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EXISTENCE OF LATTICES IN KAC–MOODY GROUPS
OVER FINITE FIELDS

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Let $\mathfrak{g}$ be a Kac–Moody Lie algebra. We give an interpretation of Tits' associated group functor using representation theory of $\mathfrak{g}$ and we construct a locally compact "Kac–Moody group" G over a finite field k. Using (twin) BN-pairs (G, B, N) and (G, B^-, N) for G we show that if k is "sufficiently large", then the subgroup B^- is a non-uniform lattice in G. We have also constructed an uncountably infinite family of both uniform and non-uniform lattices in rank 2. We conjecture that these form uncountably many distinct conjugacy classes in G. The basic tool for the construction of non-uniform lattices in rank 2 is a spherical Tits system for G which we also construct.

0. Introduction

Let $\mathfrak{g}$ be a Kac–Moody Lie algebra. In [29, 30], Tits associated to $\mathfrak{g}$ a group functor on the category of rings. In this work we give an interpretation of Tits' group functor using representation theory of $\mathfrak{g}$ (Secs. 5 and 6). We produce a "Kac–Moody group" G that is a completion of Tits' "minimal" group in [29, 30]. Let G denote such a group over a *finite field*. The group G is then locally compact and totally disconnected (Sec. 6).

Theorem 0.1. *Let G be a* (*complete*) *Kac–Moody group over a finite field k. The group G has* (*twin*) *BN-pairs (G, B, N) and (G, B^-, N) where B is* (*the completion of*) *the subgroup of G generated by a "diagonal subgroup" H and all positive real root groups, and B^- is the subgroup of G generated by H and all negative real root groups.*

The group G admits a cocompact action on its corresponding Bruhat–Tits building, which is an affine or hyperbolic building in rank greater than 2, and a homogeneous tree in rank 2. Our aim is to determine if in analogy with Lie groups, G contains lattices, that is discrete subgroups whose quotients carry a finite invariant measure. We describe a construction of non-uniform "arithmetic" lattices in Kac–Moody groups over sufficiently large finite fields using the Tits system for such groups (Theorem 8.1). B. Remy has obtained an equivalent result for the more general class of almost split Kac–Moody groups [25, 26].

Theorem 0.2. *Let G be a Kac–Moody group over a finite field k. If k is "sufficiently large", then the subgroup B^- is a non-uniform lattice in G.*

It follows that if $P^- \subseteq G$ is a proper standard parabolic subgroup for the BN-pair (G, B^-, N), $(B^- \subseteq P^- \subsetneq G)$, if the submatrix of A corresponding to P^- is positive-definite and if k is "sufficiently large", then P^- is a non-uniform lattice in G (Corollary 8.1).

We have also constructed an uncountably infinite family of both uniform and non-uniform lattices in the rank 2 Kac–Moody case (Theorems 11.1 and 15.1), by generalizing A. Lubotzky's construction of non-uniform lattices in SL_2 over a Laurent series field, using the Bruhat–Tits tree [18].

Theorem 0.3. *If G has rank 2, then G contains uncountably infinite families of both uniform and non-uniform lattices.*

The *rank* of G is defined to be $l = |I|$ as in Sec. 1. The basic tool for the construction of non-uniform lattices in rank 2 is a (new) spherical Tits system for the Kac–Moody group in rank 2 (Sec. 14).

Theorem 0.4. *Let G be a Kac–Moody group over a finite field. If G has rank 2, then G has a spherical Tits-system.*

The corresponding spherical building is 0-dimensional, and may be identified with ∂X, the boundary of the Bruhat–Tits tree X.

It remains to determine whether, as in the case of SL_2, we have constructed uncountably many distinct conjugacy classes of uniform (respectively non-uniform) lattices within the Kac–Moody group (Conjectures 12.1 and 16.1), while in contrast we show that these uniform (respectively non-uniform) lattices are all conjugate in the automorphism group of the Bruhat–Tits tree (see Proposition 12.1, respectively Proposition 16.1).

Using the action of G on its Bruhat–Tits tree in rank 2, we may deduce (Proposition 17.1):

Proposition 0.1. *If G has rank 2, G does not have property T.*

In contrast to this, in the higher rank case, results of Dymara and Januszkiewicz [10, 11] imply that certain Kac–Moody groups do have property T. Hence, the

lattices that we construct in these cases are finitely generated and have finite commutator quotients.

Regarding cohomology we have the following (Theorem 18.1 and Corollary 18.1):

Theorem 0.5. *Let l be the rank of G, $q = |k|$, let $\Gamma \leq G$ be a uniform lattice of G, and let $\rho : \Gamma \longrightarrow \mathrm{Aut}(V)$ be a unitary representation of Γ in a complex Hilbert space V. If $l \geq 3$, and if q is sufficiently large, then $H^i(\Gamma, \rho) = 0$, $0 < i < l - 1$.*

Corollary 0.1. *If G contains a uniform lattice, then G has property T.*

The paper is organized as follows. In Secs. 1–6, we describe the basic setting of Kac–Moody algebras and Kac–Moody groups. In Sec. 7, we consider some basic results about lattices in locally compact groups, and in Sec. 8 we describe our construction of non-uniform "arithmetic" lattices Kac–Moody groups over sufficiently large finite fields.

Sections 9–17 then focus on the case of rank 2. In Sec. 9 we outline the structure theory of the Bruhat–Tits tree in rank 2, and in Sec. 10 we describe A. Lubotzky's adaptation of the classical notion of a Schottky group of automorphisms of the upper half plane to the non-archimedean case, a notion central to our construction of lattices in rank 2. In Secs. 11 (uniform case) and 15 (non-uniform case) we generalize A. Lubotzky's construction of lattices in [18] to the Kac–Moody case, and in Secs. 12 (uniform case) and 16 (non-uniform case) we show how these constructions are flexible enough so as to admit uncountably infinite families of lattices in rank 2.

In Sec. 14 we introduce a spherical BN-pair for Kac–Moody groups of rank 2, and in Sec. 13, we outline some unexpected results regarding commensurability of uniform lattices in rank 2. In Secs. 17 and 18 we outline our results on the failure of property T in rank 2, and cohomology of lattices. In an appendix we give explicit computations for symmetric, 2×2, generalized Cartan matrices.

1. The Setting

We let $A = (A_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix; that is, $I = \{1, 2, \ldots, l\}$ is a finite set, and $A = (A_{ij})_{i,j \in I}$ satisfies:

(1) $A_{ij} \in \mathbb{Z}$, $i, j \in I$,
(2) $A_{ii} = 2$, $i \in I$,
(3) $A_{ij} \leq 0$ if $i \neq j$,
(4) There exist positive rational numbers $q_1, \ldots, q_l$, such that the matrix DA is symmetric, where $D = \mathrm{diag}(q_1, \ldots, q_l)$.

We remark that (4) implies:

$$A_{ij} = 0 \iff A_{ji} = 0.$$

By a proper submatrix of A, we mean a matrix of the form:

$$A_\theta = (A_{ij})_{i,j \in \theta},$$

where θ is a proper subset of $\{1, \ldots, l\}$.

We say that the matrix A is *indecomposable* if there is no partition of the set $\{1, \ldots, l\}$ into two non-empty subsets so that $A_{ij} = 0$ whenever i belongs to the first subset, while j belongs to the second.

An indecomposable symmetrizable generalized Cartan matrix A is of:

(1) *Classical type* if, for D as above, DA is positive definite. In this case, A is the Cartan matrix of a finite dimensional complex semi-simple Lie algebra.
(2) *Affine type* if for D as above, the matrix DA is positive semi-definite, but not positive definite.
(3) *Hyperbolic type* if A is neither of classical nor affine type, but every proper, indecomposable submatrix is either of classical or of affine type.

If A is of hyperbolic type, we say that A is of *compact hyperbolic type* if every proper, indecomposable submatrix is of classical type.

Our conventions are consistent with those of [2, Chapt. 4] which give a characterization hyperbolicity of A.

2. Kac–Moody Algebras over $\mathbb{Q}$

Following [16, Chapt. 1], we introduce the notion of a *realization*

$$(\mathfrak{h}, \Pi, \Pi^\nu) \tag{2.1}$$

of a symmetrizable generalized Cartan matrix A. Here $\mathfrak{h} = \mathfrak{h}_{\mathbb{Q}}$ is a vector space over $\mathbb{Q}$,

$$\Pi = \{\alpha_1, \ldots, \alpha_l\} \subseteq \mathfrak{h}_{\mathbb{Q}}^*, \quad \text{the dual space of } \mathfrak{h}_{\mathbb{Q}},$$

and the elements of Π are called *simple roots*,

$$\Pi^\nu = \{\alpha_1^\nu, \ldots, \alpha_l^\nu\} \subseteq \mathfrak{h}_{\mathbb{Q}},$$

and the elements of Π^ν are called *simple coroots*. The data $(\mathfrak{h}_{\mathbb{Q}}, \Pi, \Pi^\nu)$ is subject to the conditions

(1) The sets Π and Π^ν are each linearly independant,
(2) $\alpha_j(\alpha_i^\nu) = A_{ij}$, $i, j = 1, \ldots, l$,
(3) If $n = \operatorname{rank}(A)$, then

$$l - n = \dim(\mathfrak{h}_{\mathbb{Q}}) - l.$$

Following [16, Chapt. 1], two realizations $(\mathfrak{h}, \Pi, \Pi^\nu)$ and $(\mathfrak{h}_1, \Pi_1, \Pi_1^\nu)$ of A are said to be *isomorphic* if there is a vector space isomorphism $\phi : \mathfrak{h} \longrightarrow \mathfrak{h}_1$ such that $\phi(\Pi^\nu) = \Pi_1^\nu$, $\phi^*(\Pi_1) = \Pi$. In [16, Proposition 1.1] Kac proves that a symmetrizable generalized Cartan matrix, A, has a realization that is unique up to isomorphism.

Given a symmetrizable generalized Cartan matrix A, one can associate a Kac–Moody algebra $\mathfrak{g}_{\mathbb{Q}} = \mathfrak{g}_{\mathbb{Q}}(A)$, a Lie algebra over $\mathbb{Q}$, generated by $\mathfrak{h} = \mathfrak{h}_{\mathbb{Q}}$ and elements $(e_i)_{i \in I}$, $(f_i)_{i \in I}$ subject to relations:

(1) $[\mathfrak{h}, \mathfrak{h}] = 0$,

(2) $[h, e_i] = \alpha_i(h)e_i$, $h \in \mathfrak{h}$,

(3) $[h, f_i] = -\alpha_i(h)f_i$, $h \in \mathfrak{h}$,

(4) $[e_i, f_i] = \alpha_i^{\nu}$,

(5) $[e_i, f_j] = 0$, $i \neq j$,

(6) $(ad\ e_i)^{-A_{ij}+1}(e_j) = 0$, $i \neq j$,

(7) $(ad\ f_i)^{-A_{ij}+1}(f_j) = 0$, $i \neq j$.

If A is of classical type, the algebra $\mathfrak{g}_{\mathbb{Q}}$ is finite dimensional. If A is of affine type, $\mathfrak{g}_{\mathbb{Q}}$ is infinite dimensional but of polynomial growth. If A is of hyperbolic type, $\mathfrak{g}_{\mathbb{Q}}$ is infinite dimensional and of exponential growth.

Let $\mathfrak{g}_{\mathbb{Q},+}$ be the Lie subalgebra of $\mathfrak{g}_{\mathbb{Q}}$ generated by $(e_i)_{i \in I}$ with defining relation (6), and let $\mathfrak{g}_{\mathbb{Q},-}$ be the Lie subalgebra of $\mathfrak{g}_{\mathbb{Q}}$ generated by $(f_i)_{i \in I}$ with defining relation (7).

Theorem 2.1 ([29, p. 77]). *The canonical map*

$$\mathfrak{g}_{\mathbb{Q},-} \oplus \mathfrak{h}_{\mathbb{Q}} \oplus \mathfrak{g}_{\mathbb{Q},+} \longrightarrow \mathfrak{g}_{\mathbb{Q}}$$

is a bijection.

