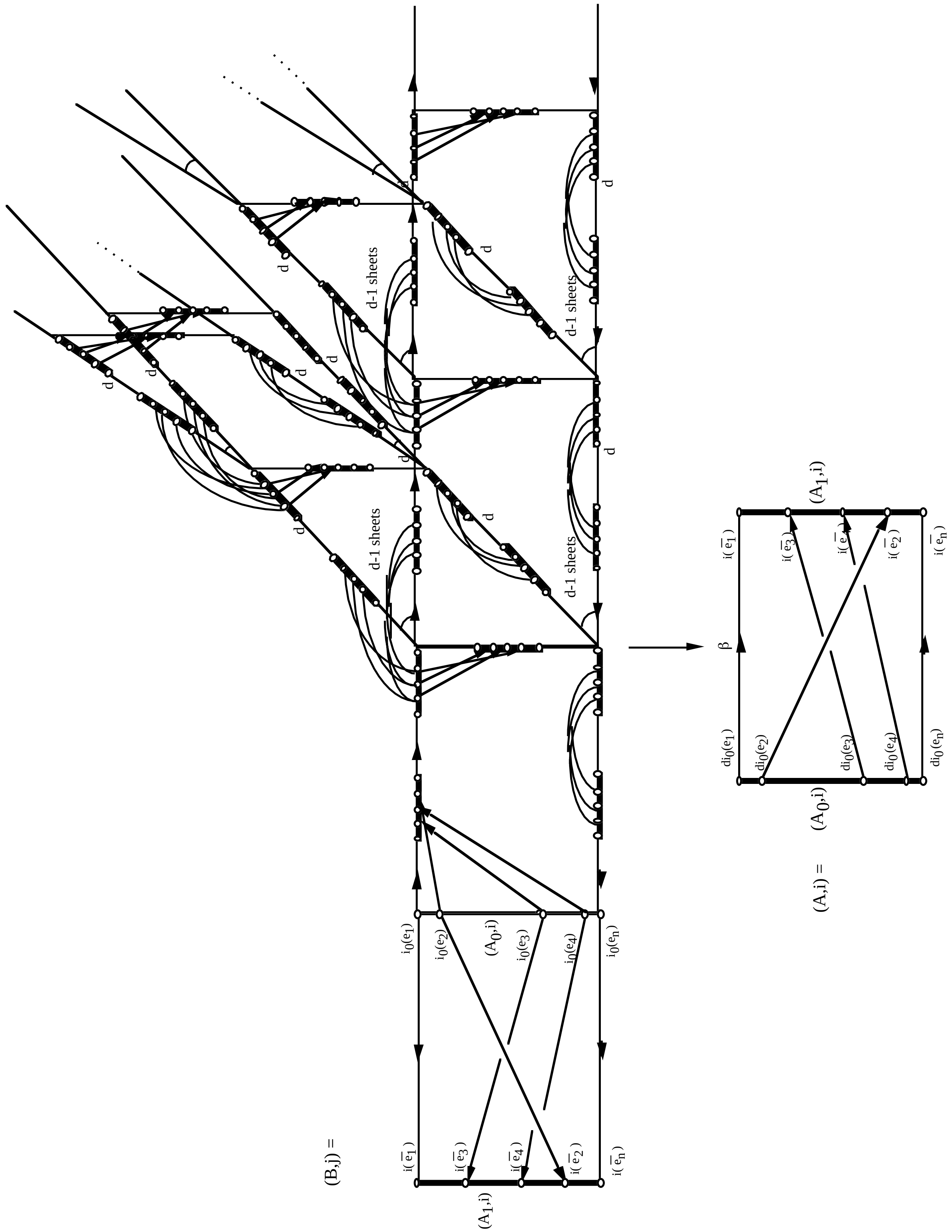



Fig (7.14)

