TREE LATTICE SUBGROUPS

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Abstract. Let X be a locally finite tree and let $G = Aut(X)$. Then G is naturally a locally compact group. A discrete subgroup $\Gamma \leq G$ is called an *X-lattice*, or a *tree lattice* if Γ has finite covolume in G. The lattice Γ is encoded in a graph of finite groups of finite volume. We describe several methods for constructing a pair of X-lattices (Γ', Γ) with $\Gamma \leq \Gamma'$, starting from 'edge-indexed graphs' (A', i') and (A, i) which correspond to the edge-indexed quotient graphs of their (common) universal covering tree by Γ' and Γ respectively. We determine when finite sheeted topological coverings of edge-indexed graphs give rise to a pair of lattice subgroups (Γ, Γ') with an inclusion $\Gamma \leq \Gamma'$. We describe when a 'full graph of subgroups' and a 'subgraph of subgroups' constructed from the graph of groups encoding a lattice Γ' gives rise to a lattice subgroup Γ and an inclusion $\Gamma \leq \Gamma'$. We show that a nonuniform X-lattice Γ contains an infinite chain of subgroups $\Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \dots$ where each Λ_k is a uniform X_k-lattice and X_k is a subtree of X. Our techniques, which are a combination of topological graph theory, covering theory for graphs of groups, and covering theory for edge-indexed graphs, have no analog in classical covering theory. We obtain a local necessary condition for extending coverings of edge-indexed graphs to covering morphisms of graphs of groups with abelian groupings. This gives rise to a combinatorial method for constructing lattice inclusions $\Gamma \leq \Gamma' \leq H \leq G$ with abelian vertex stabilizers inside a closed and hence locally compact subgroup H of G. We give examples of lattice pairs $\Gamma \leq \Gamma'$ when H is a simple algebraic group of K-rank 1 over a nonarchimedean local field K and a rank 2 locally compact complete Kac-Moody group over a finite field. We also construct an infinite descending chain of lattices $\dots \Gamma_2 \leq \Gamma_1 \leq \Gamma \leq H \leq G$ with abelian vertex stabilizers.

1. Introduction

Let X be a locally finite tree and let $G = Aut(X)$. Then G is naturally a locally compact group. A discrete subgroup $\Gamma \leq G$ is called an *X-lattice*, or a *tree lattice* if

$$Vol(\Gamma \backslash\backslash X) \quad := \quad \sum_{x \in V(\Gamma \backslash X)} \frac{1}{|\Gamma_x|}$$

is finite, and a *uniform X-lattice* if $\Gamma \backslash X$ is a finite graph, *non-uniform* otherwise.

In [C2] and [BCR] the authors gave the necessary and sufficient conditions for the existence of X-lattices. In this work we determine how to construct pairs (Γ, Γ') of tree lattice subgroups with $\Gamma \leq \Gamma' \leq G$. We also extend these methods to construct pairs (Γ, Γ') with $\Gamma \leq \Gamma' \leq H$ where H is a closed and hence locally compact subgroup of G, as well as infinite descending chain of lattices $\dots \Gamma_2 \leq \Gamma_1 \leq \Gamma \leq H \leq G$ with abelian vertex stabilizers.

In the setting of topological covering theory, there is a correspondence between coverings, $p : X \to A$, and subgroups of $\pi_1(A)$, and this gives essentially 'one' subgroup up to conjugacy. Here with lattices given by their quotient graphs of groups, we are in an 'orbifold' setting where coverings have an extra ingredient, namely isotropy groups. We make use of this additional data

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to construct lattice subgroups by using subgroups of isotropy (vertex) groups. This will give rise to zero, one, finitely many, or infinitely many possible subgroups up to isomorphism. In fact in [CR1] the authors exhibited infinite ascending chain of lattice subgroups, all with the same quotient graph.

We describe several methods (Sections 2 and 3) for constructing a pair of X-lattices (Γ', Γ) with $\Gamma \leq \Gamma'$, starting from 'edge-indexed graphs' (A', i') and (A, i) which will correspond to the edge-indexed quotient graphs of their (common) universal covering tree by Γ' and Γ respectively. We determine when finite sheeted topological coverings of edge-indexed graphs give rise to a pair of lattice subgroups (Γ, Γ') with an inclusion $\Gamma \leq \Gamma'$. We describe when a 'full graph of subgroups' and a 'subgraph of subgroups' constructed from the graph of groups encoding a lattice Γ' gives rise to a lattice subgroup Γ and an inclusion $\Gamma \leq \Gamma'$.

Our techniques are a combination of topological graph theory, covering theory for graphs of groups ([B]), and covering theory for edge-indexed graphs developed in [C1] and [BCR]. As an application, we show (Section 4) that a non-uniform X-lattice Γ contains an infinite chain of subgroups $\Lambda_1 < \Lambda_2 < \Lambda_3 < \dots$ where each Λ_k is a uniform X_j-lattice, X_k a subtree of X.

Let $\phi : (A, i) \to (A', i')$ be a covering of edge-indexed graphs (defined in Section 2). We wish to determine if it is possible to extend ϕ to a covering morphism of graphs of groups as in [B]. In Sections 5 and 6 we give a local necessary condition for extending ϕ to a covering $\Phi : \mathbb{A} \longrightarrow \mathbb{A}'$ of graphs of groups, where $\mathbb{A}'$ and $\mathbb{A}$ are abelian groupings of (A', i') and (A, i) respectively. For coverings $\phi : (A, i) \longrightarrow (A', i')$ satisfying suitable conditions (Sections 5 and 6 and [BL], [BCR], [C1]) these will give pairs of lattice subgroups $\Gamma \leq \Gamma'$ with abelian stabilizers. In Section 7 we provide some examples to illustrate some of the constructions proposed here.

In Section 8 we consider the problem of constructing lattices in a closed, hence locally compact subgroup H of $G = Aut(X)$ for X a locally finite tree. Our methods apply to the case where X is the Tits building of a BN-pair for H and is a homogeneous tree. Examples of such a group are H a simple algebraic group of K-rank 1 over a nonarchimedean local field K or a rank 2 locally compact Kac-Moody group over a finite field. We discuss several known techniques for constructing cocompact lattices Γ in H with quotient $\Gamma \backslash X$ a simplex. We then use a combination of methods to construct examples of lattice pairs $\Gamma' \leq \Gamma \leq H$ where Γ and Γ' both have abelian vertex stabilizers.

Our methods can be extended to construct an infinite descending chain of lattices $\dots \Gamma_2 \leq \Gamma_1 \leq \Gamma \leq H$ where the Γ_i, $i \geq 1$ all have abelian vertex stabilizers. We exhibit such an infinite descending chain when H is a rank 2 locally compact Kac-Moody group over the field $\mathbb{F}_2$ with 2 elements.

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2. Notation and preliminaries

2.1. **Tree lattices, edge-indexed graphs, volumes and coverings.** An *edge-indexed graph* (A, i) consists of an underlying graph A, assumed to be locally finite, and an assignment of a positive integer $i(e) > 0$ to each oriented edge $e \in EA$. We use $\partial_0 e$ and $\partial_1 e$ to denote the initial and terminal vertices of an edge $e \in EA$. Let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups, with underlying graph A, vertex groups $(\mathcal{A}_a)_{a \in VA}$, edge groups $(\mathcal{A}_e = \mathcal{A}_{\overline{e}})_{e \in EA}$ and monomorphisms $\alpha_e \colon \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}$. Then $I(\mathbb{A}) = (A, i)$ where $i : EA \to \mathbb{Z}_{>0}$ is defined as $i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e \mathcal{A}_e]$, which we assume to be finite, for all $e \in EA$.

Given an edge-indexed graph (A, i), a graph of groups $\mathbb{A}$ such that $I(\mathbb{A}) = (A, i)$, is called a *grouping* of (A, i). We call $\mathbb{A}$ a *finite grouping* if the vertex groups $\mathcal{A}_a$ are finite and a *faithful grouping* if $\mathbb{A}$ is a faithful graph of groups, that is if $\pi_1(\mathbb{A}, a)$ acts faithfully on $X = \widetilde{(\mathbb{A}, a)}$.