3. Roots and Weyl Group

The algebra $\mathfrak{g}_{\mathbb{Q}}$ decomposes into eigenspaces, called root spaces, under the simultaneous adjoint action of $\mathfrak{h}_{\mathbb{Q}}$. Here, a non-zero element $\alpha \in \mathfrak{h}_{\mathbb{Q}}^*$ is called a *root* of $\mathfrak{g}_{\mathbb{Q}}$ if there is a non-zero element $x \in \mathfrak{g}_{\mathbb{Q}}$, such that

$$[h, x] = \alpha(h)x, \quad h \in \mathfrak{h}.$$

We set $\Delta = \Delta(A)$ to be the set of all roots of $\mathfrak{g}_{\mathbb{Q}}$. Then $\mathfrak{g}_{\mathbb{Q}}$ has a direct sum decomposition

$$\mathfrak{g}_{\mathbb{Q}} = \mathfrak{h}_{\mathbb{Q}} \oplus \bigsqcup_{\alpha \in \Delta(A)} \mathfrak{g}_{\mathbb{Q}}^{\alpha},$$

where

$$\mathfrak{g}_{\mathbb{Q}}^{\alpha} = \{x \in \mathfrak{g}_{\mathbb{Q}} | [h, x] = \alpha(h)x, \ h \in \mathfrak{h}_{\mathbb{Q}}\}.$$

Each $\mathfrak{g}_{\mathbb{Q}}^{\alpha}$ is finite dimensional. In particular,

$$\pm\alpha_i \in \Delta(A), \quad i \in I$$

and $\mathfrak{g}_{\mathbb{Q}}^{\alpha_i} = \mathbb{Q}e_i$, $\mathfrak{g}_{\mathbb{Q}}^{-\alpha_i} = \mathbb{Q}f_i$, $i = 1, \ldots, l$. The elements $\alpha_1, \ldots, \alpha_l$ are the *simple roots*. Every root $\alpha \in \Delta(A)$ has a unique expression

$$\alpha = \sum_{i=1}^{l} k_i \alpha_i,$$

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where the k_i are either all in $\mathbb{Z}_{\geq 0}$, in which case we say that α is *positive*, or the k_i are all in $\mathbb{Z}_{\leq 0}$, in which case we say that α is *negative*. We set

$$\Delta^{\pm} = \Delta^{\pm}(A)$$

equal to the set of $\pm$ roots. Then

$$\Delta = \Delta^+ \cup \Delta^- \quad \text{(disjoint union)}.$$

We set

$$\mathfrak{u}_{\pm} := \mathfrak{u}_{\pm}(A) = \bigsqcup_{\alpha \in \Delta^{\pm}} \mathfrak{g}_{\mathbb{Q}}^{\alpha} \, .$$

Then $\mathfrak{u}_{\pm}$ is a subalgebra of $\mathfrak{g}_{\mathbb{Q}}$, and in fact equals $\mathfrak{g}_{\mathbb{Q},\pm}$. Moreover $\mathfrak{g}_{\mathbb{Q}}$ admits an invariant, symmetric bilinear form $\sigma(,)$, and σ induces a non-degenerate pairing between $\mathfrak{g}_{\mathbb{Q}}^{\alpha}$ and $\mathfrak{g}_{\mathbb{Q}}^{-\alpha}$, for $\alpha \in \Delta(A)$. We have the Weyl reflections

$$w_i : \mathfrak{h} \longrightarrow \mathfrak{h} \, , \quad i = 1, \ldots, l \, ,$$

where

$$w_i(h) = h - \alpha_i(h)\alpha_i^{\nu} \, , \quad h \in \mathfrak{h} \, .$$

The w_i generate a subgroup $W = W(A) \subseteq \operatorname{Aut}(\mathfrak{h})$, called the *Weyl group* of $\mathfrak{g}_{\mathbb{Q}}$, and $\sigma|_{\mathfrak{h}}$ is W-invariant. For $i, j \in I$, we set

$$c_{ii} := 1$$

and for $i \neq j$,

$$c_{ij} := 2, 3, 4, 6 \, , \quad \text{or } \infty$$

according as

$$A_{ij}A_{ji} = 0, 1, 2, 3 \, , \quad \text{or } \geq 4 \, .$$

Then $W = W(A)$ is the group with presentation (see [29, p. 81])

$$W = \langle m_i | i \in I, (m_i m_j)^{c_{ij}} = 1, \quad \text{if } c_{ij} \neq \infty \rangle \, .$$

A root $\alpha \in \Delta$ is called a *real root* (or a *Weyl root*) if there exists $w \in W$ such that $w(\alpha)$ is a simple root. A root α which is not real is called *imaginary*. We denote by Φ the Weyl roots, Φ_I the imaginary roots, $\Phi^{\pm}$ the positive and negative Weyl roots, and $\Phi_I^{\pm}$ the positive and negative imaginary roots. Then

$$\Delta = \Phi \sqcup \Phi_I \, ,$$

$$\Phi = \Phi^+ \sqcup \Phi^- \, ,$$

$$\Phi_I = \Phi_I^+ \sqcup \Phi_I^- \, ,$$

and

$$\Phi = W(\Pi) \, .$$

4. A $\mathbb{Z}$-Form of the Universal Enveloping Algebra of $\mathfrak{g}_{\mathbb{Q}}$

Let $\mathcal{U}_{\mathbb{Q}}$, $\mathcal{U}_{\mathbb{Q},+}$ and $\mathcal{U}_{\mathbb{Q},-}$ be the universal enveloping algebras of $\mathfrak{g}_{\mathbb{Q}}$, $\mathfrak{g}_{\mathbb{Q},+}$ and $\mathfrak{g}_{\mathbb{Q},-}$ respectively. The following result follows from the Birkhoff–Witt theorem.

Theorem 4.1 ([29, p. 77]). *Let $\mathcal{S}(\mathfrak{h}_{\mathbb{Q}})$ be the symmetric algebra of $\mathfrak{h}_{\mathbb{Q}}$. The canonical map*

$$\mathcal{U}_{\mathbb{Q},-} \otimes \mathcal{S}(\mathfrak{h}_{\mathbb{Q}}) \otimes \mathcal{U}_{\mathbb{Q},+} \longrightarrow \mathcal{U}_{\mathbb{Q}}$$

is a bijection.

For any associative algebra A, let $x \in A$ and let $m \in \mathbb{Z}_{\geq 0}$. We set

$$x^{(m)} := \frac{x^m}{m!},$$

and

$$\binom{x}{m} := \frac{x(x-1)(x-2)\cdots(x-m+1)}{m!}.$$

We choose $\Lambda \subseteq \mathfrak{h}^*$ to be the linear span of α_i, for $i \in I$, and $\Lambda' \subseteq \mathfrak{h}$ to be the linear span of h_i, (where $h_i = \alpha_i^\nu$), for $i \in I$. Let

$\mathcal{U}_{\mathbb{Z},+} \subseteq \mathcal{U}_{\mathbb{Q},+}$ be the $\mathbb{Z}$-subalgebra generated by $e_i^{(m)}$ for $i \in I$ and $m \geq 0$,

$\mathcal{U}_{\mathbb{Z},-} \subseteq \mathcal{U}_{\mathbb{Q},-}$ be the $\mathbb{Z}$-subalgebra generated by $f_i^{(m)}$ for $i \in I$ and $m \geq 0$,

$\mathcal{U}_{\mathbb{Z},0} \subseteq \mathcal{S}(\mathfrak{h}_{\mathbb{Q}})$ be the $\mathbb{Z}$-subalgebra generated by $\binom{\lambda'}{m}$, for $\lambda' \in \Lambda'$ and $m \geq 0$,

and let

$\mathcal{U}_{\mathbb{Z}} \subseteq \mathcal{U}_{\mathbb{Q}}$ be the $\mathbb{Z}$-subalgebra generated by $e_i^{(m)}$, $f_i^{(m)}$ for $i \in I$ and $\binom{\lambda'}{m}$, for $\lambda' \in \Lambda'$ and $m \geq 0$. By a $\mathbb{Z}$-*form* of a $\mathbb{Q}$-algebra, $X_{\mathbb{Q}}$, we mean a subring X of $X_{\mathbb{Q}}$ such that the canonical map

$$X \otimes \mathbb{Q} \longrightarrow X_{\mathbb{Q}}$$

is bijective. Using the standard grading of $\mathcal{U}_{\mathbb{Q}}$ defined by

$$\deg(e_i) = -\deg(f_i) = 1,$$

it follows that $\mathcal{U}_{\mathbb{Z},+}$ is a $\mathbb{Z}$-form of $\mathcal{U}_{\mathbb{Q},+}$ [29, p. 78].

We set

$$\mathfrak{g}_{\mathbb{Z}} = \mathfrak{g}_{\mathbb{Q}} \cap \mathcal{U}_{\mathbb{Z}},$$

$$\mathfrak{g}_{\mathbb{Z},\pm} = \mathfrak{g}_{\mathbb{Q},\pm} \cap \mathcal{U}_{\mathbb{Z}}.$$

The following theorem then implies that $\mathcal{U}_{\mathbb{Z}}$ is a $\mathbb{Z}$-form of $\mathcal{U}_{\mathbb{Q}}$.

Theorem 4.2 ([29, p. 78]). *The product map*

$$\mathcal{U}_{\mathbb{Z},-} \otimes \mathcal{U}_{\mathbb{Z},0} \otimes \mathcal{U}_{\mathbb{Z},+} \longrightarrow \mathcal{U}_{\mathbb{Z}}$$

and the sum map

$$\mathfrak{g}_{\mathbb{Z},-} \oplus \Lambda' \oplus \mathfrak{g}_{\mathbb{Z},+} \longrightarrow \mathfrak{g}_{\mathbb{Z}}$$

are isomorphisms of modules.

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For a field K, we let

$$\mathcal{U}_K = \mathcal{U}_\mathbb{Z} \otimes K$$

$$\mathcal{U}_{K,0} = \mathcal{U}_{\mathbb{Z},0} \otimes K$$

$$\mathcal{U}_{K,\pm} = \mathcal{U}_{\mathbb{Z},\pm} \otimes K$$

$$\mathfrak{g}_K = \mathfrak{g}_\mathbb{Z} \otimes K$$

$$\mathfrak{g}_{K,\pm} = \mathfrak{g}_{\mathbb{Z},\pm} \otimes K \,.$$

5. Notation

We will construct Kac–Moody groups by means of representation theory. The groups obtained in this way (Sec. 6) depend on the chosen representation. Before completion, the groups constructed (in Sec. 6) are homomorphic images of the "minimal" group of Tits in [29]. The *rank* of the Kac–Moody group of Sec. 6 is defined to be $l = |I|$ as in Sec. 1.

We let λ be a dominant integral linear functional on $\mathfrak{h}$, and let V^λ be the corresponding irreducible highest weight module. Arguing as in [27] we can use the $\mathbb{Z}$-form $\mathcal{U}_\mathbb{Z}$ of $\mathcal{U}$ (as in [29, p. 78] to obtain a $\mathbb{Z}$-form $V_\mathbb{Z}^\lambda \subseteq V^\lambda$ such that the following properties ((5.1) and (5.2)) hold:

$$\frac{e_i^m}{m!} V_\mathbb{Z}^\lambda \subseteq V_\mathbb{Z}^\lambda \,,$$

$$\frac{f_i^m}{m!} V_\mathbb{Z}^\lambda \subseteq V_\mathbb{Z}^\lambda \,, \tag{5.1}$$

for $m \in \mathbb{Z}_{\geq 0}$, and $i \in I$. We remark that for an irreducible highest weight module V^λ, the e_i and f_i are "locally nilpotent" on V^λ (see [16, Chapt. 10]), that is for any $v \in V^\lambda$, $e_i^n(v) = 0$ and $f_i^n(v) = 0$ for almost all $n \in \mathbb{Z}_{\geq 0}$.

For each weight μ of V^λ, we let V_μ^λ be the corresponding weight space, and we set

$$V_{\mu,\mathbb{Z}}^\lambda := V_\mu^\lambda \cap V_\mathbb{Z}^\lambda \,.$$

We have

$$V_\mathbb{Z}^\lambda = \oplus_\mu V_{\mu,\mathbb{Z}}^\lambda \,, \tag{5.2}$$

where the sum is taken over the weights of V^λ. We let K denote a field, and we set

$$V_K^\lambda := K \otimes_\mathbb{Z} V_\mathbb{Z}^\lambda \,,$$

$$V_{\mu,K}^\lambda := K \otimes_\mathbb{Z} V_{\mu,K}^\lambda \,,$$

so that

$$V_K^\lambda = \oplus_\mu V_{\mu,K}^\lambda \,.$$

For $u \in K$, we set

$$\chi_{\alpha_i}(u) := \sum_{m \in \mathbb{Z}_{\geq 0}} u^m \frac{e_i^m}{m!} \quad (= \exp(ue_1)), \tag{5.3}$$

$$\chi_{-\alpha_i}(u) := \sum_{m \in \mathbb{Z}_{\geq 0}} u^m \frac{f_i^m}{m!} \quad (= \exp(uf_1)).$$

Then $\chi_{\alpha_i}(u)$, $\chi_{-\alpha_i}(u)$ define elements in $\mathrm{Aut}(V_K^\lambda)$ (thanks to the local nilpotence of e_i, f_i). We let $G_0 \subseteq \mathrm{Aut}(V_K^\lambda)$ be the subgroup generated by the linear automorphisms $\chi_{\pm\alpha_i}(u)$ of V_K^λ, for $i \in I$, $u \in K$. For $u \in K^\times$, $i \in I$, we set

$$w_{\alpha_i}(u) := \chi_{\alpha_i}(u)\chi_{-\alpha_i}(-u^{-1})\chi_{\alpha_i}(u),$$

$$h_{\alpha_i}(u) := w_{\alpha_i}(u)w_{\alpha_i}(1)^{-1}.$$

With a harmless abuse of notation, we set $w_{\alpha_j} := w_{\alpha_j}(1)$ for each $j \in I$.