## 8. A covering with infinite fibers for an edge-indexed graph with a ramified separating edge

We recall (section 0) that an edge-indexed graph $(A, i)$ is called *parabolic*, if $X = \widetilde{(A, i)}$ is a parabolic tree, that is, $X$ has only one end. If $(A, i)$ is parabolic, and $(e_1,\ e_2,\ e_3,\ )$ is any infinite reduced path in $(A, i)$, then we have

$$1 = i(e_1) = i(e_2) = i(e_3) = \dots$$

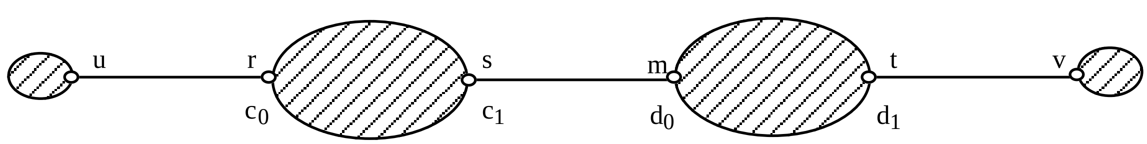
In this section we prove:

**(8.1) Theorem.** *Let $(A, i)$ be an infinite edge-indexed graph with finite volume. Assume that every ramified edge of $(A, i)$ is separating. If $(A, i)$ is not parabolic, then there exists a (necessarily non-uniform) $X$-lattice $\Gamma \leq G_{(A,i)}$, which is a non-uniform $G_{(A,i)}$ lattice.*

Theorem (8.1) follows immediately from Theorem (8.3):

**(8.2) Theorem.** *Let $(A, i)$ be an infinite, unimodular edge-indexed graph with finite volume. Suppose that*

(A, i) =




*for $u \geq 2$, $v \geq 2$, $m \geq 2$. Then there is a covering $p : (B, j) \longrightarrow (A, i)$ of edge-indexed graphs with infinite fibers such that $(B, j)$ is infinite, unimodular, has finite volume and bounded denominators.*

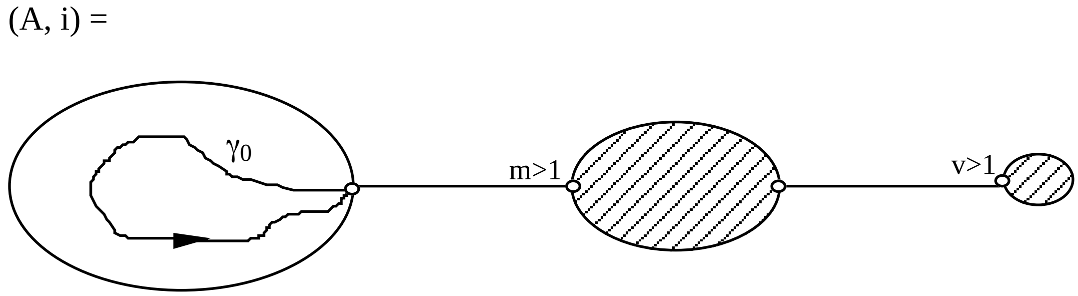
*Proof of Theorem (8.2).* The proof follows easily from ([C1], (5.21)). □

**(8.3) Theorem.** *Let $(A, i)$ be an infinite, unimodular edge-indexed graph with finite volume. Assume that $(A, i)$ is not parabolic, and that every ramified edge of $(A, i)$ is separating. Then there is a covering $p : (B, j) \longrightarrow (A, i)$ of edge-indexed graphs with infinite fibers such that $(B, j)$ is infinite, unimodular, has finite volume and bounded denominators.*

*Proof.* Choose an infinite reduced path $\gamma = (e_1, e_2, \dots)$ in $(A, i)$. Since $(A, i)$ has finite volume, $i(\overline{e}_k) > 1$ for almost all $k = 1, 2, \dots$.

Assume first that $(A, i)$ is a tree. Since $(A, i)$ is infinite and not parabolic, there is an infinite reduced path $\tau = (f_1, f_2, \dots)$ in $(A, i)$ with $i(f_k) > 1$ for some $k \geq 1$. Then we are in the case of Theorem (8.2) and we are done.