For an edge $e \in EA$, define $\Delta(e) = \dfrac{i(\overline{e})}{i(e)}$. If $\gamma = (e_1, \dots, e_n)$ is a path in A, set:

$$\Delta(\gamma) = \Delta(e_1) \dots \Delta(e_n).$$

Definition 1. *An indexed graph (A, i) is unimodular if $\Delta(\gamma) = 1$ for all closed paths γ in A.*

Now assume that (A, i) is unimodular. Pick a base point $a_0 \in VA$, and define, for $a \in VA$,

$$N_{a_0}(a) = \frac{\Delta a}{\Delta a_0} (= \Delta(\gamma) \text{ for any path } \gamma \text{ from } a_0 \text{ to } a) \in \mathbb{Q}_{>0}.$$

For $e \in EA$, put

$$N_{a_0}(e) \quad := \quad \frac{N_{a_0}(\partial_0(e))}{i(e)}.$$

Following ([BL], (2.6)), we say that (A, i) has *bounded denominators* if $\{N_{a_0}(e) \mid e \in EA\}$ has bounded denominators. This condition is automatic if A is finite. As in [BK] the functions $N : A \longrightarrow \mathbb{Q}^{\times}_{>0}$ as above are called *vertex orderings* of (A, i). We call N *integral* if for all $e \in EA$, we have $N(\partial_0(e))/i(e) \in \mathbb{Z}$ and hence $N(a) \in \mathbb{Z}$ for $a \in VA$.

Theorem 2.1. *([BK], (2.4)) The following conditions on an edge-indexed graph (A, i) are equivalent.*

(a) (A, i) admits a finite faithful grouping.
(b) (A, i) is unimodular and has bounded denominators.
(c) (A, i) admits an integral vertex ordering.

We define the *volume* of an edge-indexed graph (A, i) at a basepoint $a_0 \in VA$:

$$Vol_{a_0}(A, i) \quad := \quad \sum_{a \in VA} (\frac{\Delta a_0}{\Delta a}).$$

We write $Vol(A, i) < \infty$ if $Vol_a(A, i) < \infty$ for some, and hence every $a \in VA$. If $\mathbb{A}$ is a finite grouping of (A, i), then we have ([BL], (2.6.15)):

$$Vol(\mathbb{A}) \quad = \quad \frac{1}{|\mathcal{A}_a|} Vol_a(A, i).$$

In order to construct an X-lattice, we begin with an edge-indexed graph (A, i). Then (A, i) determines $X = \widetilde{(A, i, a_0)}$ up to isomorphism ([BL], Ch. 2). We say that (A, i) *admits a lattice* if (A, i) admits a grouping $\mathbb{A}$ such that $\pi_1(\mathbb{A}, a_0)$ is an X-lattice. This is the case if and only if $\mathbb{A}$ is a (faithful) graph of finite groups of finite volume.

Let (A, i) and $B, j)$ be edge-indexed graphs. Following ([BL], (2.5)) , a *covering* $p : (B, j) \longrightarrow (A, i)$ is a graph morphism $p : B \longrightarrow A$ such that for all $e \in EA$, $\partial_0(e) = a$, and $b \in p^{-1}(a)$, we have $i(e) = \sum_{f \in p^{-1}_{(b)}(e)} j(f)$, where $p_{(b)} : E^B_0(b) \longrightarrow E^A_0(a)$ is the local map on stars $E^B_0(b)$ and $E^A_0(a)$ of vertices $b \in VB$ and $a \in VA$. If $b \in VB$, $p(b) = a \in VA$, then we can identify

$$\widetilde{(A, i, a)} = X = \widetilde{(B, j, b)}.$$

2.2. **Lattices pairs from full graphs of subgroups.** In this subsection, we describe how to produce a pair of lattice subgroups ($\Gamma \leq \Gamma'$) by constructing a 'full graph of subgroups' encoding Γ starting with a graph of groups for a lattice Γ'. This method was used in ([B], 1.14) and ([CR], section 1) and has no analog in classical topological covering theory.

Let (A, i) be an edge-indexed graph. Let $\mathbb{A}'$ and $\mathbb{A}$ be groupings of (A, i). Then $\mathbb{A} = (A, \mathcal{A})$ is called a *full graph of subgroups* of $\mathbb{A}' = (A, \mathcal{A}')$ (as in ([B], (1.14)) if $\mathcal{A}_a \leq \mathcal{A}'_a$ for $a \in A$, and for $e \in EA$, $\mathcal{A}_e \leq \mathcal{A}'_e$, and $\alpha_e = \alpha'_e|_{\mathcal{A}_e}$. We further assume that for $e \in EA$, with $\partial_0 e = a$, $\mathcal{A}_a \cap \alpha_e \mathcal{A}'_e = \alpha_e \mathcal{A}_e$, that is $\mathcal{A}_a/\alpha_e\mathcal{A}_e \longrightarrow \mathcal{A}'_a/\alpha_e\mathcal{A}'_e$ is injective, and hence bijective. This assumption implies that $\pi_1(\mathbb{A}, a) \leq \pi_1(\mathbb{A}', a')$ ([B], (1.14)).

If $\mathbb{A}'$ is a graph of finite groups of finite volume, then this yields a lattice subgroup pair $\Gamma \leq \Gamma'$ with $\Gamma = \pi_1(\mathbb{A}, a)$ and $\Gamma' = \pi_1(\mathbb{A}', a')$, $a, a' \in VA$.

2.3. **Lattice pairs from subgraphs of subgroups.** In this subsection, we describe how to produce a pair of lattice subgroups ($\Gamma \leq \Gamma'$) by constructing a 'subgraph of subgroups' $\mathbb{A}$ of $\mathbb{A}'$ on (A', i') as in ([B], 1.14). We begin with an edge-indexed graph (A', i') that admits an X-lattice, and a lattice grouping $\mathbb{A}' = (A', \mathcal{A}')$ of (A', i'). Let $\Gamma' = \pi_1(\mathbb{A}', a')$. Let (A, i) be a subgraph of (A', i'), where $i = i'|_A$. We aim to construct a lattice subgroup $\Gamma \leq \Gamma'$ by constructing a graph of groups on (A, i) using subgroups of $\mathbb{A}'$.

It is easy to verify that if $Vol(A', i') < \infty$, then $Vol(A, i) < \infty$. Let (A', i') be a unimodular, edge-indexed graph with bounded denominators. Let (A, i) with $i = i'|_A$ be a subgraph of (A', i'). Then (A, i) also has bounded denominators. It follows that if (A', i') admits a lattice and (A, i) is a subgraph with $i = i'|_A$, then (A, i) also admits a lattice.

To construct a lattice grouping $\mathbb{A}$ of (A, i), we choose subgroups $\mathcal{A}_a \leq \mathcal{A}'_a$, $a \in VA$, $\mathcal{A}_e \leq \mathcal{A}'_e$, $e \in EA$ and $\alpha_e = \alpha'_e|_{\mathcal{A}_e}$. We further assume that for $e \in EA$, $\partial_0 e = a$, we have $\mathcal{A} \cap \alpha'_e \mathcal{A}'_e = \alpha'_e \mathcal{A}_e$ that is, $\mathcal{A}_a/\alpha_e\mathcal{A}_\alpha \to \mathcal{A}'_a/\alpha_e\mathcal{A}'_e$ is injective. Then $\mathbb{A}$ is a 'subgraph of subgroups' of $\mathbb{A}'$ in the sense of ([B],(1.14)). If $\mathcal{A}_a = \mathcal{A}'_a$ for some, hence every, vertex $a \in VA$ then we call $\mathbb{A}$ a 'subgraph of full groups' of $\mathbb{A}'$. Let $\Gamma = \pi_1(\mathbb{A}, a)$, $a \in VA$ then we obtain an inclusion $\Gamma \leq \Gamma'$ which is of finite index k, where

$$k = \frac{Vol(\Gamma \backslash\backslash X)}{Vol(\Gamma' \backslash\backslash X)} = \frac{Vol(\mathbb{A})}{Vol(\mathbb{A}')}.$$

Moreover, if A is a finite subgraph of A', then $\mathbb{A}$ is an *uniform* lattice.