We let $H \subseteq G_0$ be the subgroup generated by the elements $h_{\alpha_i}(u)$, $u \in K^\times$, $i \in I$. For Weyl roots $\alpha \in \Phi$ with $\alpha = w\alpha_j$, for some $w \in W$, $j \in I$, we define elements $\chi_\alpha(u)$, $u \in K$ by

$$\chi_\alpha(u) := \chi_{w\alpha_j}(u) = w\chi_{\alpha_j}(u)w^{-1}.$$

($\chi_\alpha(u)$ has an expression analogous to (5.3) where e_i is replaced by its w-conjugate).

We note that $V_{\mathbb{Z}}^\lambda$ admits a *coherently ordered basis* which is defined as follows. For each weight μ of V^λ, $\mu = \lambda - \sum_{i=1}^l k_i\alpha_i$, where $k_i \in \mathbb{Z}_{\geq 0}$. Define the *depth* of μ to be

$$\mathrm{depth}(\mu) = \sum_{i=1}^l k_i.$$

A basis $\Psi = \{v_1, v_2, \ldots\}$ of V^λ is called *coherently ordered* if

(1) Ψ consists of weight vectors.
(2) If $v_i \in V_\mu^\lambda$, $v_j \in V_{\mu'}^\lambda$ and $\mathrm{depth}(\mu') > \mathrm{depth}(\mu)$, then $j > i$.
(3) $\Psi \cap V_\mu^\lambda$ consists of an interval $v_k, v_{k+1}, \ldots, v_{k+m}$.

Relative to a coherently ordered basis, the elements of H are represented by diagonal matrices, the elements $\chi_\alpha(u)$, $u \in K$, $\alpha \in \Phi^+$ by upper triangular, unipotent matrices, and the elements $\chi_\alpha(u)$, $u \in K$, $\alpha \in \Phi^-$ by lower triangular, unipotent matrices.

6. Kac–Moody Groups

For each Weyl root $\alpha \in \Phi$, we define the *root group* U_α to be the group:

$$U_\alpha := \{\chi_\alpha(u)|u \in K\}, \quad \alpha \in \Phi,$$

and we let B_α be the group generated by U_α and H. Then $U_\alpha \subseteq B_\alpha$ is a normal subgroup and B_α is the semi-direct product of H and U_α. One can then establish the following axioms (see [30, p. 562]):

(RD1$'$) Let (α, β) be a *prenilpotent pair* (see [30, p. 562] and (6.1) below). Let

$$]\alpha, \beta[_{\text{lin}} = \{n\alpha + m\beta \in \Phi | n, m \in \mathbb{Z}_{>0}\}.$$

Then the commutator of U_α and U_β is contained in the group generated by the U_γ, $\gamma \in]\alpha, \beta[_{\text{lin}}$.

(RD2) For $i \in I$, and corresponding simple root α_i, we have:

$$B_{\alpha_i} \cap B_{-\alpha_i} = H.$$

(RD3) The group B_{α_i} has two double cosets in the group $\langle B_{\alpha_i}, B_{-\alpha_i} \rangle$.

(RD4) There is an element in $\langle B_{\alpha_i}, B_{-\alpha_i} \rangle$ which, for each $\beta \in \Phi$, conjugates B_β to $B_{w_i\beta}$, where w_i is the Weyl reflection corresponding to α_i.

(RD5) For each simple root α_i, B_{α_i} is not contained in B^-, and $B_{-\alpha_i}$ is not contained in B^+, where $B^\pm$ is the group generated by the B_β, $\beta \in \Phi^\pm$.

We remark that the verification of these axioms is straightforward, and we need only comment that we can obtain (RD1$'$) using the arguments of [27, Lemma 15] and its corollary (p. 24).

Recall from [30, p. 562] that $(\alpha, \beta) \in \Phi$ is a *prenilpotent pair* if there exist elements w, $w' \in W$ such that

$$\begin{aligned} w \cdot \alpha, \quad w \cdot \beta \in \Phi^+, \\ w' \cdot \alpha, \quad w' \cdot \beta \in \Phi^-. \end{aligned} \tag{6.1}$$

We let $S_I \subset W$ be the set of simple root reflections (so $w_i \in S_I$, $i \in I$ will be the reflection corresponding to the simple root α_i). We let $U_0^\pm$ be the subgroup of G_0 generated by the U_α, $\alpha \in \Phi^\pm$. We let $B_0^\pm$ be the subgroup of G_0 generated by the B_β, $\beta \in \Phi^\pm$, and we use the following notation:

$$U^- := U_0^-,$$

$$B^- := B_0^-,$$

$$B_0 := B_0^+,$$

$$U_0 := U_0^+.$$

We have

$$B_0 = HU_0,$$

$$B^- = HU^-.$$

It follows that $B_0 \cap B^- = H$ (just use a coherently ordered basis).

We fix $m_i \in \langle B_{\alpha_i}, B_{-\alpha_i} \rangle$ so that

$$m_i B_\beta m_i^{-1} = B_{w_i \cdot \beta}, \quad \beta \in \Phi,$$

as in (RD4). We let N be the subgroup of G_0 generated by H and $\{m_i\}_{w_i \in S_I}$. From [30], we have

Proposition 6.1 ([30, Sec. 5]). *(G_0, B_0, N), and (G_0, B^-, N) are BN-pairs, in fact twin BN-pairs in the sense of [31]. In particular, we have:*

$$G_0 = B_0 N B_0$$

$$= B^- N B^-$$

$$= B^- N B_0$$

$$= B_0 N B^-.$$

Next we consider the *completion* of G_0, and we take $K = \mathbb{F}_q$, a finite field. In V_K^λ we choose a coherently ordered basis $\Xi = \{v_1, v_2, \ldots\}$, so that

(i) B_0 is represented by upper triangular matrices with respect to Ξ.

(ii) B^- is represented by lower triangular matrices with respect to Ξ.

(iii) H is represented by diagonal matrices with respect to Ξ.   (6.2)

We let V_t be the span of the v_u, for $v_u \in \Xi$ with $u \le t$. Then

$$B_0 V_t \subseteq V_t, \quad \text{for each } t.$$

We let B_t be the image of B_0 in $\operatorname{Aut}(V_t)$. We then have surjective homomorphisms

$$B_{t'} \xrightarrow{\pi_{tt'}} B_t, \quad t' \ge t,$$

and we let B (the completion of B_0) be the projective limit of the projective family:

$$\{B_t, \pi_{tt'}\}.$$

Then B is profinite. On the other hand, G_0 itself can be topologized as follows: for a base of neighborhoods of the identity, we take sets

$$V_{(u_1, \ldots, u_p)}, \quad u_1, \ldots, u_p \in V_K^\lambda,$$

defined by

$$V_{(u_1, \ldots, u_p)} = \{g \in G_0 | g u_t = u_t, \ t = 1, \ldots, p\}.$$

We take the corresponding completion G of G_0, keeping groups B^-, H, N as before. Then G is a locally compact group, totally disconnected with B an open profinite neighborhood of the identity, as follows from:

Lemma 6.1. *Let $\{g_t\}_{t=1,2,\ldots}$ be a sequence in G_0 such that*

$$g_t \longrightarrow g \in G$$

(in the above topology). Then there exists $g' \in G_0$ and $t_0 \in \mathbb{Z}_{\geq 0}$ so that for all $t \geq t_0$,

$$g_t = (g') \bmod B_0 \,.$$

Proof. Let $v_\lambda \in V_K^\lambda$ be a highest weight vector $(v_\lambda = 1 \otimes v_\lambda, \, v_\lambda \in V_{\mathbb{Z}}^\lambda)$. Then the sequence $g_t v_\lambda$ is eventually constant. Say $g_t v_\lambda = g_{t_0} v_\lambda$, $t \geq t_0$. For simplicity, we assume that λ is *regular*, that is:

$$\lambda(h_i) > 0 \,, \quad i = 1, \ldots, l \,,$$

(h_i being the simple coroots). We have

$$g_t^{-1} g_{t_0} v_\lambda = v_\lambda \,, \quad t \geq t_0 \,,$$

and since λ is regular,

$$g_t^{-1} g_{t_0} \in B_0 \,, \quad t \geq t_0 \,,$$

as follows from the Bruhat decomposition $G_0 = B_0 W B_0$ obtained from [30, Proposition 1.1]. □

For $w_i \in S_I$, set

$$U_{0,w_i} = m_i U_0^+ m_i^{-1} \cap U_0^+ \,.$$

Lemma 6.2. *If $u \in U_0^+$, and*

$$u \in m_i B_0 m_i^{-1} \,, \tag{6.3}$$

then

$$u \in m_i U_0^+ m_i^{-1} \,.$$

Proof. Since

$$B_0 = H U_0 \,,$$

and m_i normalizes H, we have

$$m_i B_0 m_i^{-1} = H(m_i U_0 m_i^{-1}) \,.$$

Hence, by (6.3), u has an expression

$$u = h m_i u' m_i^{-1} \,, \quad h \in H, u' \in U_0 \,. \tag{6.4}$$

For each $q \in \mathbb{Z}_{>0}$, consider the set

$$m_i v_1, \ldots, m_i v_q \,, \tag{6.5}$$

and the subspace $V(q) \subseteq V_K^\lambda$ spanned by the vectors (6.5).

The subspace $V(q)$ is u-invariant, and u restricted to $V(q)$ is upper triangular with respect to the basis (6.5), thanks to the expression (6.4). Enlarge $V(q)$ to a finite dimensional subspace $\tilde{V}(q)$ by adding all weight spaces $V_{\mu,K}^\lambda$ with μ of smaller

depth than the weight of some $m_i v_j$, $j = 1, \ldots, q$. Then $\tilde{V}(q)$ is u-invariant, and u acts on $\tilde{V}(q)$ as a unipotent linear transformation. Hence u restricted to $V(q)$ is unipotent, and hence $h|_{V(q)}$ is the identity. Since q was arbitrary, h is the identity on V_K^λ, and $u = m_i u' m_i^{-1}$. $\qquad\blacksquare$

We now prove

Lemma 6.3. *Assume that λ is regular $(\lambda(\alpha_i^\nu) > 0$, $i = 1, \ldots, q)$. Then U_0 is the semi-direct product*

$$U_0 = U_{\alpha_i} \ltimes U_{0,w_i} \,.$$

Proof. We let $U' \subseteq U_0$ be the *normal* subgroup generated by the U_α, $\alpha \in \Phi^\pm \{\alpha_i\}$, that is, U' is the smallest normal subgroup of U_0 containing all such U_α. Then it is clear that

$$U_0 = U_{\alpha_i} U' \,. \tag{6.6}$$

We consider an element

$$u = \chi_{\alpha_i}(s)\chi_\beta(s')\chi_{\alpha_i}(-s) \,, \quad s, s' \in K \,,$$

where β is a positive Weyl root distinct from α_i. For v_1, we take a highest weight vector v_λ of V_K^λ. We have

$$m_i v_\lambda = v_{w_i \lambda} \,,$$

a weight vector corresponding to the weight

$$w_i \lambda = \lambda - k\alpha_i \,, \quad \text{some } k \in \mathbb{Z}_{>0} \,,$$

then

$$\chi_{\alpha_i}(-s)m_i v_\lambda = \chi_{\alpha_i}(-s)v_{\lambda - k\alpha_i}$$

$$\in V' \,,$$

where

$$V' = \bigoplus_{j=0}^{k} V_{\lambda - j\alpha_i, K}^\lambda \,.$$

We observe that $\chi_\beta(s')$ leaves each element of V' invariant. Hence

$$\chi_{\alpha_i}(s)\chi_\beta(s')\chi_{\alpha_i}(-s)m_i v_\lambda = m_i v_\lambda \,,$$

and so

$$m_i^{-1}\chi_{\alpha_i}(s)\chi_\beta(s')\chi_{\alpha_i}(-s)m_i \in B_0$$

(λ being regular, we apply the Bruhat decomposition).

By Lemma 6.2,

$$m_i^{-1}\chi_{\alpha_i}(s)\chi_\beta(s')\chi_{\alpha_i}(-s)m_i \in U_0\,,$$

and so

$$u \in U_0 \cap m_i U_0 m_i^{-1} = U_{0,w_i}\,.$$

It follows that

$$U' \subseteq U_{0,w_i}\,, \tag{6.7}$$

and hence from (6.6), we have

$$U_0 = U_{\alpha_i} U_{0,w_i}\,. \tag{6.8}$$

Now we have from the above argument

$$u' v_{w_i\lambda} = v_{w_i\lambda}\,, \quad u' \in U'$$

(indeed we have verified this for a generating set of $U\prime$), while for $s \neq 0$

$$\chi_{\alpha_i}(s)v_{w_i\lambda} \neq v_{w_i\lambda}\,.$$

Hence

$$U_0 = U_{\alpha_i} \ltimes U' \tag{6.9}$$

If $u \in U_{0,w_i}$, then

$$u = u_1 u_2\,, \quad u_1 \in U_{\alpha_i},\ u_2 \in U'\,,$$

and by (6.7), $u_2 \in U_{0,w_i}$. Hence $u_1 \in U_{0,w_i}$, which is impossible, unless u_1 is the identity.