Now suppose that $(A, i)$ is not a tree. Since every ramified edge is separating, $(A, i)$ contains an (unramified) closed path $\gamma_0 = (g_1, \dots, g_n)$ of length $n \geq 1$. Since $(A, i)$ is infinite, we can find an infinite reduced path $\gamma = (e_1, e_2, \dots)$ in $(A, i)$ such that $\partial_0 e_1 = \partial_0 g_k$ for some $k \in \{1, \dots n\}$:




Once again, since $(A, i)$ has finite volume, $i(\overline{e}_k) > 1$ for almost all $k = 1, 2, \dots$. We take a 2-sheeted topological cover $(D, j)$ of $(A, i)$ and then $(D, j)$ satisfies the hypotheses of Theorem (8.2) and we are done. □

**(8.4) Lemma.** *Let $(A, i)$ be an infinite, unimodular edge-indexed graph with finite volume. Let $e$ be a ramified non-separating edge with $\Delta(e) = c/d$ (in lowest terms). Then either $d > 1$, (in which case $e$ satisfies the conditions of Theorem (0.6) (Theorem (9.4))), or $c > 1$ (in which case $\overline{e}$ satisfies the conditions of Theorem (0.6) (Theorem (9.4))), or $\Delta(e) = 1$ (that is, $i(e) = i(\overline{e})$), in which case $e$ satisfies the conditions of Theorem (8.5) below.*

*Proof.* Immediate. □

**(8.5) Theorem.** *Let $(A, i)$ be an infinite, unimodular edge-indexed graph with finite volume. If there exists $e \in EA$ such that $i(e) = i(\overline{e}) > 1$, then there is a covering $p : (B, j) \longrightarrow (A, i)$ of edge-indexed graphs with infinite fibers such that $(B, j)$ is infinite, unimodular, has finite volume and bounded denominators.*

*Proof.* Let $e \in EA$ be such that $i(e) = i(\overline{e}) > 1$. Let $(A', i')$ be obtained from $(A, i)$ by subdividing the single edge $e$. That is, the edge pair $(e, \overline{e})$ is replaced by two edge pairs $(f_1, \overline{f_1})$ and $(f_2, \overline{f_2})$ and a new vertex $m$ with

$$\partial_0(f_1) = \partial_0(e),\ \partial_0(f_2) = \partial_1(e),\ \partial_1(f_1) = \partial_1(f_2) = m$$

and

$$i'(f_1) = i'(f_2) = i(e) = i(\overline{e})$$

$$i'(\overline{f_1}) = i'(\overline{f_2}) = 1.$$

Then $(A', i')$ is infinite, unimodular and has finite volume. Moreover, $(A', i')$ contains an arithmetic bridge consisting of the edges $f_1$ and $f_2$ with ramification factor $i(e) > 1$. By Theorem (7.2), there is a covering $p' : (B', j') \longrightarrow (A', i')$ with infinite fibers such that $(B', j')$ is infinite, unimodular, has finite volume and bounded denominators. We modify $(B', j')$ to obtain a covering of $(A, i)$ in the following way. Each local fiber above the vertex $m$ in $(B', j')$ consists of 2 edges with index 1 emanating from vertices that lie over $m$. We replace each of these stars by a single edge pair (hence removing the subdivision that was induced by subdividing

$e$). The resulting edge-indexed graph, denoted $(B, j)$, has all of the desired properties and is a covering of $(A, i)$. □

## 9. Existence of arithmetic bridges

In this section, we prove Theorem (0.6). We make use of the following notation to provide an alternate way of considering bridges. Let $A$ be a connected graph, finite or infinite. Let $X \subseteq VA$ and $Y \subseteq VA$ be subsets of vertices of $A$ then we define:

$$(X, Y) = \{e \in EA \mid \partial_0 e \in X,\ \partial_1 e \in Y\},$$

$$\hat{X} = \text{subgraph of } A \text{ with vertices } X \text{ and edges } (X, X).$$

**(9.1) Lemma.** *Let $A$ be a connected graph with $EA \neq \varnothing$. Let $X \subset VA$ and $Y \subset VA$ be proper subsets of vertices of $A$ satisfying the following conditions:*

(1) $X \cap Y = \varnothing$, $X \cup Y = VA$,

(2) $\hat{X}$ *has $p$ path components and* $\hat{Y}$ *has $q$ path components.*

*Then $(X, Y)$ is a $(p, q)$-geometric bridge for $A$.*

*Proof.* Let $X$ and $Y$ satisfy the conditions above. Then $(X, Y) \neq \varnothing$ as $A$ is connected. Further, $(X, Y) \cap (Y, X) = \varnothing$ as $X \cap Y = \varnothing$. Finally, $A \setminus ((X, Y) \cup (Y, X))$ consists of $\hat{X}$ and $\hat{Y}$. □