3. Finite sheeted topological coverings

In this section we will construct a lattice subgroup pair $\Gamma \leq \Gamma'$ using topological coverings of the underlying edge-indexed graphs. We begin with an edge-indexed graph (A', i') that admits an X-lattice, where $X = \widetilde{(A', i')}$. Let $\mathbb{A}'$ be a finite faithful grouping of finite volume of (A', i'). Let $\Gamma' = \pi_1(\mathbb{A}', a')$, $a' \in VA'$, then Γ' is an X-lattice, uniform if A' is finite, non-uniform if A' is infinite. We assume that $\pi_1(A) \neq \{1\}$. Let $p : (A, i) \to (A', i')$ be a finite sheeted (n-fold) topological covering; that is, locally (on the star of each vertex) p is an isomorphism, and p is index-preserving. Then $X = \widetilde{(A, i)}$ (section 2 and [BL]). We seek a lattice grouping of (A, i). As in ([R], p108) we observe that the grouping $\mathbb{A}'$ on (A', i') induces a finite faithful grouping $\mathbb{A}$ on (A, i) with $Vol(\mathbb{A}) = n \cdot \mathbb{A}'$. It follows ([BK], 2.4) that (A, i) admits a lattice, namely $\Gamma = \pi_1(\mathbb{A}, a)$ for $a \in VA$. Since $\Gamma' = \pi_1(\mathbb{A}', a')$ and $\Gamma = \pi_1(\mathbb{A}, a)$ are X-lattices, the inclusion

$\Gamma \leq \Gamma'$ is automatically of finite index k where

$$k = \frac{Vol(\Gamma\backslash\backslash X)}{Vol(\Gamma'\backslash\backslash X)} = \frac{Vol(\mathbb{A})}{Vol(\mathbb{A}')}.$$

The following lemma is easily verified directly, and also follows from the observations above.

Lemma 3.1. *Let (A', i') be an edge-indexed graph. If (A', i') admits a lattice and $p : (A, i) \to (A', i')$ is a finite sheeted topological covering, then (A, i) admits a lattice. That is,*
(a) if (A', i') is unimodular, then (A, i) is unimodular

(b) if (A', i') has finite volume, then (A, i) has finite volume

(c) if (A', i') has bounded denominators, then (A, i) has bounded denominators.

4. Approximating non-uniform tree lattices by uniform ones.

To prove Theorem 4.2, will make use of the following lemma.

Lemma 4.1. *Let (A, i) be an infinite edge-indexed graph. Let $\mathbb{A}$ be a finite faithful grouping of (A, i). Then there is a finite subgraph (A_0, i_0) of (A, i) on which the restriction $\mathbb{A} \mid_{A_0}$ is faithful.*

Proof Write $A = \cup_{k=1}^{\infty} A_k$ where $A_1 \subset A_2 \subset A_3 \subset \dots$ are all finite graphs. Choose a basepoint $a \in A_1$. For each $k \geq 1$ let N_k be the kernel of the action of $\pi_1(\mathbb{A} \mid_{A_k}, a) < \pi_1(\mathbb{A}, a)$ on $(\widetilde{\mathbb{A} \mid_{A_k}, a})$. Then N_k fixes the subtree $(\widetilde{\mathbb{A} \mid_{A_k}, a})$ of $(\widetilde{\mathbb{A}, a})$. So $N_1 \supset N_2 \supset N_3 \supset \dots$. Since $\mathbb{A}$ is a finite grouping, all N_k are finite. As $\mathbb{A}$ is faithful we must have $\cap_{k=1}^{\infty} N_k = \{1\}$. Thus $N_k = \{1\}$ for some k. We now take A_0 to be A_k for such k. □

Theorem 4.2. *Let Γ be a non-uniform X-lattice.Then Γ contains an infinite chain of subgroups $\Lambda_1 < \Lambda_2 < \Lambda_3 < \dots$ where each Λ_k is a uniform X_k-lattice, X_k a subtree of X.*

Proof Let $\mathbb{A}$ be the graph of groups for Γ and let $(A, i) = I(\mathbb{A})$. Then $\mathbb{A}$ is a finite faithful grouping of (A, i). Let (A_0, i_0) be as in Lemma 4.1. Let

$$A_0 \subset A_1 \subset A_2 \subset \dots$$

be any infinite sequence of finite subgraphs of A. We define an indexing on each A_k for $k \geq 0$ by $i_k = i \mid_{A_k}$. Then each (A_k, i_k) is unimodular, since a closed path in (A_k, i_k) is closed in (A, i).

The universal covering X_k of (A_k, i_k) is a subtree of X. This defines a tower of subtrees of X:

$$X_0 \subset X_1 \subset X_2 \subset \dots$$

For groupings of (A_k, i_k), choose subgraphs of full groups of $\mathbb{A}$, that is take $\mathbb{A}_k$ to be $\mathbb{A} \mid_{A_k}$. We get inclusions of fundamental groups

$$\pi_1(\mathbb{A}_0, a) < \pi_1(\mathbb{A}_1, a) < \pi_1(\mathbb{A}_2, a) < \dots < \Gamma.$$

Set $\Lambda_k = \pi_1(\mathbb{A}_k)$. Then we get a tower of inclusions:

$$\Lambda_0 < \Lambda_1 < \Lambda_2 < \dots < \Gamma.$$

As in Lemma 4.1 the kernels of the actions of $\pi_1(\mathbb{A}_k, a)$ are descending and $\mathbb{A}_0$ is faithful hence we must have that each $\mathbb{A}_k$ is faithful. Thus each Λ_k is a uniform X_k - lattice, where X_k is the universal cover of (A_k, i_k), and X_k is a subtree of X. Of course Λ_k may not be uniform or even a lattice as a subgroup of Γ.□

5. Coverings of abelian groupings

Let $\phi : (A, i) \to (A', i')$ be a covering of edge-indexed graphs, and assume that (A, i) and (A', i') admit finite groupings. In this section we consider the following:

Question 5.1. *Is it possible to find faithful finite groupings $\mathbb{A}$ and $\mathbb{A}'$ of (A, i) and (A', i') respectively in such a way that ϕ extends to a covering morphism $\Phi : \mathbb{A} \longrightarrow \mathbb{A}'$.*

A positive answer to Question 1 would then give rise to a pair $\Gamma \leq \Gamma'$ of discrete subgroups of $Aut(X)$, where $X = \widetilde{(A, i)} = \widetilde{(A', i')}$ and $\Gamma = \pi_1(\mathbb{A}, a)$, $\Gamma' = \pi_1(\mathbb{A}', a')$, $a \in VA$, and $a' = \phi(a)$.

To answer this question let us first recall what is meant by a covering morphism of graphs of groups originally defined by Bass in [B]. In [B] Bass noted that if there is a general covering morphism between two graphs of groups there exists a special one (which he referred to as $\delta\Phi$) which requires less information. Since we only care about the existence of a covering morphism we will make use of this fact and say that a covering morphism $\Phi = (\phi, (\delta)) : \mathbb{A} \to \mathbb{A}'$ consists of:

(1) a graph morphism $\phi : A \to A'$;

(2) monomorphisms

$$\phi_a : \mathcal{A}_a \to \mathcal{A}'_{\phi(a)} \quad (a \in A), \qquad \phi_e = \phi_{\bar{e}} : \mathcal{A}_e \to \mathcal{A}'_{\phi(e)} \quad (e \in EA);$$

(3) For each $e \in EA$ with $a = \partial_0 e$ an element $\delta_e \in \mathcal{A}'_{\phi(a)}$ such that the following two conditions hold:

(a) the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_e & \xrightarrow{\alpha_e} & \mathcal{A}_a \\ {\scriptstyle \phi_e}\downarrow & & \downarrow{\scriptstyle \phi_a} \\ \mathcal{A}'_{\phi(e)} & \xrightarrow{ad(\delta_e)\cdot\alpha'_{\phi(e)}} & \mathcal{A}'_{\phi(a)} \end{array}$$

where $ad(x)(s) = xsx^{-1}$.