This shows that $U_{0,w_i} = U'$, and hence by (6.9)

$$U_0 = U_{\alpha_i} \ltimes U_{0,w_i}\,,$$

which proves the lemma.   □

From now on, we will assume that λ is regular.

Now let U^+ be the closure of U_0 in G. If

$$u_t \longrightarrow u \in U^+\,,$$

for $u_t \in U_0^+$, then eventually the sequences $u_t v_\lambda$ and $u_t m_i v_\lambda$ are constant, and by the above argument, for some t_0

$$u_t = u_{t_0} \bmod U_{0,w_i}\,, \quad t \geq t_0\,.$$

(First $u_t = u_{t_0} \bmod B_0 \cap m_i B_0 m_i^{-1}$, and then apply Lemma 6.2). It follows that if U_{w_i} is the closure of U_{0,w_i}, then

$$U^+ = U_{\alpha_i} \ltimes U_{w_i}\,,$$

for each $w_i \in S_I$.

One can then verify easily

Theorem 6.1. (G, B, N) *is a BN-pair, and also*

$$G = BNB^- = B^-NB.$$

Proof. We show that

$$w_i Bw \subseteq Bw_i wB \cup BwB.$$

But since $w_i H w_i^{-1} = H$, it suffices to show that

$$w_i U^+ w \subseteq Bw_i wB \cup BwB.$$

But

$$U^+ = U_{\alpha_i} U', \quad U' = U_{w_i},$$

so if $u \in U^+$, then

$$u = u' u_{\alpha_i}, \quad u_{\alpha_i} \in U_{\alpha_i}, u' \in U',$$

and

$$w_i u w = w_i u' u_{\alpha_i} w$$

$$= (w_i u' w_i^{-1}) w_i u_{\alpha_i} w, \quad w_i u w_i^{-1} \in B,$$

(since $U' = U_{w_i}$). But then we are reduced to showing

$$w_i u_{\alpha_i} w \in Bw_i wB \cup BwB,$$

which follows from [30, Sec. 5] ((G_0, B_0, N) is a BN-pair). The last two equalities of the theorem follow from the corresponding equalities for G_0 (see [30]) together with Lemma 6.1 above. $\square$

Remark 6.1. We have

$$B = HU, \qquad (\text{where } U = U^+)$$

$$B^- = HU^-, \quad (\text{as we noted earlier})$$

and hence $B \cap B^- = H$, as follows by considering a coherently ordered basis of V_K^λ.

The BN-pair structure on G (6.13) gives an amalgam decomposition for G, where the factors are the standard parabolic subgroups.

Theorem 6.2 (Tits; [27, p. 92]). *Let (G, B, N) be a BN-pair. Let S be the standard generating set for the Weyl group $W := N/(B \cap N)$ consisting of simple root reflections. For each $s \in S$ let W_s be the subgroup of W generated by the element s. Let $G_s := BW_s B = \bigcup_{w \in W_s} BwB$ be the corresponding standard parabolic subgroup. Then G_s is a subgroup of G, and G is the free product of N and the G_s, $s \in S$, amalgamated along their intersections.*

7. Existence of Lattices

Let G be a locally compact group with (left) Haar measure μ. Let Γ be a discrete subgroup. Then $\Gamma\backslash G$ carries a measure, also denoted μ. We call Γ a *lattice* if $\mu(\Gamma\backslash G) < \infty$, and a *uniform lattice* if $\Gamma\backslash G$ is compact, a *non-uniform lattice* otherwise.

To discuss the existence of lattices in G, we begin with the following general lemma.

Lemma 7.1. *Let G be a locally compact group with an open compact subgroup B such that G is the smallest normal subgroup containing B. Let $\Gamma \subseteq G$ be a discrete subgroup of G. We set $X = G/B$. Then Γ has a natural left action on X. Let $S \subseteq X$ be a subset such that*

$$\Gamma S = X. \tag{7.1}$$

Then

(a) *The isotropy group Γ_x, $x \in X$, is finite.*
(b) *If the series*

$$\sum_{x \in S} \frac{1}{|\Gamma_x|}$$

  is convergent, then $\Gamma\backslash G$ has finite invariant volume.

(The volume refered to here is the volume with respect to some (and hence any) Haar measure on G. Our assumptions on B imply that G is unimodular.)

Proof. Let

$$\tilde{\omega} : G \longrightarrow G/B$$

denote the natural projection, and let

$$\tilde{S} = \tilde{\omega}^{-1}(S).$$

Then S is an open subset of G, and

$$\Gamma\tilde{S} = G, \tag{7.2}$$

by (7.1). Let $\pi : G \longrightarrow \Gamma\backslash G$ denote the natural projection. Then

$$\pi(\tilde{S}) = \Gamma\backslash G, \tag{7.3}$$

by (7.2), and

$$\pi(\tilde{S}) = \bigcup_{x \in S} \pi(\tilde{S}_x), \tag{7.4}$$

where $\tilde{S}_x = \tilde{\omega}^{-1}(x)$, $x \in X = G/B$. In fact for $x \in X$, each set $\tilde{S}_x$ is a coset of the form $x'B$, with $x' \in G$ and $\tilde{\omega}(x') = x$. Moreover,

$$\Gamma_x = \Gamma \cap x'B(x')^{-1},$$

which is a discrete subgroup of the compact group $x'B(x')^{-1}$, and hence is finite. This proves (a). Now (7.3) and (7.4) imply that

$$\text{volume}(\Gamma \backslash G) \leq \sum_{x \in S} \text{volume } \pi(\Gamma_x \backslash \tilde{S}_x) = \sum_{x \in S} \frac{1}{|\Gamma_x|}\,,$$

where "volume" is the volume with respect to a Haar measure on G, normalized so that the compact open subgroup B has measure 1. Assertion (b) then follows. □

8. Non-Uniform "Arithmetic" Lattices

We now let G be a complete Kac–Moody group over a finite field, denoted k, and let B, B^-, W be as in Sec. 6. In this section, we show how to construct a non-uniform lattice in G. We have the decomposition from Theorem 6.1:

$$G = B^- N B\,. \tag{8.1}$$

To apply Lemma 7.1, we take $B^- = \Gamma$, $S = WB/B$, and using (8.1), we see that condition (7.1) is satisfied. We let Γ_w, $w \in W$, denote the isotropy group of the coset wB:

$$\Gamma_w = \{\gamma \in \Gamma | \gamma wB = wB\}\,.$$

We wish to compute the orders $|\Gamma_w|$ of the groups Γ_w, $w \in W$, in particular showing that the groups Γ_w are finite. We also wish to determine if

$$\sum_{w \in W} \frac{1}{|\Gamma_w|} < \infty\,. \tag{8.2}$$

If the Γ_w are finite, and (8.2) holds, then we can apply Lemma 7.1 to conclude that Γ is a discrete subgroup of G with finite covolume. In fact, we shall show:

Lemma 8.1. *If $q = |k|$, then*

$$|\Gamma_w| = q^{l(w)}(q-1)^l\,,$$

where $l(\cdot)$ is the length function of the Weyl group W.

Assume Lemma 8.1. If G is constructed using the $l \times l$ symmetrizable generalized Cartan matrix A, then W is generated by simple root reflections $w_1, \ldots, w_l$, and the number of elements of length r is $\leq l^r$. Hence if $q > l$, then

$$\sum_{w \in W} \frac{1}{|\Gamma_w|} < \sum_{r=0}^{\infty} \frac{l^r}{q^r}\,,$$

which converges, provided $q > l$. Hence as a consequence of Lemmas 7.1 and 8.1, we have

Theorem 8.1. *If $q > l$, then $\Gamma = B^-$ is a discrete subgroup of G with finite covolume.*

In case $l = 2$, the number of elements in W of length r equals two (asuming that W is infinite). In this case, we can drop the restriction on q, and we have

Theorem 8.2. *If $l = 2$, then $\Gamma = B^-$ is a discrete subgroup of G with finite covolume (with no restriction on q).*

Proof of Lemma 8.1. First, for $w \in W$, we describe the cosets in

$$BwB/B\,.$$

We consider a *reduced expression*

$$w^{-1} = w_{i_1} \cdots w_{i_r}\,, \quad (r = l(w))$$

of w^{-1}, and the set of all roots $\alpha \in \Delta^+$ such that

$$w^{-1} \cdot \alpha \in \Delta^-\,;$$

that is, consider the set

$$\Delta^+ \cap w(\Delta^-)\,.$$

Each simple root reflection w_i corresponds to a simple root α_i of $\mathfrak{g}_{\mathbb{Q}}$. We then have (see [15, Proposition 2.2] and [22, Sec. 2]):

$$\Delta^+ \cap w(\Delta^-) = \{\alpha_{i_r}, w_{i_r} \cdot \alpha_{i_{r-1}}, \ldots, w_{i_r} \cdots w_{i_m} \cdot \alpha_{i_{m-1}}, \ldots, w_{i_r} \cdots w_{i_2} \cdot \alpha_{i_1}\}\,.$$

$$(8.3)$$

Note that in particular,

$$|\Delta^+ \cap w(\Delta^-)| = r\,.$$

The ordering of the elements in $\Delta^+ \cap w(\Delta^-)$ given in (8.6) will be singled out and called the "special ordering".

Lemma 8.2. *For every $b' \in B$, we have*

$$b'wB = bwB\,,$$

where b has the form

$$b = \Pi_{\alpha \in \Delta^+ \cap w(\Delta^-)} \chi_\alpha(s_\alpha)\,, \quad s_\alpha \in k \tag{8.4}$$

(product taken with respect to the special ordering on $\Delta^+ \cap w(\Delta^-)$). Moreover, if

$$b'' = \Pi_{\alpha \in \Delta^+ \cap w(\Delta^-)} \chi_\alpha(s''_\alpha)\,, \quad s''_\alpha \in k$$

(product again taken with respect to the special ordering on $\Delta^+ \cap w(\Delta^-)$), and if $bwB = b''wB$, then

$$s_\alpha = s''_\alpha\,, \quad \text{for all } \alpha \in \Delta^+ \cap w(\Delta^-)\,.$$

Proof. We will argue by induction on the length of w. Assume that $w = w_i$ ($l(w) = 1$). If $b' \in B$, then b' has an expression

$$b' = uh\,, \quad u \in U,\ h \in H$$

and u has an expression

$$u = u_1 u_2 \,,$$

where $u_1 = \chi_{\alpha_i}(s)$, $s \in k$, $u_2 \in U$ with $w_i^{-1} u_2 w_i \in U$ (see Sec. 6). Then

$$b'wB = u_1 u_2 h w_i B$$

$$= u_1 w_i B$$

(as $w_i^{-1} H w_i = H \subseteq B$). This proves the first assertion of the lemma for $l(w) = 1$. On the other hand, if

$$\chi_{\alpha_i}(s) w_i B = \chi_{\alpha_i}(s') w_i B \,, \quad s, s' \in k \,,$$

and if $s \neq s'$, then we have

$$\chi_{\alpha_i}(\tilde{s}) w_i B = w_i B \,,$$

for some $\tilde{s} \neq 0$. Take λ regular (as in Lemma 6.3). If V_k^λ is a highest weight module with highest weight λ, and $v_\lambda \neq 0$ is a highest weight vector, then

$$w_i \tilde{b} \cdot v_\lambda = \tilde{v}_{w_i \lambda} \,, \quad \tilde{b} \in B \tag{8.5}$$

where $\tilde{v}_{w_i \lambda}$ is a (non-zero) weight vector corresponding to the weight $w_i \lambda$. On the other hand,

$$\chi_{\alpha_i}(\tilde{s}) w_i \cdot v_\lambda = v_{w_i \lambda} + \text{weight vectors corresponding to weights higher than } w_i \lambda$$

($v_{w_i \lambda} = w_i \cdot v_\lambda$, a non-zero weight vector corresponding to the weight $w_i \lambda$). The only way this can coincide with (8.5) is if there are no higher weight components. But then $\tilde{s} = 0$, a contradiction.