**(9.2) Lemma.** *Let $A$ be a connected graph and let $X'$ and $Y'$ be proper subsets of $VA$ satisfying the conditions as in Lemma (9.1). Choose vertices $x_0 \in X'$ and $y_0 \in Y'$. Then there exists proper subsets $X$ and $Y$ of $VA$ satisfying the following conditions:*

(1) $X \cap Y = \varnothing$, $X \cup Y = VA$,

(2) $\hat{X}$ *and* $\hat{Y}$ *are connected,*

(3) $(X, Y) \subseteq (X', Y')$,

(4) $x_0 \in X$ *and* $y_0 \in Y$.

**(9.3) Remarks.**

We note the first two conditions of Lemma (9.2) imply, by Lemma (9.1), that $(X, Y)$ is a geometric bridge. Together we have, with the third condition, that if $(X', Y')$ was a $(p, q)$-arithmetic bridge then $(X, Y)$ is an arithmetic bridge. Finally the last condition implies that if $(X', Y')$ contained a chosen edge $e \in EA$, then $X$ and $Y$ can also be chosen to contain $e$ (simply let $x_0 = \partial_0 e$ and $y_0 = \partial_1 e$).

*Proof of Lemma (9.2).* First we consider the case that $\hat{X}'$ is already connected. That is, $(X', Y')$ forms a $(1, q)$-geometric bridge. If $\hat{Y}'$ is also connected there is nothing to show. Assume $\hat{Y}'$ is not connected. Let $Y$ be the vertices of the connected component of $\hat{Y}'$ containing $y_0$. Let $X = X' \cup (Y' \setminus Y)$. It follows that $X \cap Y = \varnothing$ and $X \cup Y = X' \cup Y' = VA$. We still have $x_0 \in X$ and $y_0 \in Y$. We chose $Y$ so that $\hat{Y}$ would be connected. We claim that $\hat{X}$ is also connected. As $\hat{X} \supseteq \hat{X}'$ and $\hat{X}'$ is connected it suffices to prove that for $y \in (Y' \setminus Y)$ there exists a path $\gamma$ in $\hat{X}'$ from $y$ to some vertex $x \in X'$. As $A$ is connected and $\hat{Y}'$ is not connected there is a path

$$(y = a_0, e_1, a_1, e_2, \ldots, e_n, a_n = y_0)$$

in $A$ from $y$ to $y_0$ which intersects $X'$ before intersecting $Y$. Let $a_k$ be the first vertex in the above sequence such that $a_k \in X'$. The path

$$(y = a_0, e_1, a_1, e_2, \ldots, e_k, a_k)$$

then lies entirely in $\hat{X}$ as desired thus showing $\hat{X}$ is connected. Finally let $e \in (X, Y)$. Then $\partial_1 e \in Y \subseteq Y'$. If $x = \partial_0 e \notin X'$ then we must have $x \in (Y' \backslash Y)$. Thus $e \in (Y', Y')$ contradicting our definiton of $Y$ as the vertices of a connected component of $\hat{Y}'$ thus completing the proof in the case that $\hat{X}'$ is connected.

Now let us assume that $\hat{X}'$ is not connected. By the preceeding argument it suffices to construct the sets of vertices $X$ and $Y$ satisfying only:

(1) $X \cap Y = \varnothing$, $X \cup Y = VA$,
(2) $\hat{X}$ is connected,
(3) $(X, Y) \subseteq (X', Y')$,
(4) $x_0 \in X$ and $y_0 \in Y$.

Let $X$ be the vertices of the connected component of $\hat{X}'$ containing $x_0$. Let $Y = Y' \cup (X' \setminus X)$. Then $X \cap Y = \varnothing$ and $X \cup Y = X' \cup Y' = VA$. We chose $X$ so that $\hat{X}$ is connected and $x_0 \in X$. As $Y' \subset Y$ we also have $y_0 \in Y$. Finally let $e \in (X, Y)$ with $x = \partial_0 e \in X \subset X'$ and $y = \partial_1 e \in Y$. We wish to show that $y \in Y'$. If not we must have $y \in (X' \setminus X)$. Thus $e \in (X', X')$ contradicting our definiton of $X$ as the vertices of a connected component of $\hat{X}'$. □

In the following theorem any fraction will assume to be in lowest terms unless otherwise stated.