(b) For $f \in EA'$, $a' = \partial_0 f$ and $a \in \phi^{-1}(a')$, the map

$$\Phi_{a/f} : \left(\coprod_{e \in \phi_{(a)}^{-1}(f)} \mathcal{A}_a / \alpha_e \mathcal{A}_e \right) \longrightarrow \mathcal{A}'_{\phi(a)} / \alpha'_f \mathcal{A}'_f$$

defined by

$$\Phi_{a/f}([s]_e) = [\phi_a(s)\delta_e]_f$$

is bijective (where $s \in \mathcal{A}_a$ and $[s]_e$ is the class of s in $\mathcal{A}_a/\alpha_e\mathcal{A}_e$).

To simplify matters greatly let us consider only the special case where the action $ad(\delta_e)$ is trivial. This must be the case in particular when $\mathcal{A}'_{\phi(a)'}$ is abelian. Since the maps ϕ_a and ϕ_e are monomorphisms we may identify the groups $\mathcal{A}_a$ and $\mathcal{A}_e$ with their images in $\mathcal{A}_{\phi(a)}$ and $\mathcal{A}_{\phi(e)}$ respectively. Condition $(3)(a)$ then becomes

$$(3)(a') \qquad \alpha_e = \alpha'_{\phi(e)}|_{\mathcal{A}_e}.$$

We have the following,

Theorem 5.2. *Let $a \in VA$ and let $f \in EA'$ be such that $\partial_0(f) = \phi(a)$. If $\mathcal{A}'_{\phi(a)}$ is abelian, then the subgroup $\alpha_e \mathcal{A}_e$ is independent of the choice of edge $e \in \phi_{(a)}^{-1}(f)$.*

Proof Let $s \in \alpha_e \mathcal{A}_e$ for some $e \in \phi_{(a)}^{-1}(f)$. Let e' be another edge in $\phi_{(a)}^{-1}(f)$. Since $\alpha_e = \alpha_f$ restricted to $\mathcal{A}_e$ we have that $s \in \alpha_f \mathcal{A}_f$. As $\mathcal{A}'_{\phi(a)}$ is abelian it follows that for $x \in \mathcal{A}'_{\phi(a)}$ we have $[sx]_f = [x]_f$, thus

$$\Phi_{a/f}([s]_{e'}) = [s\delta_{e'}]_f = [\delta_{e'}]_f = \Phi_{a/f}([1]_{e'}).$$

Since $\Phi_{a/f}$ is injective we must then have that $[s]_{e'} = [1]_{e'}$ and thus $s \in \alpha_{e'} \mathcal{A}_{e'}$. So we see that for any two edges e and e' in $\phi_{(a)}^{-1}(f)$ we have

$$\alpha_e \mathcal{A}_e \subset \alpha_{e'} \mathcal{A}_{e'} \subset \alpha_e \mathcal{A}_e. \ \square$$

In terms of conditions on the edge-indexed graphs we get the following corollary.

Corollary 5.3. *If $\phi : (A, i) \to (A', i')$ is a covering of edge-indexed graphs and we wish to find finite abelian groupings of (A, i) and (A', i') in such a way as to extend ϕ to a covering morphism of graphs of groups then we must have the following necessary condition on ϕ which we will call locally constant fibers:*

(LCF) If for $e_1, e_2 \in EA$, $\phi(e_1) = \phi(e_2)$ and $\partial_0 e_1 = \partial_0 e_2$, then $i(e_1) = i(e_2)$.

The condition (LCF) is necessary in order to extend ϕ to a covering morphism of finite abelian graphs of groups. We also have the following sufficient condition on the finite abelian graphs of groups themselves.

Theorem 5.4. *Let $\phi : (A, i) \to (A', i')$ be a covering of edge-indexed graphs. Let $\mathbb{A}$ and $\mathbb{A}'$ be finite abelian groupings of (A, i) and (A', i') respectively. Suppose further that*

(i) $\mathcal{A}_a \leq \mathcal{A}'_{\phi(a)}$, $a \in VA$ and $\mathcal{A}_e \leq \mathcal{A}'_{\phi(e)}$, $e \in EA$ and $\mathcal{A}_a$ and $\mathcal{A}_e$ are identified with their images in $\mathcal{A}'_{\phi(a)}$ and $\mathcal{A}'_{\phi(e)}$ respectively.

(ii) $\alpha_e = \alpha'_{\phi(e)}|_{\mathcal{A}_e}$, $e \in EA$.

(iii) $\mathcal{A}_a \cap \alpha'_f \mathcal{A}'_f = \alpha_e \mathcal{A}_e$, where $a \in VA$ and $f \in EA'$ is such that $\partial_0(f) = \phi(a)$,

Then ϕ extends to a covering morphism $\Phi = (\phi, (\delta)) : \mathbb{A} \longrightarrow \mathbb{A}'$.

Proof The monomorphisms ϕ_a and ϕ_e are naturally just the inclusion maps. We have seen that since we are using abelian groups condition (3)(a) of a covering morphism reduces to condition (3)(a') which is what we require by (ii). It remains only to define the elements δ_e for each $e \in EA$ and show that condition (3)(b) of a covering morphism is satisfied. So let $a \in VA$ and let $f \in EA'$ be such that $\partial_0(f) = \phi(a)$. We take $\{\delta_e\}_{e \in \phi_{(a)}^{-1}(f)}$ to be the distinct coset representatives of

$$\left(\mathcal{A}'_{\phi(a)}/\alpha'_f \mathcal{A}'_f\right) / \left(\mathcal{A}_a/\alpha_e \mathcal{A}_e\right) = \left(\mathcal{A}'_{\phi(a)}/\alpha'_f \mathcal{A}'_f\right) / \left(\mathcal{A}_a/\mathcal{A}_a \cap \alpha'_f \mathcal{A}'_f\right).$$

We first note that this defintion makes sense. $\mathcal{A}_a/\alpha_e \mathcal{A}_e$ is naturally a subgroup of $\mathcal{A}'_{\phi(a)}/\alpha'_f \mathcal{A}'_f$ by the map which takes $[s]_e$ to $[s]_f$. That map is well-defined since $\alpha_e \mathcal{A}_e$ is a subgroup of $\alpha'_f \mathcal{A}'_f$. It is clearly a homomorphism and it is in fact a monomorphism by (iii).

We also must show that the number of cosets above is precisely the number of edges $e \in p_{(a)}^{-1}(f)$. Recall that if K and L are subgroups of H then the index $[H/K : L/K \cap L] = [H : K]/[L : K \cap L]$. Hence the number of cosets we have is $i'(f)/i(e)$. However $i(e)$ is constant on

$\phi^{-1}_{(a)}(f)$ and ϕ is a covering, so we have that the number of edges in $\phi^{-1}_{(a)}(f)$ must be exactly $i'(f)/i(e)$ as well.

The map $\Phi_{a/f}$ is then just the map that for each $e \in \phi^{-1}_{(a)}(f)$ sends the subgroup $\mathcal{A}_a/\alpha_e\mathcal{A}_e$ to the coset $(\mathcal{A}_a/\alpha_e\mathcal{A}_e)\delta_e$ which is clearly bijective as we have exactly one representative from each coset. □

We have noted that a necessary condition for a covering ϕ to extend to a covering morphism of graphs of finite abelian groups is that ϕ satisfy (LCF). One special case of coverings which satisfy this condition are the 'denominator clearing' coverings constructed in [BCR].

6. Lattice subgroups with abelian stabilizers

We now obtain sufficient conditions on a covering $\phi : (B, j) \longrightarrow (A, i)$ of unimodular bounded denominator edge-indexed graphs so that it will extend to a covering morphism of finite faithful abelian graphs of groups. We have already seen one necessary condition in Section 6 which we called locally constant fibers (LCF). We now observe a further necessary condition.

Assume that $\phi : (B, j) \longrightarrow (A, i)$ is a covering of unimodular bounded denominator edge-indexed graphs satisfying (LCF). It follows that $j(e)$ divide $i(\phi(e))$ and we may thus form a new edge-indexed graph (B, k) with underlying graph B and indexing $k(e) = \frac{i(\phi(e))}{j(e)}$.

Definition 2. *We say that ϕ satisfies the relatively bounded denominator condition (RBD) if the edge-indexed graph (B, k) satisfies the bounded denominator condition.*

See example 2 of section 9 for an example of the graph (B, k) and a covering which does not sayify (RBD).