Now we assume that we have proven the lemma for Weyl group elements of length $\leq r - 1$, and assume that $l(w) = r$. Assume that we have a coset

$$b'wB \,,$$

with $w^{-1} = w_{i_1} \cdots w_{i_r}$, $w = w_{i_r} \cdots w_{i_1}$, $b' \in B$. Then

$$b'wB = b'w_{i_r} w'B \,,$$

where $w' = w_{i_{r-1}} \cdots w_{i_1}$. By the above case (Weyl group elements of length one), we have

$$b'w_{i_r} = \chi_{\alpha_{i_r}}(s) w_{i_r} b'' \,,$$

for some $s \in k$, $b'' \in B$. Therefore,

$$b'w_{i_r} w'B = \chi_{\alpha_{i_r}}(s) w_{i_r} b'' w'B \,, \tag{8.6}$$

and by our inductive hypothesis,

$$b''w'B = \Pi_{\alpha \in \Delta^+ \cap w'(\Delta^-)} \chi_\alpha(s'_\alpha) w'B$$

for some $s'_\alpha \in k$ (the product taken with respect to the special ordering). Hence, (continuing with (8.6)), we obtain

$$b'w_{i_r}w'B = \chi_{\alpha_{i_r}}(s)w_{i_r}\Pi_{\alpha \in \Delta^+ \cap w'(\Delta^-)} \chi_\alpha(s'_\alpha) w'B$$

$$= [\chi_{\alpha_{i_r}}(s)\Pi_{\alpha \in \Delta^+ \cap w'(\Delta^-)} \chi_{w_{i_r} \cdot \alpha}(\pm s'_\alpha)]w'B \,,$$

and the expression in brackets is exactly of the desired form, so the induction is complete. Finally we must prove the uniqueness assertion of the lemma. Let

$$
\begin{aligned}
u_1 &= \Pi_{\alpha \in \Delta^+ \cap w(\Delta^-)} \chi_\alpha(s_\alpha) \\
u_2 &= \Pi_{\alpha \in \Delta^+ \cap w(\Delta^-)} \chi_\alpha(s''_\alpha)
\end{aligned}
\tag{8.7}
$$

(products with respect to the special order on $\Delta^+ \cap w(\Delta^-)$), and assume

$$u_1 w = u_2 w b \,, \tag{8.8}$$

for some $b \in B$. Then

$$w^{-1}u_i w \in U^- \,,$$

for $i = 1, 2$, and so,

$$w^{-1}u_2^{-1}u_1 w \in U^- \,. \tag{8.9}$$

From (8.8), we obtain

$$w^{-1}u_2^{-1}u_1 w = b \,, \tag{8.10}$$

where, acting on a highest weight module V_k^λ (λ dominant integral), b is upper triangular, and $w^{-1}u_2^{-1}u_1 w$ is lower triangular with ones on the diagonal (with respect to a coherently ordered basis) of V_k^λ). Hence both sides of (8.10) act as the identity on V_k^λ, and so as automorphisms on V_k^λ,

$$u_2 = u_1 \,. \tag{8.11}$$

Now we consider

$$w_{i_r}^{-1}u_1 w_{i_r} \cdot v_\lambda = w_{i_r}^{-1}\chi_{\alpha_{i_r}}(s_{\alpha_{i_r}})w_{i_r} \cdot v_\lambda = v_\lambda + s_{\alpha_{i_r}} \cdot v_{\lambda - \alpha_{i_r}} + \cdots$$

where $v_{\lambda - \alpha_{i_r}} \neq 0$ is a vector in the weight space $V_{\lambda - \alpha_{i_r}}^\lambda$ and similarly

$$w_{i_r}^{-1}u_2 w_{i_r} \cdot v_\lambda = v_\lambda + s''_{\alpha_{i_r}} \cdot v_{\lambda - \alpha_{i_r}} + \cdots \,,$$

and hence by (8.11)

$$s_{\alpha_{i_r}} = s''_{\alpha_{i_r}} \,.$$

To see that $v_{\lambda-\alpha_{i_r}} \neq 0$, we consider $w_{i_r} \cdot v_\lambda$, which equals

$$\pm \left(\frac{f_{i_r}^n}{n!} \right) \cdot v_\lambda \,, \quad n = \lambda(\alpha_{i_r}^\vee) \,,$$

thanks to [14], and the argument in the proof of [14, Lemma 2.9] therein. It follows that

$$f_{i_r} \cdot v_\lambda = v_{\lambda-\alpha_{i_r}} \neq 0$$

as well.

Hence from the expression (8.7), we have

$$w_{i_r}^{-1} \chi_{\alpha_{i_r}} (s_{\alpha_{i_r}})^{-1} u_1 w_{i_r}^{-1} = u_1' \,,$$

$$w_{i_r}^{-1} \chi_{\alpha_{i_r}} (s_{\alpha_{i_r}})^{-1} u_2 w_{i_r}^{-1} = u_2' \,,$$

where

$$u_1' = \Pi_{\alpha \in \Delta^+ \cap w'(\Delta^-)} \chi_\alpha (\pm s_\alpha) \,,$$

$$u_2' = \Pi_{\alpha \in \Delta^+ \cap w'(\Delta^-)} \chi_\alpha (\pm s_\alpha'') \,,$$

(where the $\pm$ signs in the two expressions match up "α by α"). By induction, $s_\alpha = s_\alpha''$, $\alpha \in \Delta^+ \cap w'(\Delta^-)$, and since $s_{\alpha_{i_r}} = s_{\alpha_{i_r}}''$, we obtain $s_\alpha = s_\alpha''$, $\alpha \in \Delta^+ \cap w(\Delta^-)$, which completes the induction, and hence the proof of uniqueness, and the proof of Lemma 8.2. □

We are now ready to consider the equality

$$|\Gamma_w| = q^{l(w)} (q-1)^l \,, \tag{8.12}$$

of Lemma 8.1. If $b^- \in B^-$, and

$$Bwb^- = Bw \,,$$

then for some $b \in B$,

$$wb^- w^{-1} = b \,.$$

Equivalently,

$$w^{-1} bw = b^- \in B^- \,.$$

Now by Lemma 8.2,

$$bwB = uwB \,,$$

for

$$u = \Pi_{\alpha \in \Delta^+ \cap w(\Delta^-)} \chi_\alpha (s_\alpha) \,,$$

$s_\alpha \in k$, (product taken with respect to the special ordering), and the s_α are uniquely determined. Hence

$$b^- = w^{-1}bw = w^{-1}uwb'\,, \tag{8.13}$$

for some $b' \in B$. Now b^- has an expression

$$b^- = u^- h\,,$$

$u^- \in U^-$, $h \in H$, and so (8.13) becomes

$$u^- h = w^{-1}uwb'\,.$$

Then

$$(u^-)^{-1}w^{-1}uw = h(b')^{-1} \in B \cap U^-\,.$$

By Remark 6.1, we have

$$B \cap B^- = H\,,$$

so

$$(u^-)^{-1}(w^{-1}uw) \in H\,,$$

and hence

$$u^- = w^{-1}uw\,.$$

There are then exactly $q^{l(w)}$ choices for u^-, and then $(q-1)^l$ choices for h. This concludes the proof of Lemma 8.1, and hence of Theorems 8.1 and 8.2. □

Let $P^- \subseteq G$ be a proper standard parabolic subgroup for the BN-pair (G, B^-, N) (thus $B^- \subseteq P^- \subsetneq G$). Assume that the submatrix of A corresponding to P^- is positive-definite. Then $B^-\backslash P^-$ is finite. Hence, if B^- is a lattice, then so is P^-. Thus we obtain

Corollary 8.1 ([7, (1.7)]). *If P^- corresponds to a positive-definite submatrix of A and $q > l$, then P^- is a non-uniform lattice in G.*

9. Bruhat–Tits Tree in Rank 2

We now take $l = 2$ and we assume that the Cartan matrix A is symmetric and given by

$$A = \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix} \tag{9.1}$$

for $m \geq 2$. We let (G, B, N) be the BN-pair as in Theorem 6.1. We have fixed subgroups $B \subseteq G$, $N \subseteq G$, and the Weyl group $W = N/H$, where $H = N \cap B$ is a normal subgroup of N. We let $S = \{w_1, w_2\}$ be the generators of W corresponding to simple roots α_1 and α_2 respectively. Then W is the infinite dihedral group

$$W = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z} \rtimes \{\pm 1\}\,.$$

If we consider the Bruhat cells Bw_1B and Bw_2B from (6.13), we obtain subgroups

$$P_1 := B \sqcup Bw_1B \,,$$
$$P_2 := B \sqcup Bw_2B$$

(9.2)

of G, which will be called *standard parabolic subgroups*. The subgroup $B \subseteq G$ will be called a *minimal parabolic subgroup*.

The Bruhat–Tits building of (G, B, N) is a simplicial complex X of dimension

$$\dim(X) = |S| - 1 = 1 \,.$$

Since W is infinite, by the Solomon–Tits theorem (see [12, Theorem 2.2]) X is contractible, and so X is a tree. The vertices of X are the conjugates of P_1 and P_2 in G. If Q_1 and Q_2 are vertices, then there is an edge connecting Q_1 and Q_2 if and only if $Q_1 \cap Q_2$ contains a conjugate of B. We have an action of G on X by conjugation. The vertex set of X will be denoted VX and the set of oriented edges by EX. We will make use of the following basic facts:

(1) P_1 and P_2 are not conjugate in G.
(2) Each P_i is its own normalizer in G, $i = 1, 2$, and B is its own normalizer in G.

Let $g \in G$. Then (1) and (2) imply that the maps:

$$f_0 : gB \longrightarrow gBg^{-1} \,,$$
$$f_1 : gP_1 \longrightarrow gP_1g^{-1} \,,$$
$$f_2 : gP_2 \longrightarrow gP_2g^{-1}$$

are bijections. Thus

$$G/P_1 \cong \text{all conjugates of } P_1 \text{ in } G \,,$$
$$G/P_2 \cong \text{all conjugates of } P_2 \text{ in } G \,,$$
$$G/B \cong \text{all conjugates of } B \text{ in } G \,.$$

Conjugates of P_1, P_2 will be called *parabolic subgroups*. (In the affine case, these are often called *parahoric subgroups*.) Applying the fundamental theory of Bass–Serre [28], we obtain

$$G = P_1 *_B P_2 \,.$$

We obtain the following description of the Bruhat–Tits tree X. We have bijective correspondences

$$VX \cong G/P_1 \sqcup G/P_2$$
$$EX \cong G/B \sqcup \overline{G/B} \,,$$

where $\overline{G/B}$ is a copy of the set G/B, giving an orientation to EX so that positively oriented edges come from G/B, and negatively oriented edges come from $\overline{G/B}$.

There is a *standard simplex* $\mathcal{C}_0$ corresponding to the identity coset of B. Let P be a parabolic subgroup. The corresponding simplex (vertex or edge) will be denoted σ_P. We may at times identify a simplex with it's corresponding parabolic subgroup.

The group G acts by left multiplication on cosets. There are natural projections on cosets induced by the inclusion of B in P_1 and P_2:

$$\pi : G/B \longrightarrow G/P_i\,, \quad i = 1, 2\,.$$

If $v_i \in G/P_i$ is a vertex, and $St^X(v_i) = \pi^{-1}(v_i)$ is the set of edges with origin v_i, then we may index $St^X(v_i)$ by $P_i/B \subseteq G/B$, $i = 1, 2$ (however, the choice of indexing is not unique). The Bruhat–Tits tree X is a *homogeneous*, bipartite tree of degree

$$[P_1 : B] = [P_2 : B] = |k| + 1\,.$$

Adapting the discussion in the previous section to the case $l = 2$, the following describes how the cosets Bw_1B and Bw_2B are indexed modulo B:

Lemma 9.1.

$$Bw_1B/B = \{\chi_{\alpha_1}(s)w_1B/B | s \in k\}\,,$$

$$Bw_2B/B = \{\chi_{\alpha_2}(s)w_2B/B | s \in k\}\,,$$

where α_1 and α_2 are the simple roots corresponding to w_1 and w_2 respectively.

It follows that the edges emanating from P_1 and P_2 may be indexed as follows:

$$St^X(P_1) = \{B\} \cup \{\chi_{\alpha_1}(s)w_1B/B | s \in k\}\,,$$

$$St^X(P_2) = \{B\} \cup \{\chi_{\alpha_2}(s)w_2B/B | s \in k\}\,,$$

where B denotes the identity coset, and the stars of other vertices are obtained by translating (conjugating) these.