**(9.4) Theorem.** *Let $(A, i)$ be a connected locally finite unimodular edge-indexed graph. Let $e \in EA$ be an edge with $\Delta(e) = \frac{c}{d}$ and $p \geq 2$ a prime number such that $p \mid d$. If $e$ is not separating, then there exists an arithmetic bridge $\beta$ of $(A, i)$ with $n \geq 2$ edges,and with ramification factor $p$ such that $e \in \beta$.*

*Proof.* Let $a_0 = \partial_0 e$ and $a_1 = \partial_1 e$. By Lemmas (9.1) and(9.2) above, it suffices to show that there exists set $X_0 \subset VA$ and $X_1 \subset VA$ such that

(i) $X_0 \cap X_1 = \varnothing$, $X_0 \cup X_1 = VA$,
(ii) $p \mid i(e)$ for all $e \in (X_0, X_1)$,
(iii) $a_0 \in X_0$ and $a_1 \in X_1$.

As $(A, i)$ is unimodular, the rational numbers $\dfrac{\Delta b}{\Delta a}$ are well-defined for all $a, b \in VA$. Let the vertices $VA$ be labeled $\{a_0, a_1, a_2, \ldots, a_n\}$ if $A$ is finite and labeled $\{a_0, a_1, a_2, \ldots\}$ if $A$ is infinite. For $1 \leq k \leq |VA|$ we define

$$A^k := \text{the full subgraph with vertices } \{a_0, a_1, \ldots, a_k\}.$$

We prove by induction on $k$ that for $1 \leq k \leq |VA|$ there exists sets $X_0^k \subset VA^k$ and $X_1^k \subset VA^k$ such that

(a) $X_0^k \cap X_1^k = \varnothing$, $X_0^k \cup X_1^k = VA^k$,

(b) For $x_0 \in X_0^k$ and $x_1 \in X_1^k$ with $\dfrac{\Delta x_1}{\Delta x_0} = \dfrac{c}{d}$ we have $p \mid d$,

(c) $a_0 \in X_0^k$ and $a_1 \in X_1^k$.

Note that condition (b) implies that $p \mid i(e)$ for all $e \in (X_0^k, X_1^k)$. Thus this suffices to prove the existence of our arithmetic bridge. If $k = 1$ then we let $A_0^1 = a_0$ and $A_1^1 = a_1$. Then conditions (a) and (c) are satisfied. As $\dfrac{\Delta a_1}{\Delta a_0} = \Delta(e)$, condition (b) follows from the hypotheses of our theorem.

For the finite induction step, assume we have sets $X_0^k$ and $X_1^k$ satisfying the inductive hypotheses. We claim either

$$\{X_0^{k+1} = X_0^k \cup \{a_{k+1}\}, \quad X_1^{k+1} = X_1^k\}$$

or

$$\{X_0^{k+1} = X_0^k, \quad X_1^{k+1} = X_1^k \cup \{a_{k+1}\}\}$$

also satisfy the inductive hypotheses. Certianly both possibilities satisfy conditions (a) and (c). Suppose, however that neither satisfy condition (b). Then there exists $x_0 \in X_0^k$ and $x_1 \in X_1^k$ such that writing

$$\frac{c_0}{d_0} = \frac{\Delta a_{k+1}}{\Delta x_0} \text{ and } \frac{c_1}{d_1} = \frac{\Delta x_1}{\Delta a_{k+1}}$$

we have $p \nmid d_0$ and $p \nmid d_1$. But

$$\frac{\Delta x_1}{\Delta x_0} = \frac{\Delta a_{k+1}}{\Delta x_0} \cdot \frac{\Delta x_1}{\Delta a_{k+1}} = \frac{c_0 c_1}{d_0 d_1}.$$

While the latter is not in lowest terms, it follows that $p \nmid d$ where $d$ is the denominator of $\dfrac{\Delta x_1}{\Delta x_0}$ written in lowest terms, thus contradicting our assumptions on the sets $X_0^k$ and $X_1^k$.