Theorem 6.1. *Let $\phi : (B, j) \longrightarrow (A, i)$ be a covering of edge-indexed graphs satisfying (LCF). Then a necessary condition for ϕ to extend to a covering morphism of finite graphs of groups is that ϕ satisfy (RBD).*

Proof Suppose there exist finite groupings $\mathbb{A} = (A, \mathcal{A})$ of (A, i) and $\mathbb{B} = (B, \mathcal{B})$ of (B, j) and a covering morphism $\Phi = (\phi, (\delta)) : \mathbb{B} \longrightarrow \mathbb{A}$. Then it follows that for $b \in VB$ we have $\phi_b(\mathcal{B}_b) \leq \mathcal{A}_{\phi(b)}$ and so $|\mathcal{B}_b|$ divides $|\mathcal{A}_{\phi(b)}|$. We then have that

$$N(b) = \frac{|\mathcal{A}_{\phi(b)}|}{|\mathcal{B}_b|}$$

is an integral vertex ordering of (B, k) and thus necessarily (B, k) satisfies the bounded denominator condition and by definition ϕ satisfies (RBD). □

We have seen that a necessary condition to extend ϕ to a covering morphism of finite faithful abelian graphs of groups is that ϕ satisfy (LCF) and (RBD). We would like to say that the conditions are also sufficient but we are unable to do so. Even if we drop the requirement that the groupings be abelian, (RBD) alone is not sufficient, as can be seen in example 3. If we strengthen the (LCF) condition, though, we are able to obtain sufficient conditions.

We say that a covering $\phi : (B, j) \longrightarrow (A, i)$ satisfies the globally constant fibers condition if

(GCF) For $e_1, e_2 \in EB$ with $\phi(e_1) = \phi(e_2)$ we have $j(e_1) = j(e_2)$.

Lemma 6.2. *Let $\phi : B \longrightarrow A$ be a graph morphism and let (B, j) be a unimodular edge-indexed graph with bounded denominators. Suppose further the indexing j is constant on the fibers of ϕ. That is $j(e_1) = j(e_2)$ if $\phi(e_1) = \phi(e_2)$. Then any vertex ordering $N : (B, j) \longrightarrow \mathbb{Q}^x_{>0}$ is constant on the fibers of ϕ. That is if $\phi(b_1) = \phi(b_2)$ then $N(b_1) = N(b_2)$.*

Proof Let $\gamma = (e_1, e_2, \ldots, e_n)$ be a path in (B, j) from b_1 to b_2. Then by assumption we have that $j(e_i)$ and hence $\Delta(e_i)$ and hence $\Delta(\gamma)$ depends only on its image by ϕ in A. If $\Delta(\gamma) = 1$ then it follows that $N(b_1) = N(b_2)$. So assume instead that $\Delta(\gamma) \neq 1$. If necessary switch b_1 and b_2 and replace γ by γ^{-1} to insure that $\Delta(\gamma) = q$ for some rational q which is not an integer.

Notice that $\phi(\gamma)$ is a path in A from $a = \phi(b_1) = \phi(b_2)$ to itself. So let γ_2 be a lifting of $\phi(\gamma)$ starting at b_2 and ending at some vertex b_3 with $\phi(b_3) = a$. Inductively we define γ_n to be a lifting of $\phi(\gamma)$ starting at b_n and ending at some vertex b_{n+1}. Then $\Delta(\gamma_n) = q$ for all n and thus $\Delta(\gamma \circ \gamma_2 \circ \cdots \circ \gamma_n) = q^n$ thus contradicting our assumption that (B, j) has bounded denominators. $\square$

Theorem 6.3. *Let $\phi : (B, j) \longrightarrow (A, i)$ be a covering of unimodular bounded denominator edge-indexed graphs satisfying (GCF) and (RBD). Then there exist finite faithful groupings $\mathbb{A} = (A, \mathcal{A})$ of (A, i) and $\mathbb{B} = (B, \mathcal{B})$ of (B, j) and a covering morphism $\Phi = (\phi, (\delta)) : \mathbb{B} \longrightarrow \mathbb{A}$.*

Proof Let (B, k) be the edge-indexed graph constructed as in the definition of (RBD). Let $\mathbb{B} = (B, \mathcal{B})$ be a faithful finite abelian grouping of (B, j) such that the groups are all isomorphic on each (global) fiber of ϕ. That is $\mathcal{B}_{b_1} = \mathcal{B}_{b_2}$ if $\phi(b_1) = \phi(b_2)$. We can find such a grouping since (B, j) is unimodular with bounded denominators and by the previous lemma any vertex ordering of (B, j) is constant on the fibers of ϕ. Likewise let $\mathbb{B}' = (B, \mathcal{B}')$ be a faithful finite abelian grouping of (B, k) with isomorphic groups along each fiber. Again the previous lemma and the (RBD) condition guarantee the existence of such a grouping. We will define a grouping $\mathbb{A} = (A, \mathcal{A})$ on (A, i) by taking the group at a vertex or edge $a \in VA \cup EA$ to be the direct product of the groups $\mathcal{B}_b$ and $\mathcal{B}'_b$ for some $b \in \phi^{-1}(a)$. Recall the choice of b in the fiber does not mattter as all groups in the fiber are isomorphic in both $\mathbb{B}$ and $\mathbb{B}'$.

The edge monomorphisms in $\mathbb{A}$ are those obtained naturally from the edge monomorphisms in $\mathbb{B}$ and $\mathbb{B}'$. First let us verify that $\mathbb{A}$ is consistant with the indexing on (A, i). If e is an edge in EA then

$$[\mathcal{A}_{\partial_0 e} : \alpha_e \mathcal{A}_e] = j(e) \cdot k(e) = i(e).$$

Since each grouping $\mathbb{B}$ and $\mathbb{B}'$ is faithful it follows that $\mathbb{A}$ is faithful (see for example section 5.5 of [R]). The groups of $\mathbb{B}$ are naturally subgroups of the groups of $\mathbb{A}$ and the edge monomorphisms of $\mathbb{A}$ were defined to be extensions of the edge monomorphisms of $\mathbb{B}$. All that remains is to check condition *(iii)* of Theorem 4 to establish the existence of our desired covering morphism. We see that if $\phi(e) = f$ with $\partial_0 e = b$ and $\partial_0 f = a$ then

$$\mathcal{B}_b \cap \alpha_f \mathcal{A}_f = \mathcal{B}_e$$

as required. $\square$

7. Examples

In this section we provide a number of examples of the construction of lattice subgroup pairs using the methods introduced in the previous sections.

Example 1. Consider the covering $p : (B, j) \longrightarrow (A, i)$ shown in Figure 1. Then $p : (B, j) \longrightarrow (A, i)$ is a denominator clearing covering (as in [BCR]). Hence there are abelian groupings $\mathbb{B}$ of (B, j) and $\mathbb{A}$ of (A, i), and a covering morphism $\Phi : \mathbb{B} \longrightarrow \mathbb{A}$ as shown in Figure 2. Note also that $p : (B, j) \longrightarrow (A, i)$ satisfies (GCF) and clearly also (RBD). Let

$$\Lambda = C_{q+1} * \cdots * C_{q+1}$$

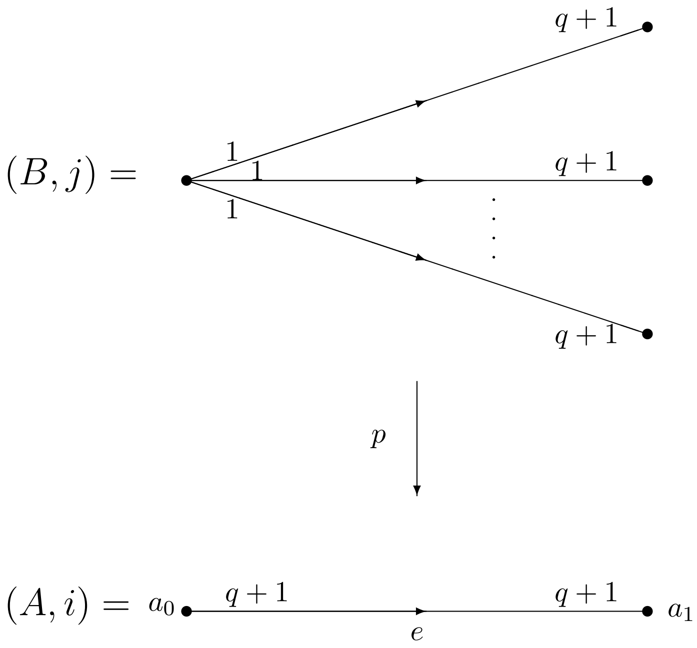


FIGURE 1. Covering $p : (B, j) \longrightarrow (A, i)$ of Example 1

($(q+1)$-factors), and let $\Gamma = C_{q+1} * C_{q+1}$, where C_{q+1} denotes the cyclic group of order $q+1$. We obtain the inclusion $\Lambda \leq \Gamma$ of discrete subgroups of $G = Aut(X_{q+1})$. In Section 8 we show how to embed this lattice pair into a locally compact subgroup H of G, such as a K-rank 1 simple algebraic groups over nonarchimedean local fields K with residue class field $\mathbb{F}_q$ with $q = 2^s$, or when $q = 2^s$, a rank 2 locally compact Kac-Moody groups over $\mathbb{F}_q$.