Apartments in X are infinite lines. The *standard apartment*, denoted by $\mathcal{A}_0$, in X consists of all Weyl group translates of the standard simplex. That is

$$V\mathcal{A}_0 = W \cdot \sigma_{P_1} \cup W \cdot \sigma_{P_2}$$

$$E\mathcal{A}_0 = W \cdot \sigma_B\,.$$

It is worthwhile to picture $\mathcal{A}_0$. In the diagram below, it is to be understood that all products

$$w'w_i\,, \quad w' \in W\,, \quad i = 1, 2\,,$$

have the property that

$$l(w'w_i) = l(w') + 1\,.$$

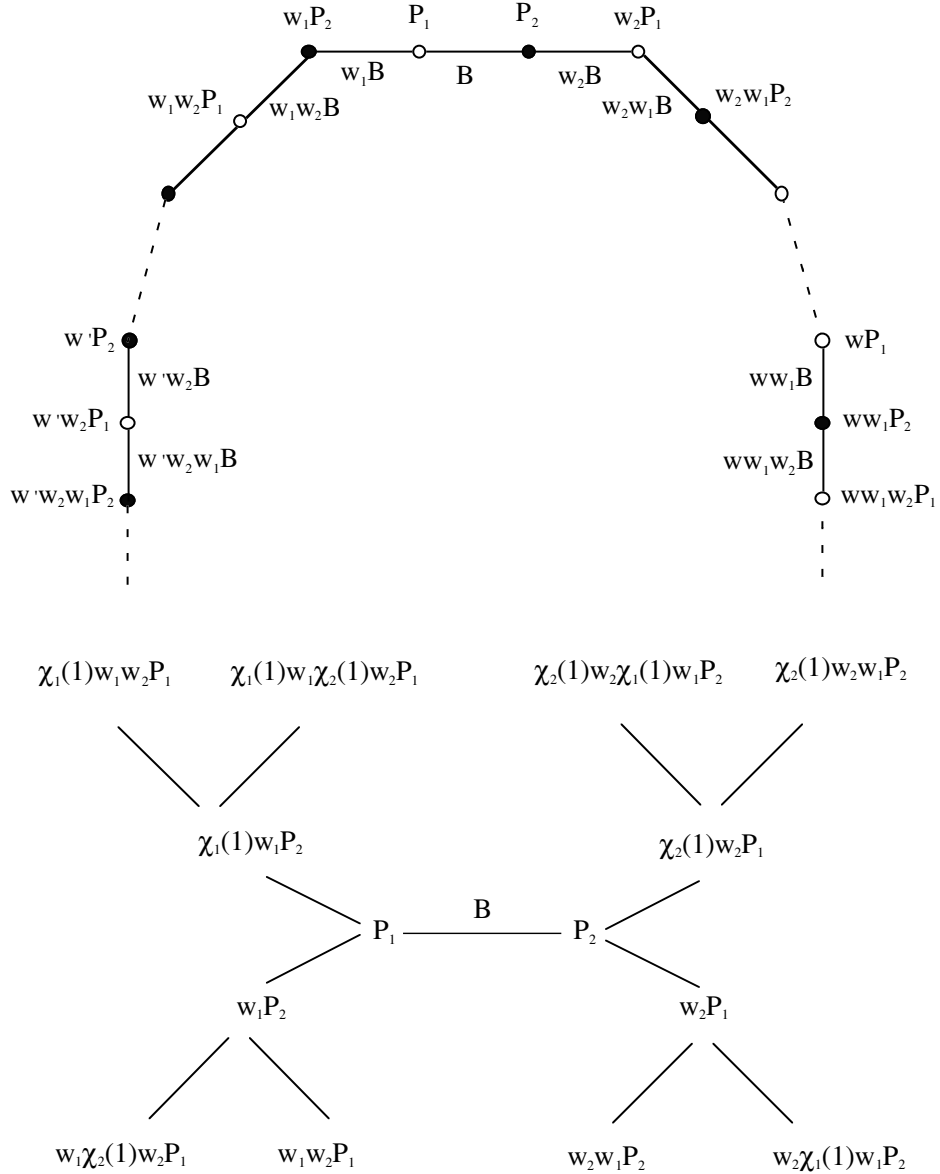We then have the following diagram of the apartment $\mathcal{A}_0$:

Fig. 9.1.   The tree of a field with 2 elements.

Since the subgroups P_1 and P_2 are not conjugate in G, the group G acts without inversions on X. Also, G acts transitively on edges of X, and has 2 orbits for the vertices, corresponding to vertices that come from P_1 and from P_2. Figure 9.1 shows the tree of the field with 2 elements.

The group

$$G \Big/ \left(\bigcap_{g \in G} gBg^{-1} \right)$$

acts faithfully on X, and the defining homomorphism

$$\rho : G \Big/ \left(\bigcap_{g \in G} gBg^{-1} \right) \hookrightarrow \mathrm{Aut}(X)$$

is continuous and the image is closed. The quotient topology on $G/(\bigcap_{g \in G} gBg^{-1})$ coincides with that induced by the natural topology on $\mathrm{Aut}(X)$, where the stabilizers of finite sets of vertices form a fundamental system of compact open neighborhoods of the identity. The group B of the Tits system is the stabilizer of the standard simplex $\mathcal{C}_0$, and N is contained in the stabilizer of the standard apartment $\mathcal{A}_0$.

10. Schottky Groups of Tree Automorphisms

In [18], Lubotzky adapted the classical notion of a Schottky group of automorphisms of the upper half plane to the non-archimedean case. We describe Lubotzky's Schottky groups of tree automorphisms in this section.

Let X be any locally finite tree. Then $\mathrm{Aut}(X)$ contains three possible types of automorphisms. Let $g \in \mathrm{Aut}(X)$. Then [32] g is either:

(1) an inversion, (g^2 fixes a vertex, but g fixes none),
(2) elliptic (g fixes a vertex),
(3) hyperbolic (both g and g^2 fix no vertex).

Lubotzky constructed uniform lattices out of hyperbolic tree automorphisms. We have the following:

Proposition 10.1 ([32]). *Let X be a tree and let $g \in \mathrm{Aut}(X)$ be hyperbolic. Let $m = \min_{x \in VX} d(x, gx)$. Let $L = \{x \in VX | d(x, gx) = m\}$. Then L is the vertex set of a bi-infinite path in X, and g acts as a translation of length (amplitude) m along L (L is the "axis" of g).*

Lubotzky has given the following criterion for recognizing hyperbolic elements:

Lemma 10.1 ([18, Lemma 1.2]). *Let $x, y \in VX$ be such that $d(x, y) = 1$ and suppose that there exists $\gamma \in \mathrm{Aut}(X)$ with*

$$d(x, \gamma x) = d(y, \gamma y) > 0 \,.$$

Then γ is hyperbolic and x and y lie on the axis of γ.

Let γ be hyperbolic of amplitude $m = m(\gamma)$ and with axis $L = L(\gamma)$. We may choose a labelling of the vertices of $L(\gamma)$ as $\{x_n, \ n \in \mathbb{Z}\}$ in such a way that

$d(x_n, x_{n+1}) = 1$, and $\gamma(x_n) = x_{n+m}$. We choose such a labelling and set

$$A(\gamma) = \{x \in VX | d(x, x_0) < d(x, x_1)\},$$
$$B(\gamma) = \{x \in VX | d(x, x_{m+1}) < d(x, x_m)\}. \tag{10.1}$$

Let $\gamma_1, \ldots, \gamma_s$ be hyperbolic elements in $\mathrm{Aut}(X)$. Assume that the axes $L(\gamma_1), \ldots, L(\gamma_s)$ can be labelled in such a way that the sets $A(\gamma_i)$, $B(\gamma_i)$, $i = 1, \ldots, s$ are mutually disjoint. Let Γ be the group generated by $\{\gamma_1, \ldots, \gamma_s\}$. Then $\{\gamma_1, \ldots, \gamma_s\}$ is called a *Schottky basis* for Γ. We also call Γ a *Schottky group*. We have the following:

Theorem 10.1 ([18, Proposition 1.6]). *Let Γ be generated by a Schottky basis $\gamma_1, \ldots, \gamma_s$. Then*

(1) *Γ is discrete and every element $\neq 1$ of Γ is hyperbolic.*
(2) *Γ is a free group and $\gamma_1, \ldots, \gamma_s$ are free generators of Γ.*
(3) *$F = X - \left(\bigcup_{i=1}^{s}(A(\gamma_i) \cup B(\gamma_i))\right)$ is a fundamental domain for Γ on X.*

Remark 10.1. (1) (1) in the theorem implies that Γ acts freely on X since no element $\neq 1$ of Γ fixes a vertex.

(2) A Schottky group Γ may be non-uniform, but if Γ is a lattice, it is necessarily uniform.

Let $F = \langle s_1, \ldots, s_r \rangle$ be free of rank r. Let X be a locally finite tree. Let $\rho : F \longrightarrow \mathrm{Aut}(X)$ be a representation. We say that $\rho(F)$ admits a *Schottky basis in* $\mathrm{Aut}(X)$ if ρ is faithful, and $\rho(s_i) = \gamma_i$, where $\langle \gamma_1, \ldots, \gamma_r \rangle$ is a Schottky group. We also call ρ a *Schottky representation*.

Theorem 10.2 ([18, Proposition 1.7]). *Let X be a locally finite tree. Let Γ be any torsion free discrete subgroup of $\mathrm{Aut}(X)$. Then Γ admits a Schottky basis in $\mathrm{Aut}(X)$.*

Proof. Since Γ is torsion free, Γ contains no elements of finite order, so Γ must act without inversions. Let $p : X \longrightarrow \Gamma\backslash X$ be the quotient morphism. Then Γ is discrete if and only if each vertex stabilizer Γ_x is finite, for $x \in VX$. Hence Γ discrete and torsion free implies that

$$\Gamma_x = \{1\}, \quad \text{for all } x \in VX.$$

Thus Γ acts freely on X, and so $\Gamma \cong \pi_1(\Gamma\backslash X)$, a free group, say Γ is free of rank $r \leq \infty$. To find a Schottky basis for Γ in $\mathrm{Aut}(X)$, we form an *opening* (as in [28, Proposition 14]) of $\Gamma\backslash X$ in X as follows: since the quotient morphism $p : X \longrightarrow \Gamma\backslash X$ is locally surjective (surjective on stars of vertices), there exist connected subgraphs $T \subseteq S \subseteq X$ such that:

(i) T is a tree,
(ii) $p_T : T \longrightarrow \Gamma\backslash X$ is bijective on vertices and $p_S : S \longrightarrow \Gamma\backslash X$ is bijective on edges.

(iii) For each $e \in ES$, either $\partial_0 e$ or $\partial_1 e$ belongs to T.

(iv) $|VS - VT| = r = \operatorname{rank}(\Gamma)$.

Denote the inverses of the bijections $p_T : VT \longrightarrow V(\Gamma \backslash X)$ and $p_S : ES \longrightarrow E(\Gamma \backslash X)$ by

$$a \mapsto a^T,$$

$$e \mapsto e^S,$$

respectively.

For each $x \in VS$, choose $g_x \in \Gamma$ so that $g_x x \in VT$; that is, $g_x x = p(x)^T$, and so that $g_x = 1$, if $x \in VT$. Then Γ is free (of rank r) on

$$\{g_x | x \in VS - VT\}.$$

We claim that Γ is a Schottky group. For each $x \in VS - VT$, $g_x \in \Gamma$ which acts freely on X, hence without fixed points, so each g_x is hyperbolic. Let $m_x = m(g_x)$ be the amplitude of g_x and let $L_x = L(g_x)$ be the axis of g_x. Choose a labelling of the vertices of L_x as in (10.1), and define sets $A(g_x)$ and $B(g_x)$, as in (10.1). Since T is a fundamental domain for vertices and S is a fundamental domain for edges, it is a routine check that the subsets $A(g_x)$ and $B(g_x)$ are all disjoint, for $x \in VS - VT$. $\qquad\square$

Let $\Gamma \langle s_1, \ldots, s_r \rangle$ be free of rank r. Let X be a locally finite tree, and let $\operatorname{Aut}(X)$. Theorem 10.2 gives rise to the following discussion of representation spaces of Γ in G. Since Γ is free,

$$\operatorname{Hom}(\Gamma, \operatorname{Aut}(X)) \cong \operatorname{Aut}(X) \times \cdots \times \operatorname{Aut}(X)$$

(r factors). Let $\rho_0 : \Gamma \longrightarrow \operatorname{Aut}(X)$ be a faithful discrete representation. Hence $\rho_0(\Gamma)$ acts freely on X. Choose $T \subseteq S \subseteq X$ such that $p_T : T \longrightarrow \Gamma \backslash X$ is bijective on vertices and $p_S : S \longrightarrow \Gamma \backslash X$ is bijective on edges. For each $x \in VS - VT$, choose $g_x \in \rho_0(\Gamma)$ such that $g_x x = p(x)^T$. Then $|VS - VT| = r$, and $\{g_{x_1}, \ldots, g_{x_r}\}$ generates $\rho_0(\Gamma)$ freely, and is a Schottky basis for $\rho_0(\Gamma)$. Let $L_i = L(g_i^{-1})$ be the axis of g_i^{-1}. Let $f_i \in L_i$ and let $g_i^{-1}(f_i) = e_i$. Let

$$\operatorname{Hom}_0(\Gamma, \operatorname{Aut}(X)) = \{\phi \in \operatorname{Hom}(\Gamma, \operatorname{Aut}(X)) | \phi \text{ is faithful}$$
$$\text{and } \phi(s_i)(e_i) = f_i, \ i = 1, \ldots, r\},$$

and let

$$\operatorname{Hom}_1(\Gamma, \operatorname{Aut}(X)) = \{\phi \in \operatorname{Hom}(\Gamma, \operatorname{Aut}(X)) | \phi \text{ is faithful}$$
$$\text{and } \phi(s_i) \in B_i g_i, \ i = 1, \ldots, r\},$$

where $B_i = \operatorname{Stab}_{\operatorname{Aut}(X)}(f_i)$.

Proposition 10.2. $\operatorname{Hom}_1(\Gamma, \operatorname{Aut}(X)) = \operatorname{Hom}_0(\Gamma, \operatorname{Aut}(X))$

Proof. Let $\psi \in \text{Hom}_1(\Gamma, \text{Aut}(X))$. Thus

$$\psi(s_i)(e_i) = b_i g_i(e_i), \ b_i \in B_i$$

$$= b_i f_i$$

$$= f_i \,,$$

and so $\psi \in \text{Hom}_0(\Gamma, \text{Aut}(X))$.