Finally for the transfinite induction step, let $X_0^\infty = \bigcup_{k=1}^\infty X_0^k$ and $X_1^\infty = \bigcup_{k=1}^\infty X_1^k$. Then $X_0^\infty \cap X_1^\infty = \varnothing$. If $a \in A^\infty = VA$ then $a \in A^k$ for some finite $k$ and hence $a \in X_0^\infty \cup X_1^\infty$. Likewise if $x_0 \in X_0^\infty$ and $x_1 \in X_1^\infty$ then choosing $k$ large enough but finite we have $x_0 \in X_0^k$ and $x_1 \in X_1^k$. Thus writing $\dfrac{\Delta x_1}{\Delta x_0} = \dfrac{c}{d}$ we have $p \mid d$ as desired. Finally we clearly have $a_0 \in X_0^\infty$ and $a_1 \in X_1^\infty$, thus completing our proof. □

**(9.5) Theorem.** *Let $X$ be a locally finite tree with more than one end, and let $H \leq G = Aut(X)$ be a closed, saturated subgroup. If $H$ contains a non-uniform $X$-lattice then $H$ contains a non-uniform $H$-lattice.*

*Proof.* Let $(A, i) = I(H \backslash\backslash X)$. Since $H$ is saturated, we may translate the assumptions on $H$ into properties of $(A, i)$. Since $H$ contains a non-uniform $X$-lattice, $Vol_{a_0}(A, i) < \infty$, for $a_0 \in VA$ so $(A, i)$ contains a ramified edge. By Theorem (8.3), if every ramified edge of $(A, i)$ is separating, then we obtain a covering $p : (B, j) \longrightarrow (A, i)$ with the desired properties to give rise to a non-uniform $X$-lattice which is also a non-uniform $H$-lattice, that is, $(B, j)$ is unimodular, has finite volume, bounded denominators, and the projection $p$ has infinite fibers. Otherwise there exists a ramified non-separating edge $e$, and by Lemma (8.4) either $e$ or $\overline{e}$ satisfies the hypothesis of Theorem (0.6) (Theorem (9.4)) and hence $e$ (or $\overline{e}$) is contained in an arithmetic bridge $\beta$ in $(A,i)$ of $n \geq 2$ edges. By Theorem (0.4) (Theorem (7.2)) we obtain a covering with the desired properties to give rise to a non-uniform $H$-lattice. The only remaining possibility is that $E$ satisfies the conditions of Theorem (8.5) and hence we obtain a covering with infinite fibers that gives rise to a non-uniform $H$-lattice. □

Applying Theorem (0.1) we also have $\mu(H\backslash\backslash X) < \infty$ and we obtain the implication $(a) \implies (b)$ of Theorem (0.2) as a corollary of Theorem (9.5).

We obtain Corollary (0.7) from Theorem (9.5). This is a generalization of Corollary (0.9) in [BCR].

**(9.6) Corollary.** *Let $X$ be a locally finite tree, $G = Aut(X)$, $\mu$ a (left) Haar measure on $G$, $H \leq G$ a unimodular closed subgroup acting without inversions, $p_H : X \longrightarrow A = H\backslash X$, and $(A,i) = I(H\backslash\backslash X)$. Assume that $H = G_{(A,i)}$ and that $\mu(H\backslash\backslash X) < \infty$. If $X$ has more than one end, and $H\backslash X$ is infinite, then there exists a (necessarily non-uniform) $X$-lattice $\Gamma \leq H$, which is a non-uniform $H$-lattice.*

*Proof.* Since $H$ is unimodular, $\mu(H\backslash\backslash X) < \infty$ and $H\backslash X$ is infinite, $H$ contains a (necessarily non-uniform) $X$-lattice by ([BCR], Theorem (0.5)). Since $X$ has more than one end and $H = G_{(A,i)}$, that is $H$ is saturated, we apply Theorem (9.5) to obtain a non-uniform $X$-lattice which is also a non-uniform $H$-lattice. □

## 10. Towers of coverings with infinite fibers and finite volume

In this section we prove:

**(10.1) Theorem.** *Let $(A,i)$ be an infinite, unimodular edge-indexed graph with finite volume. If $(A,i)$ contains an arithmetic bridge with $n \geq 2$ edges, then $(A,i)$ has an infinite sequence of coverings:*

$$(B_0,j_0) \xrightarrow{p_0} (B_1,j_1) \xrightarrow{p_1} (B_2,j_2) \xrightarrow{p_2} \dots \quad \longrightarrow \quad (A,i)$$

*with infinite fibers such that each $(B_k,j_k)$ is infinite, unimodular, has finite volume and bounded denominators. Hence we obtain an infinite ascending chain of closed subgroups of $Aut(X)$:*

$$G_{(B_0,j_0)} \quad \leq \quad G_{(B_1,j_1)} \quad \leq \quad G_{(B_2,j_2)} \quad \leq \quad \dots \quad \leq \quad G_{(A,i)},$$

*and non-uniform $G_{(A,i)}$-lattices $\Gamma_k$ with $\Gamma_k \leq G_{(B_k,j_k)}$, $k = 0,1,2\dots$.*

*Proof.* Theorem (7.2) provides us with a single covering $p : (B,j) \longrightarrow (A,i)$ with the desired properties. We refer to the notation of Fig (7.14) to describe the sequence of coverings $(B_0,j_0) \xrightarrow{p_0} (B_1,j_1) \xrightarrow{p_1} (B_2,j_2) \xrightarrow{p_2} \dots \longrightarrow (A,i)$ indicated above. We set $(B_0,j_0) = (B,j)$, and the projection $p_0 : (B_0,j_0) \longrightarrow (B_1,j_1)$ projects the first 'vertical sheet' onto the 'bottom sheet', as indicated in Fig (10.2). Continuing to project the next vertical sheet onto the bottom sheet defines the next projection $p_1 : (B_1,j_1) \longrightarrow (B_2,j_2)$, and so on.

It is obvious that each $p_k$, $k = 0,1,2\dots$ is a covering of edge-indexed graphs with infinite fibers, and it is clear that each $(B_k,j_k)$ is infinite, unimodular and has bounded denominators.

An easy application of the Bass-Rosenberg volume formula ([R]) shows that each $(B_k, j_k)$ has finite volume $k = 0, 1, 2 \ldots$, and so we obtain a sequence of coverings with the desired properties.

The existence of the infinite ascending chain of closed subgroups of $Aut(X)$

$$G_{(B_0,j_0)} \quad \leq \quad G_{(B_1,j_1)} \quad \leq \quad G_{(B_2,j_2)} \quad \leq \quad \ldots \quad \leq \quad G_{(A,i)}$$

follows from (1.6).

Finally, each $(B_k, j_k)$ admits a finite faithful grouping, denoted $\mathbb{B}_k$, $k = 0, 1, 2, \ldots$ which gives rise to a non-uniform lattice

$$\Gamma_k = \pi_1(\mathbb{B}_k, b_k) \leq G_{(B_k,j_k)}, \quad k = 0, 1, 2, \ldots,$$

for $b_k \in VB_k$, $k = 0, 1, 2, \ldots$. □

In [CC] we prove Theorem (10.1) in the case that $(A, i)$ is parabolic and hence has no arithmetic bridge with $n \geq 2$ edges.