Example 2. Consider the covering $p : (B, j) \longrightarrow (A, i)$ shown in Figure 3. This is an example of a covering of edge-indexed graphs where both (A, i) and (B, j) are unimodular and have bounded denominator. However no covering morphism of faithful graphs of finite groups exists. To see this we can look at the orders of the groups at each vertex relative to the orders of the groups at the base vertices a_0 and b_0. In the graph (A, i) a group at a vertex a_n must be of order $2^{\frac{n+1}{2}}$ times the order of the group at a_0. The order of a group at a vertex b_n lying above a_n must be of the order 2^n times the order of the group at b_0. Since the order of the group a_i must be at least as large as the order of the group at b_i we must have the order of the group at a_0 is infinite. Note that the covering here satisfies (GCF) but does not satisfy (RBD).

Example 3. For each $n \geq 1$ consider the covering $p_n : (B_n, j_n) \longrightarrow (A, i)$ shown in Figure 4. For each $n \geq 1$ we have a covering of edge-indexed graphs where both (A, i) and (B, j) are unimodular and have bounded denominator. Yet when $n \geq 4$ no covering of faithful graphs of finite groups exist. Unlike the last example, both graphs here are finite. The problem, though, can still be seen by examing forced requirements for the orders of groups. Since the order of the group at b_0 is at least 2, the order of the group at b_k is at least 2^{2k+1} for $k < n$ and the order of the group at b_n is at least $3 \cdot 2^n$. Thus the order of the group at a_0 is also at least $3 \cdot 2^n$ which is not possible for a faithful grouping of (A, i) when $n \geq 4$ by a result of Goldschmidt [G].

We note that although (RBD) and (GCF) together are sufficient for the existence of our desired graph of group coverings,(GCF) is not a necessary condition. This leads to an interesting

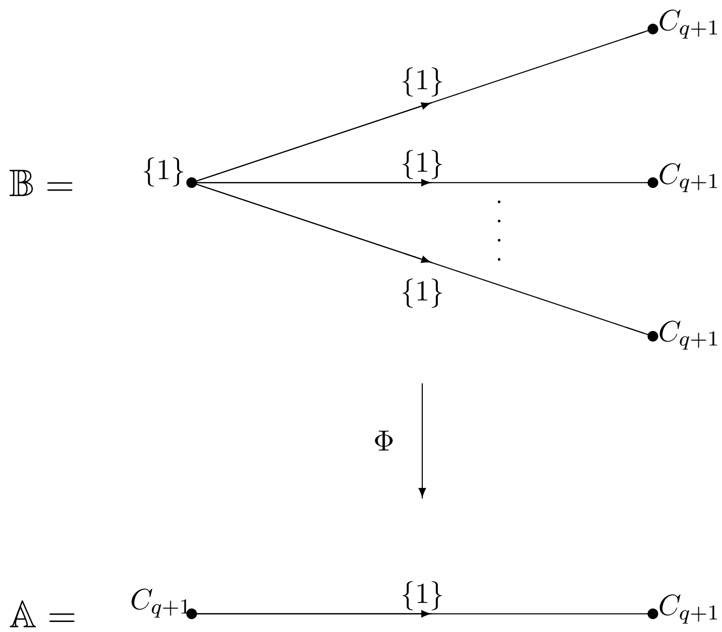


FIGURE 2. Covering morphism $\Phi : \mathbb{B} \longrightarrow \mathbb{A}$ of Example 1

question. While our proof of theorem 6 certainly made use of (GCF) (via lemma 5), perhaps it could be proven without this condition.

Question 7.1. *If we are given a covering of unimodular edge-indexed graphs having bounded denominator, is (LCF) + (RBD) sufficient for the existence of an extension to a covering morphsim of faithful graphs of finite groups?*

8. LATTICES IN $H \leq G = Aut(X)$

We now turn our attention to the problem of constructing lattices in a closed, hence locally compact subgroup H of $G = Aut(X)$ for X a locally finite tree. Our methods will apply to the case where X is homogeneous. It is well known that if $H = SL_2(\mathbb{F}_q((t^{-1})))$, then H has Bruhat-Tits tree $X = X_{q+1}$. If instead H is a rank 2 locally compact complete affine or hyperbolic Kac-Moody group over a finite field $\mathbb{F}_q$ corresponding to any 2×2 generalized Cartan matrix, then H has Tits building the homogeneous tree $X = X_{q+1}$ ([CG]). The group

$$H/\left(\cap_{g\in H} gBg^{-1}\right)$$

acts faithfully on X, the defining homomorphism

$$\rho : H/\left(\cap_{g\in H} gBg^{-1}\right) \hookrightarrow Aut(X)$$

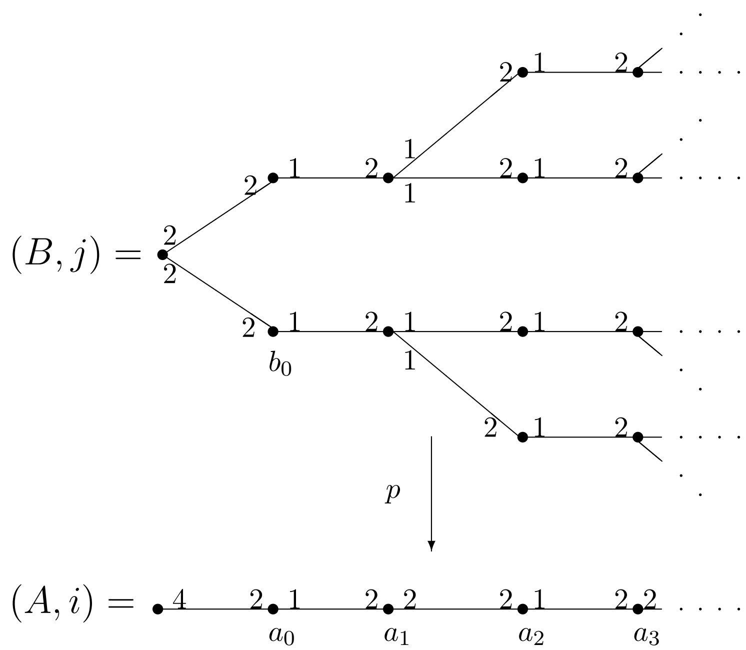


FIGURE 3. Covering $p : (B, j) \longrightarrow (A, i)$ of Example 2

is continuous and the image is closed. The quotient topology on $H/\left(\cap_{g\in H}\, gBg^{-1}\right)$ coincides with that induced by the natural topology on $Aut(X)$, where the stabilizers of finite sets of vertices form a fundamental system of compact open neighborhoods of the identity ([CG]).

We now make some remarks about the existence of cocompact lattices in the rank 2 locally compact complete affine or hyperbolic Kac-Moody group H over a finite field $\mathbb{F}_q$ corresponding to any 2×2 generalized Cartan matrix. In [CC], the authors realized $\Gamma = C_{q+1} * C_{q+1}$ as a cocompact lattice in H when $q = 2^s$. In [CT], the authors classified the finite subgroups of H. Capdeboscq (Korchagina) and Thomas also gave a classification up to isomorphism of all cocompact lattices with torsion and with quotient a simplex in rank 2 locally compact Kac-Moody groups H over finite fields when the generalized Cartan matrix is symmetric. They showed that if $q = 2^s$, then H contains a lattice $\Gamma = \Gamma_1 *_{\Gamma_0} \Gamma_2$ where $\Gamma_i = \Gamma_0 \times H_i$ with $H_i \cong C_{q+1}$ and Γ_0 a cyclic subgroup of a maximal split torus T in H whose order divides $(q-1)$. Thus when $q = 2$, Γ is the free product $C_3 * C_3$.