Conversely, let $\psi \in \text{Hom}_0(\Gamma, \text{Aut}(X))$. Then $\psi(s_i)^{-1}(f_i) = e_i$, but $g_i^{-1}(f_i) = e_i$, so

$$g_i \psi(s_i)^{-1}(f_i) = f_i \,,$$

and so $g_i \psi(s_i)^{-1} \in B_i = \text{Stab}_{\text{Aut}(X)}(f_i)$. Therefore

$$\psi(s_i) = b_i g_i, \ \text{some } b_i \in B_i \,,$$

and so $\psi \in \text{Hom}_1(\Gamma, \text{Aut}(X))$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 10.3. $\text{Hom}_0(\Gamma, \text{Aut}(X))$ *is open in* $\text{Hom}(\Gamma, \text{Aut}(X))$. *In particular,* $\text{Hom}_0(\Gamma, \text{Aut}(X))$ *contains all representations in an open neighbourhood of the inclusion map* $\rho_0 : \Gamma \longrightarrow \text{Aut}(X)$.

Proof. Let $\psi : \Gamma \longrightarrow \text{Aut}(X)$ be a faithful representation in an open neighbourhood of the inclusion map $\rho_0 : \Gamma \longrightarrow \text{Aut}(X)$. Using the topology on $\text{Aut}(X)$ [3, Chapt. 3], we see that ψ and ρ_0 agree on the axis of $\rho_0(s_i)$, $i = 1, \ldots, r$. Thus $\psi \in \text{Hom}_0(\Gamma, \text{Aut}(X))$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Thus $\text{Hom}_0(\Gamma, \text{Aut}(X)) = \text{Hom}_1(\Gamma, \text{Aut}(X))$ is open in $\text{Hom}(\Gamma, \text{Aut}(X))$ and consists of Schottky representations of Γ in $\text{Aut}(X)$ with the same fundamental domain as Γ.

## 11. Construction of Uniform Lattices in Rank 2

Let X be a bihomogeneous, bipartite tree. Let $G \leq \text{Aut}(X)$ be a closed subgroup with a Tits system (G, B, N). Examples of such subgroups are:

(a) $G = K$-rank 1 simple algebraic group over a non-archimedean local field K.
(b) $G = $ group of a rank 2 Kac–Moody algebra over a finite field k as in Sec. 6.

Theorem 11.1 ([18, (2.1)]). *Let X be any locally finite bihomogeneous bipartite tree. Let $G \leq \text{Aut}(X)$ be an edge-transitive closed subgroup. Then G contains hyperbolic elements. Moreover, there is a uniform Schottky group $\Gamma \leq G$.*

Proof. Lubotzky's stratgey is to first construct a Schottky group in $\text{Aut}(X)$ with finite fundamental domain, and then to check that such a construction can be made within G.

Choose an edge $e \in EX$, the "standard simplex", and label $\partial_0 e = P_1$ and $\partial_1 e = P_2$. Let $P_2^1, \ldots P_2^{r_2}$, be the neighbors of P_1 other than P_2. For every $i = 1, \ldots, r_2$, let $P_1^{i1}, \ldots, P_1^{ir_1}$ be the neighbors of P_2^i, other than P_1. Similarly, Let $P_1^1 \cdots P_1^{r_1}$ be the neighbors of P_2 other than P_1. For every $i = 1, \ldots, r_1$, let $P_2^{i1}, \ldots, P_2^{ir_2}$ be the neighbors of P_1^i, other than P_2:
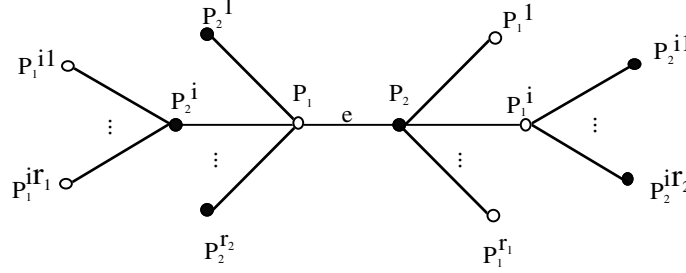
Then P_1 has $r_2 \cdot r_1$ "grandchildren" $\{P_1^{ij}|i = 1, \ldots, r_2, \ j = 1, \ldots, r_1\}$, and P_2 has $r_1 \cdot r_2$ "grandchildren" $\{P_2^{ij}|i = 1, \ldots, r_1, \ j = 1, \ldots, r_2\}$. Choose a one to one correspondence between them, say P_1^{ij} corresponds to P_2^{ji}. Let γ_{ij} be an automorphism of X which satisfies:

$$\gamma_{ij}(P_1^{ij}) = P_1^j \quad \text{and} \quad \gamma_{ij}(P_2^i) = P_2^{ji}.$$

Since $d(P_1^{ij}, P_1^j) = d(P_2^i, P_2^{ji}) = 4$, Lemma 10.1 implies that γ_{ij} is hyperbolic, the axis of γ_{ij} passes through

$$P_1^{ij}, P_2^i, P_1, P_2, P_1^j, P_2^{ji},$$

and the amplitude of γ_{ij} is $m(\gamma_{ij}) = 4$. By labelling the axis $L(\gamma_{ij})$ of γ_{ij} such that $x_0 = P_1^{ij}$, we can see easily that the $2r_2r_1$ sets $A(\gamma_{ij})$ and $B(\gamma_{ij})$ are all disjoint.

By Theorem 10.1 the group Γ generated by $\{\gamma_{ij}|i = 1, \ldots, r_2, \ j = 1, \ldots, r_1\}$ is therefore a free discrete group on r_2r_1 generators, and a fundamental domain for Γ is the set

$$\{P_1, P_2, P_1^j, P_2^i|i = 1, \ldots, r_2, \ j = 1, \ldots, r_1\}$$

with $r_2 + r_1 + 2$ vertices.

Since G acts transitively on the edges of X, *the elements γ_{ij} can be found inside the group G.* This shows that G contains hyperbolic elements. Therefore, there exists $\gamma_{ij} \in G$ which takes the edge $\{P_1^{ij}, P_2^i\}$ to the edge $\{P_1^j, P_2^{ji}\}$. The element $\gamma_{ij} \in G$ necessarily takes P_1^{ij} to P_1^j since X is bihomogeneous and bipartite, and both vertices are of the same type, and similarly, $\gamma_{ij} \in G$ takes P_2^i to P_2^{ji} since both vertices are of the same type. The group Γ is a (free) uniform Schottky subgroup of G. $\qquad\square$

Corollary 11.1. *Let G be a K-rank 1 simple algebraic group over a non-archimedean local field K or a group of a rank 2 Kac–Moody algebra over a finite field k. Then there is a (free) uniform Schottky group $\Gamma \leq G$.*

12. Deformations of Uniform Lattices in Rank 2

Using the notation in the previous section, and under the hypotheses of Theorem 11.1, let $\Gamma = \langle \gamma_{ij} | i = 1, \ldots, r_1, \; j = 1, \ldots, r_2 \rangle$ be free of rank $r = r_1 r_2$. We apply Theorem 11.1 to the Kac–Moody group of rank 2, as in Sec. 9. For our Kac–Moody group G, we have $r_1 = |k| = r_2$. Let ρ_0 be a faithful representation:

$$\rho_0 : \Gamma \longrightarrow G$$

$$\gamma_{ij} \mapsto g_{ij} \,.$$

We set

$$\mathrm{Hom}_0(\Gamma, G) := \mathrm{Hom}_0(\Gamma, \mathrm{Aut}(X)) \cap \mathrm{Hom}(\Gamma, G) \,.$$

Then $\mathrm{Hom}_0(\Gamma, G)$ is an open neighborhood of ρ_0 in $\mathrm{Hom}(\Gamma, G)$ consisting of Schottky representations of Γ with the same fundamental domain as Γ.

Relative to the fundamental domain F for Γ as in the proof of Theorem 11.1, and relative to the coordinate system for X as in Fig. 9.1, we can determine the structure of $\mathrm{Hom}_0(\Gamma, G)$. Choose the distinguished edge e in the proof of Theorem 11.1 to be the standard simplex. Using the notation in the proof of Theorem 11.1, let

$$G_{ij} = \{ \gamma \in G | \gamma(P_1^{ij}, P_2^i) = (P_1^j, P_2^{ji}) \} \,.$$

Then relative to the fundamental domain F for Γ as in the proof of Theorem 11.1, and relative to the coordinate system for X as in Fig. 9.1, for G, we have:

$$G_{ij} = \chi_2(s_j) w_2 \chi_1(s_i) w_1 B w_2^{-1} \chi_2(s_j)^{-1} w_1^{-1} \chi_1(s_i)^{-1} \,, \tag{12.1}$$

where $s_i, \, s_j \in k$, B is group of the (B, N) pair as in Theorem 6.1, and w_1, w_2 are Weyl group generators. Moreover, relative to the fundamental domain F for Γ as in the proof of Theorem 11.1, we have

$$\mathrm{Hom}_0(\Gamma, G) = \prod_{i=1,\ldots,r_1, j=1,\ldots,r_2} G_{ij} \,, \tag{12.2}$$

where $r_1 = |k| = r_2$. Thus any choice of

$$(b_1, \ldots, b_r) \in B \times \cdots \times B \,,$$

($r = |k|^2$ factors) gives rise to generating set for a free uniform lattice of rank $r = |k|^2$ in G with the same fundamental domain as Γ. We have

Conjecture 12.1.

$$\mathrm{Hom}_0(\Gamma, G)/\mathrm{conjugation}$$

is uncountable.

Conjecture 12.1 will follow from the following:

Conjecture 12.2. *The orbits for the conjugation action on* $\mathrm{Hom}_0(\Gamma, G)$ *are nowhere dense.*

(cf. [7, Conjecture 3.3]). The following result shows that the uniform Schottky groups we have constructed are all conjugate in $\mathrm{Aut}(X)$.

Proposition 12.1. *Let X be a locally finite tree, let $\rho_0 : \Gamma \longrightarrow \mathrm{Aut}(X)$ and $\rho_1 : \Gamma \longrightarrow \mathrm{Aut}(X)$ be faithful free representations of a group Γ in $\mathrm{Aut}(X)$ (that is, $\rho_0(\Gamma)$ and $\rho_1(\Gamma)$ act freely on X). If there is an isomorphism of graphs*

$$h : A_0 = \rho_0(\Gamma)\backslash X \xrightarrow{\cong} A_1 = \rho_1(\Gamma)\backslash X \,,$$

then $\rho_0(\Gamma)$ and $\rho_1(\Gamma)$ are conjugate in $\mathrm{Aut}(X)$.

Proof. Since $\rho_0(\Gamma)$ and $\rho_1(\Gamma)$ act freely on X, we have

$$\rho_0(\Gamma) = \pi_1(A_0, *)\,, \quad \rho_1(\Gamma) = \pi_1(A_1, *)\,,$$

both free groups. Since

$$h : A_0 \xrightarrow{\cong} A_1 \,,$$

there is a lift

$$f = \tilde{h} : X \xrightarrow{\cong} X$$

with the property

$$f(\rho_0(\gamma)x) = \rho_1(\gamma)f(x)$$

for $x \in X$, that is

$$\rho_0(\cdot) = f^{-1} \circ \rho_1(\cdot) \circ f$$

for $f \in \mathrm{Aut}(X)$. $\qquad\square$

Corollary 12.1. *Let Γ_1 and $\Gamma_2 \leq G$ be uniform Schottky subgroups with the same fundamental domain. Let X be the Bruhat–Tits tree of G. Then Γ_1 and Γ_2 are conjugate in $\mathrm{Aut}(X)$.*

13. Uniform Commensurability in Rank 2

Let X be a locally finite tree. A group Γ is a uniform lattice in $\mathrm{Aut}(X)$ if and only if Γ is finitely generated and virtually free [1, Theorem 8.4]. Furthermore, results of Bass–Kulkarni and Liu give us certain *commensurability* theorems for uniform lattices our in rank 2 Kac–Moody group. We have the following

Theorem 13.1 ([3, Corollary 4.8]). *Let X be a tree that covers a finite connected graph. If Γ_1, Γ_2 are uniform lattices in $\mathrm{Aut}(X)$, then $g\Gamma_1 g^{-1}$ and Γ_2 are commensurable for some $g \in \mathrm{Aut}(X)$.*

Let Γ be a uniform lattice in $\mathrm{Aut}(X)$. Let

$$C_{\mathrm{Aut}(X)}(\Gamma) = \{g \in \mathrm{Aut}(X) | g\Gamma g^{-1} \text{ and } \Gamma \text{ are commensurable}\}$$

be the *commensurability group* of Γ in $\mathrm{Aut}(X)$. A corollary to Theorem 13.1 is the following:

Corollary 13.1 ([3]). *If Γ_1, Γ_2 are uniform lattices in $\mathrm{Aut}(X)$, then their commensurability groups $C_{\mathrm{Aut}(X)}(\Gamma_1)$ and $C_{\mathrm{Aut}(X)}(\Gamma_2)$ are conjugate in $\mathrm{Aut}(X)$.*

We also have the following:

Theorem 13.2 ([19]). *For any uniform lattice Γ, in $\mathrm{Aut}(X)$, $C_{\mathrm{Aut}(X)}(\Gamma)$ is dense in $\mathrm{Aut}(X)$.*

Let $G_0 = G_0(\mathbb{F}_p)$ be the group of a rank 2 Kac–Moody algebra over the finite field $\mathbb{F}_p$. Let $G_1 = SL_2(\mathbb{Q}_p)$, $G_2 = SL_2(\mathbb{F}_p((t^{-1})))$. Then G_0, G_1, and G_2 have Bruhat–Tits tree X_{p+1}. Theorem 13.1 implies:

(1) Any two uniform lattices in the *same* group (G_0, G_1, or G_2) are commensurable to each other (up to conjugation in $\mathrm{Aut}(X_{p+1})$).
(2) Any two uniform lattices in *different* groups (G_0, G_1, or G_2) are commensurable to each other (up to conjugation in $\mathrm{Aut}(X_{p+1})$).
(3) Any uniform lattice in G_0, G_1, or G_2 is commensurable to a uniform lattice in $\mathrm{Aut}(X_{p+1})$ (up to conjugation in $\mathrm{Aut}(X_{p+1})$).