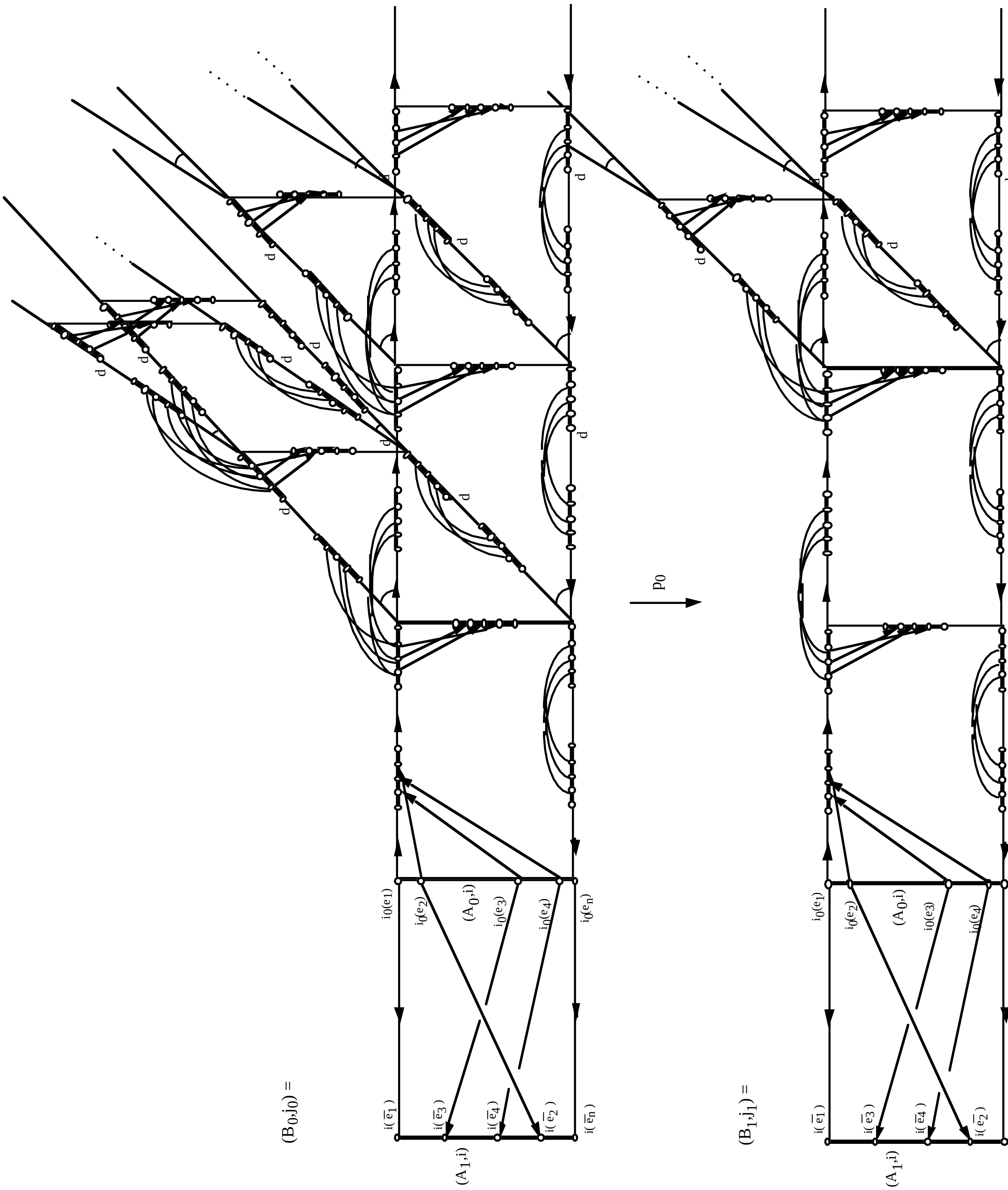



Fig (10.2)

## References


[BCR] Bass, H, Carbone L, and Rosenberg, G, *The existence theorem for tree lattices*, Appendix [BCR], '*Tree Lattices*' by Hyman Bass and Alex Lubotzky (2000), Progress in Mathematics 176, Birkhauser, Boston.

[BK] Bass H and Kulkarni R, *Uniform tree lattices*, Journal of the Amer Math Society **3 (4)** (1990).

[BL] Bass H and Lubotzky A, *Tree lattices*, Progress in Mathematics 176, Birkhauser, Boston (2000).

[BT] Bass H and Tits J, *A Discreteness Criterion for Certain Tree Automorphism Groups*, Appendix [BT], '*Tree Lattices*' by Hyman Bass and Alex Lubotzky (2000), Progress in Mathematics 176, Birkhauser, Boston.

[C1] Carbone, L, *Non-uniform lattices on uniform trees*, Memoirs of the AMS, vol. 152, no. 724 (July 2001).

[C2] Carbone, L, *Non-minimal tree actions and the existence of non-uniform tree lattices*, Preprint (2001).

[CC] Carbone, L and Clark, D, *Lattices on parabolic trees*, Communications in Algebra, Vol. 30, Issue 4 (2002).

[R] Rosenberg, G, *Towers and covolumes of tree lattices*, PhD. Thesis, Columbia University (2000).



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