So far we have only discussed the construction of lattices with quotient a simplex. In this setting, a detailed analysis of the possible lattice subgroups can be obtained. In order to construct cocompact lattices with other quotient graphs, there are many issues to consider. We refer the reader to [BCR] and [C2] for a discussion of the existence of tree lattices.

We now give example of lattice subgroups pairs $\Gamma' \leq \Gamma \leq H$ of a given cocompact lattice Γ. We first build on Example 1 of Section 7. In Section 9 we describe how to extend this to give infinite descending chains of cocompact lattices over Γ.

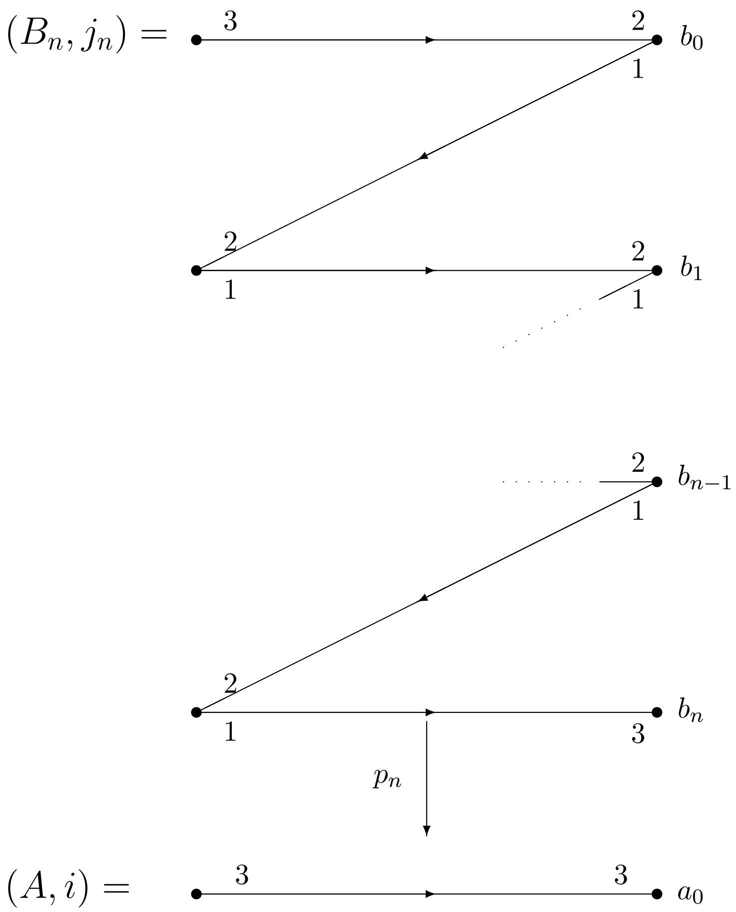
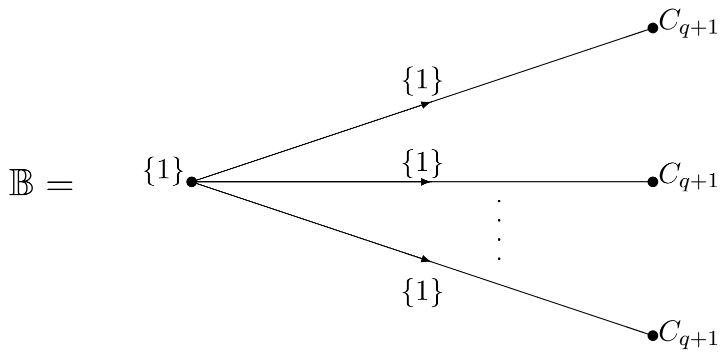


FIGURE 4. Covering $p_n : (B_n, j_n) \longrightarrow (A, i)$ of Example 3

Proposition 8.1. *Let H denote the group $SL_2(\mathbb{F}_q((t^{-1})))$. Let $q = 2^s$. Let Γ denote the lattice subgroup $C_{q+1} * C_{q+1}$ of H. Then there is a lattice $\Gamma' \leq \Gamma \leq H$ with graph of groups $\mathbb{B}$:*



Thus $\Gamma' \cong \pi_1(\mathbb{B}) \cong C_{q+1} * \cdots * C_{q+1}$, *((q+1)-factors), is a cocompact lattice in* H.

Proof In Example 1 of Section 7 we proved that there is a covering morphism of graphs of groups $\Phi : \mathbb{B} \longrightarrow \mathbb{A}$ where $\mathbb{A}$ is the graph of groups for $\Gamma \cong C_{q+1} * C_{q+1} \leq H$. Thus

$$\Gamma' \cong \pi_1(\mathbb{B}) \cong C_{q+1} * \cdots * C_{q+1} \ \leq \ \Gamma \cong C_{q+1} * C_{q+1} \ \leq \ H. \ \square$$

When $q = 2$ and A is any 2×2 generalized Cartan matrix, it was shown in [CC] that a rank 2 locally compact affine or hyperbolic Kac-Moody group $H = H_A(\mathbb{F}_2)$ over the field $\mathbb{F}_2$ contains the lattice pair $\Gamma \leq \Gamma'$, where Γ' is the free product $C_3 * C_3$ and $\Gamma \cong C_3 * C_3 * C_3$.

Proposition 8.2. *Let* H *denote the group* $SL_2(\mathbb{F}_2((t^{-1})))$ *or a rank 2 locally compact affine or hyperbolic Kac-Moody group over the finite field* $\mathbb{F}_2$ *corresponding to any* 2×2 *generalized Cartan matrix. Let* Γ *denote the lattice subgroup* $C_3 * C_3$ *of* H. *Then there is a lattice* $\Gamma' \leq \Gamma \leq H$ *with graph of groups* $\mathbb{B}$ *as in Figure 5. Thus* $\Gamma' \cong \pi_1(\mathbb{B}) \cong C_3 * C_3 * \mathbb{Z}$ *is a cocompact lattice in* H.

Proof Let (A, i) be the edge indexed graph corresponding to the lattice $C_3 * C_3$ of H:

Let (B, j) be the edge indexed graph corresponding to $\mathbb{B}$. The covering $p : (B, j) \longrightarrow (A, i)$ of edge indexed graphs is evident. We have the abelian grouping $C_3 * C_3$ of (A, i). This may be lifted to an abelian grouping of (B, j). Thus there is a covering morphism of graphs of groups $\Phi : \mathbb{B} \longrightarrow \mathbb{A}$, and

$$\Gamma' \cong \pi_1(\mathbb{B}) \cong C_3 * C_3 * \mathbb{Z} \ \leq \ \Gamma \cong C_3 * C_3 \ \leq \ H. \ \square$$

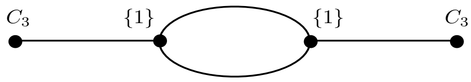
$\mathbb{B}=$ 

FIGURE 5. The graph of groups $\mathbb{B}$ as in Proposition 8.2

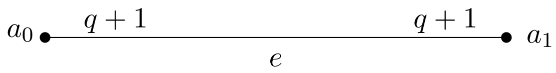
9. A NEW EXAMPLE OF INFINITE DESCENDING CHAINS OF LATTICES IN $H \leq G = Aut(X)$

We take H to be a rank 2 locally compact complete affine or hyperbolic Kac-Moody group H over a finite field $\mathbb{F}_q$ corresponding to any 2×2 generalized Cartan matrix so that the Tits building of H is the homogeneous tree $X = X_{q+1}$. We now describe how to construct infinite descending chains of cocompact lattices over a cocompact lattice $\Lambda \leq H$. The method we give here was first described in [CC] and [Co]. We refer the reader to these references for detailed examples.

In general, by a result of Bass and Kulkarni ([BK]), all cocompact tree lattices are finitely generated and virtually free. Thus any cocompact lattice in G has a residually finite subgroup of finite index. It is known that residually finite groups can admit descending chains of subgroups. However, we do not use residual finiteness to construct infinite descending chains over the lattices in this section.

Our strategy here is to extend coverings of edge-indexed graphs to covering morphisms of graphs of groups with abelian groupings. This provides a new tool for constructing descending chains of subgroups in locally compact groups that act on trees. This method is constructive and allows us to determine the quotient graphs of groups, the group presentations and the covolumes of all sublattices in the descending chain. This method is not specific to cocompact lattices in Kac-Moody groups and may be used to produce finite or infinite descending chains of subgroups acting on trees with abelian stabilizers in a general setting. For example, our method also applies to lattices that are not residually finite, such as nonuniform lattices, though we do not consider this case here.

The main observation is the following. If H contains a cocompact lattice Γ with quotient $\Gamma\backslash X$ a simplex, with abelian vertex groups and edge-indexed graph $(A, i) = I(\Gamma\backslash\backslash X)$

$(A, i) =$ 

with $q \geq 3$, then there is an infinite descending chain of cocompact lattices

$$\ldots \Gamma_3 \leq \Gamma_2 \leq \Gamma_1 \leq \Gamma \leq H \leq G,$$

where the Γ_i, $i \geq 1$, have abelian vertex groups. The infinite descending chain is constructed by iterating a method known as 'open fanning' which provides an infinite sequence of edge-indexed coverings and then extending these coverings of edge-indexed graphs to covering morphisms with abelian groupings. This method was used extensively in [CC] and [Co]. We give a specific example here, inside a locally compact complete rank 2 Kac-Moody group H over the field of two elements.

Theorem 9.1. *Let H be a locally compact complete rank 2 Kac-Moody group over the field $\mathbb{F}_2$. Then H contains an infinite descending chain $\ldots \Gamma_3 \leq \Gamma_2 \leq \Gamma_1$ of cocompact lattices with distinct fundamental domains, with*

$$\Gamma_k \cong *_3\mathbb{Z}/3\mathbb{Z} *_{m_k} \mathbb{Z} \ \text{ where}$$

$$m_k = \begin{cases} 0 & \text{if } k = 1 \\ 2 & \text{if } k = 2 \\ 8\sum_{j=0}^{\frac{k-3}{2}} 3^{2j} & \text{if } k \text{ odd}, \ k \geq 3 \\ 24\sum_{j=0}^{\frac{k-4}{2}} 3^{2j} + 2 & \text{if } k \text{ even}, \ k \geq 4, \end{cases}$$

and $Vol(\Gamma_k) = 2 \cdot 3^{k-1}$.

Proof. We recursively construct a sequence of edge-indexed coverings as follows:

(1) Let (B_1, i_1) be the edge-indexed tripod (3-star) as in Figure 6 which is an edge indexed covering of the simplex (A, i). Choose two edges e_1, e_2 of B_1 with index $i_1(e_1) = i_1(e_2) = 3$.
(2) Let (B_2, i_2) be an open fanning of (B_1, i_1) on the bridge $\{e_1, e_2\}$.
(3) For each $k \geq 2$, choose two edges e_1, e_2 of B_k with index 3 and let (B_{k+1}, i_{k+1}) be an open fanning of B_k on $\{e_1, e_2\}$.

The general scheme for the open fanning on two edges is given in Figure 7. Note that no graph in the sequence (after the initial tripod) is a tree. In particular these cycles introduce copies of $\mathbb{Z}$ in the free product decomposition of the fundamental groups. It is easy to see from the schematic that for $k \geq 1$, if B_k has m_k edges outside its spanning tree then B_{k+1} has $3m_k + 2$ edges outside its spanning tree. An easy induction combined with the recursive relationship gives the general formula

$$m_k = \begin{cases} 0 & \text{if } k = 1 \\ 2 & \text{if } k = 2 \\ 8\sum_{j=0}^{\frac{k-3}{2}} 3^{2j} & \text{if } k \text{ odd}, \ k \geq 3 \\ 24\sum_{j=0}^{\frac{k-4}{2}} 3^{2j} + 2 & \text{if } k \text{ even}, \ k \geq 4, \end{cases}$$

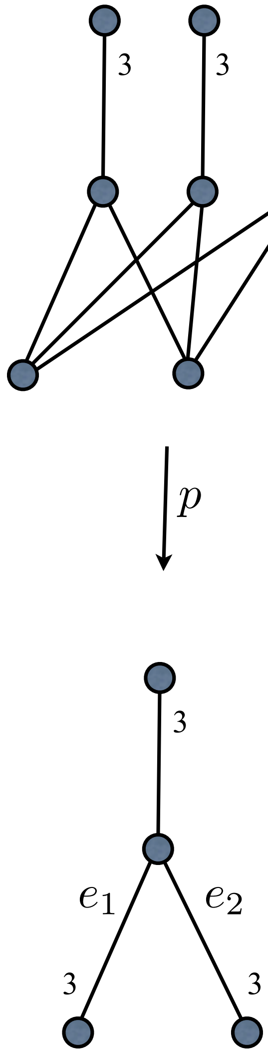


FIGURE 6. covering of edge-indexed graphs $p : (B_2, i_2) \longrightarrow (B_1, i_1)$

Note also from the scheme above that if B_k has three edges of index 3 (one in the center subgraph plus the two edges of the bridge), then B_{k+1} also has three edges of index 3 (one in each of copies of the central subgraph of B_k). Since the tripod has three edges of index 3, it follows that every graph in the sequence has exactly three edges of index 3. Then the corresponding groupings $\mathbb{B}_k$ consist of three copies of $\mathbb{Z}/3\mathbb{Z}$ at each initial vertex of these index 3 edges and trivial groups elsewhere. Each $\mathbb{B}_k$ is an abelian grouping with trivial edge groups, and thus for $k \geq 2$ the edge-indexed covering $p_k : (B_k, i_k) \rightarrow (B_{k-1}, i_{k-1})$ extends to a covering morphism $\phi_k : \mathbb{B}_k \rightarrow \mathbb{B}_{k-1}$. This gives the desired sequence of corresponding covering morphisms of graphs of groups and

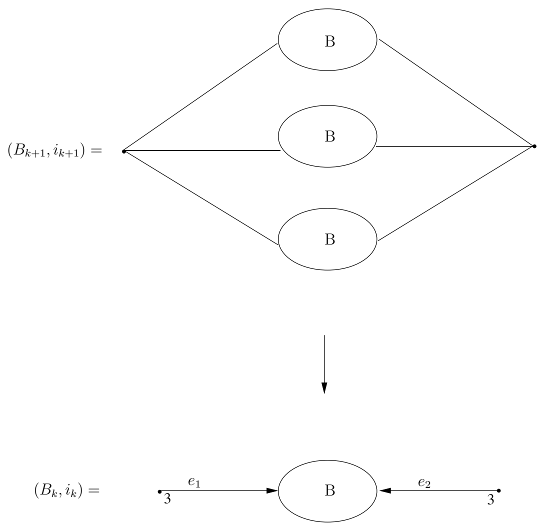


FIGURE 7. schematic of open fanning on two edges

the embeddings of their fundamental groups

$$\Gamma_k \;=\; \pi_1(\mathbb{B}_k) \;\cong\; *_3\mathbb{Z}/3\mathbb{Z} *_{m_k} \mathbb{Z}$$

These Γ_k form an infinite descending chain of cocompact lattices.

Finally, note that the open fanning of (B_k, j_k) produces three copies of the middle subgraph with the same grouping on these vertices. The two initial vertices of the bridge fan from degree 1 vertices to degree three vertices, changing the groups at these vertices from $\mathbb{Z}/3\mathbb{Z}$ to trivial groups. This observation yields a recursive definition of the covolume. Since $Vol(\Gamma_1) = 2$, another easy induction shows for $k \geq 2$,

$$Vol(\Gamma_k) \;=\; 3(Vol(\Gamma_{k-1}) - 2(\frac{1}{3})) \;+\; 2 \;=\; 3Vol(\Gamma_{k-1}) \;=\; 2 \cdot 3^{k-1}.$$

□

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