Since any uniform lattice in G_0, G_1, or G_2 is a uniform lattice in $\mathrm{Aut}(X_{p+1})$, we have:

(5) If Γ_1 and Γ_2 are uniform lattices in G_0, G_1, or G_2, then $C_{\mathrm{Aut}(X_{p+1})}(\Gamma_1)$ and $C_{\mathrm{Aut}(X_{p+1})}(\Gamma_2)$ are conjugate in $\mathrm{Aut}(X_{p+1})$.

Theorem (13.3) implies

(6) For any uniform lattice Γ in G_0, G_1, or G_2, $C_{\mathrm{Aut}(X_{p+1})}(\Gamma)$ is dense in $\mathrm{Aut}(X_{p+1})$.

We pose the following

Question 13.1. *Let Γ be a uniform lattice in G_0. Is $C_{G_0}(\Gamma)$ dense in G_0?*

14. A Spherical BN-Pair in Rank 2

Using the notation of Sec. 9, we let $\mathcal{A}$ be any apartment in X. Also fix a "base edge" $\sigma(\mathcal{A})$ of $\mathcal{A}$, and $g \in G$, such that

$$g \cdot \sigma(\mathcal{A}) = \sigma_B \tag{14.1}$$

(so $\sigma(\mathcal{A}) = \sigma_{g^{-1}Bg}$). Replacing $\mathcal{A}$ by $g \cdot \mathcal{A}$, we may in fact *assume*

$$\sigma(\mathcal{A}) = \sigma_B. \tag{14.2}$$

For an apartment $\mathcal{A}$ with base edge $\sigma(\mathcal{A})$, let $\sigma_1(\mathcal{A})$, $\sigma_2(\mathcal{A})$ be the two vertices of $\sigma(\mathcal{A})$. Interchanging $\sigma_1(\mathcal{A})$, $\sigma_2(\mathcal{A})$ if necessary, we may assume that for some $g \in G$

(we define $g(H) = gHg^{-1}$, for a subset $H \subseteq G$)

$$\sigma_1(\mathcal{A}) = \sigma_{g(P_1)},$$

$$\sigma_2(\mathcal{A}) = \sigma_{g(P_2)},$$

$$\sigma(\mathcal{A}) = \sigma_{g(B)}.$$

We let $\mathcal{A}^+$ be the subcomplex of $\mathcal{A}$ consisting of the ray joined to $\sigma(\mathcal{A})$ at $\sigma_2(\mathcal{A})$, and not containing $\sigma(\mathcal{A})$. We let $\mathcal{A}^-$ be the subcomplex of $\mathcal{A}$ consisting of $\mathcal{A} - (\mathcal{A}^+ \cup \sigma(\mathcal{A}))$. For the standard apartment $\mathcal{A}_0$ we take $\sigma(\mathcal{A}_0) = \sigma_B$, and then have $\mathcal{A}_0^{\pm}$ defined. Hence

$$\mathcal{A} = \mathcal{A}^+ \sqcup \mathcal{A}^- \sqcup \sigma(\mathcal{A}).$$

Assume $(\mathcal{A}, \sigma(\mathcal{A}))$ satisfies (14.2), so $\sigma_1(\mathcal{A}) = \sigma_{P_1}$, $\sigma_2(\mathcal{A}) = \sigma_{P_2}$. Then we have
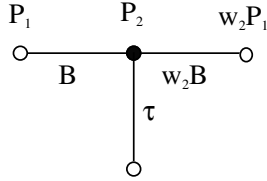
Lemma 14.1. *We can find elements b_1, $b_2 \in U$ such that*

$$b_1 \cdot \mathcal{A}^+ = \mathcal{A}_0^+, \tag{14.3}$$

$$b_2 b_1 \cdot \mathcal{A}^- = \mathcal{A}_0^-, \tag{14.4}$$

and furthermore b_2 stabilizes $\mathcal{A}_0^+$ pointwise.

Proof. For (14.3), we first consider:

where τ is the (unique) edge of $\mathcal{A}^+$ with σ_{P_2} as vertex. Moreover, τ and $w_2 \cdot \sigma_B$ are two edges with σ_{P_2} as a vertex, and both distinct from σ_B. We may describe the set of all such edges as follows:

$$\{u \cdot w_2 \cdot \sigma_B\}_{u \in U_{\alpha_2}},$$

where $U_{\alpha_2} = \{\chi_{\alpha_2}(s)\}_{s \in k}$ (α_2 being the simple root corresponding to w_2). Hence

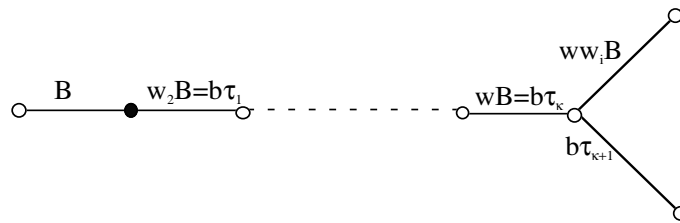$$\tau = u w_2 \cdot \sigma_B = \sigma_{u w_2 B w_2^{-1} u^{-1}},$$

for some $u \in U_{\alpha_2}$ and, since u stabilizes σ_B ($u \in B$), we may assume, replacing $\mathcal{A}^+$ by $u^{-1}\mathcal{A}^+$, that $\tau = w_2 \cdot \sigma_B$. Let

$$(\tau =)\tau_1, \tau_2 \cdots$$

be the edges of $\mathcal{A}^+$, going out to ∞. Assume inductively, that for some $b \in U \subseteq B$, we have

$$b \cdot \tau_1 = w_2 \cdot \sigma_B \,,$$

$$b \cdot \tau_2 = w_2 w_1 \cdot \sigma_B \,,$$

$$\vdots$$

$$b \cdot \tau_k = w \cdot \sigma_B, \ w = w_2 w_1 \cdots \quad (k \text{ factors}), \ l(w) = k \,.$$

(We have begun the induction, by proving such a b exists for τ_1.) We have
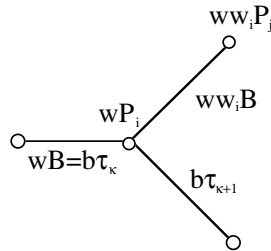

$$l(ww_i) = 1 + l(w) \,,$$

where $i = 1$ or 2. Now

$$w = w_2 w_1 \cdots w_j, \quad j \neq i, \ (\text{reduced expression}) \,,$$

(k factors). We have, filling in the vertices at the right of the above diagram,

The edges other than $w \cdot \sigma_B$, having $w \cdot \sigma_{P_i}$ as a vertex are just the w-transforms of edges other than σ_B, having σ_{P_i} as a vertex, and this is just the set:

$$w \cdot u \cdot w_i \cdot \sigma_B \,, \quad u \in U_{\alpha_i} \,,$$

which is the set of all

$$wuw^{-1} \cdot ww_i \sigma_B \,.$$

We claim that

$$wuw^{-1} \in U \subseteq B \,.$$

Indeed,

$$wU_{\alpha_i} w^{-1} = U_{w \cdot \alpha_i} \,,$$

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and since

$$l(ww_i) = 1 + l(w),$$

$w \cdot \alpha_i$ is positive. We have that

$$b \cdot \tau_{k+1} = uww_i \cdot \sigma_B, \quad \text{for some } u \in U_{w \cdot \alpha_i},$$

and

$$u^{-1} \cdot \sigma_B = \sigma_B,$$

$$u^{-1} \cdot w_2 \cdot \sigma_B = w_2 \cdot \sigma_B,$$

$$\vdots$$

$$u^{-1} \cdot w \cdot \sigma_B = w \cdot \sigma_B.$$

Moreover, inductively assume

$$b \in U_{\beta_1} \dots U_{\beta_k},$$

where

$$\beta_1 = \alpha_{i_1} = \alpha_2,$$

$$\beta_2 = w_{i_1} \cdot \alpha_{i_2},$$

$$\vdots$$

$$\beta_k = w_{i_1} \cdots w_{i_{k-1}} \cdot \alpha_{i_k}$$

$$= w_{i_1} \cdots w_{i_{k-1}} \cdot \alpha_j,$$

$$w = w_{i_1} \cdots w_{i_k} = w_2 w_1 \cdots w_j;$$

then

$$u^{-1}b \in \prod_{\alpha \in \Delta_+ \cap ww_i(\Delta_-)} U_\alpha.$$

In summary, we have proved that there is a sequence of elements

$$u_1, \dots, u_k, \dots \in U,$$

so that

$$u_k \cdot \tau_q = w_2 w_1 \cdots w_j \cdot \sigma_B, \quad q \leq k,$$

(q factors), with

$$u_k \in \prod_{\alpha \in \Delta_+ \cap w(\Delta_-)} U_\alpha, \quad w = w_1 w_2 \cdots w_j \text{ (reduced expression)}$$

and

$$u_{k+1} = uu_k, \quad u \in U_{w \cdot \alpha_i}, \ i \neq j.$$

In particular, the sequence u_k must then converge to a limit u, and we may take $b_1 = u$ (in (14.3)).

We now have (replacing $\mathcal{A}$ by $b_j\mathcal{A}$))

where $w' = w_1 w_2 \cdots w_1$, and we assume now (as we may) that

$$\tau_q = w_2 w_1 \cdots \sigma_B\,,$$

q factors, $q = 1, 2, \ldots$.

We now let $\tau_{-1}, \tau_{-2}, \ldots$ be the edges of $\mathcal{A}^-$ going out to ∞, and we let

$$w(r) = w_1 w_2 \cdots\,,$$

(r factors) and we manufacture a sequence

$$\bar{u}_r \in \prod_{\alpha \in \Delta_+ \cap w(r)\Delta_-} U_\alpha$$

as before. But

$$\Delta_+ \cap w(r)\Delta_- = \{\alpha_1, w_1 \cdot \alpha_2, \ldots\}\,,$$

and each U_α, $\alpha \in \Delta_+ \cap w(r)\Delta_-$, leaves each simplex of $\mathcal{A}_0^+$ fixed, and we may take b_2 to be the limit of the $\bar{u}_r$. $\qquad\square$

Corollary 14.1. *G acts doubly transitively on the ends of X.*

Corollary 14.2. *Let $\mathcal{B}^\pm$ be the stabilizer of the end $\mathcal{E}_0^\pm$ determined by $\mathcal{A}_0^\pm$; then $(\mathcal{B} = \mathcal{B}^+)$ we have*

$$G = \mathcal{B} \sqcup \mathcal{B}w_1\mathcal{B} = \mathcal{B} \sqcup \mathcal{B}w_2\mathcal{B}\,.$$

Proof. We note that $w_1 \cdot (\sigma_B \cup \mathcal{A}_0^+) = \mathcal{A}_0^-$. We let $g \in G$, and consider $g \cdot \mathcal{A}_0$; then $g \cdot \mathcal{A}_0^+$, $g \cdot \mathcal{A}_0^-$ determine two ends $g \cdot \mathcal{E}_0^\pm$ of X and by Corollary 14.1, if $g \notin \mathcal{B}$, there exists $b \in G$ such that $b \cdot \mathcal{E}_0^+ = \mathcal{E}_0^+$ and $b \cdot g\mathcal{E}_0^+ = \mathcal{E}_0^-$; then $b \in \mathcal{B}$ and

$$w_1 b \cdot g\mathcal{E}_0^+ = w_1 \cdot \mathcal{E}_0^- = \mathcal{E}_0^+\,;$$

Hence

$$w_1 bg = b' \in \mathcal{B};$$

or

$$g = b^{-1} w_1 b'.\qquad\qquad\square$$
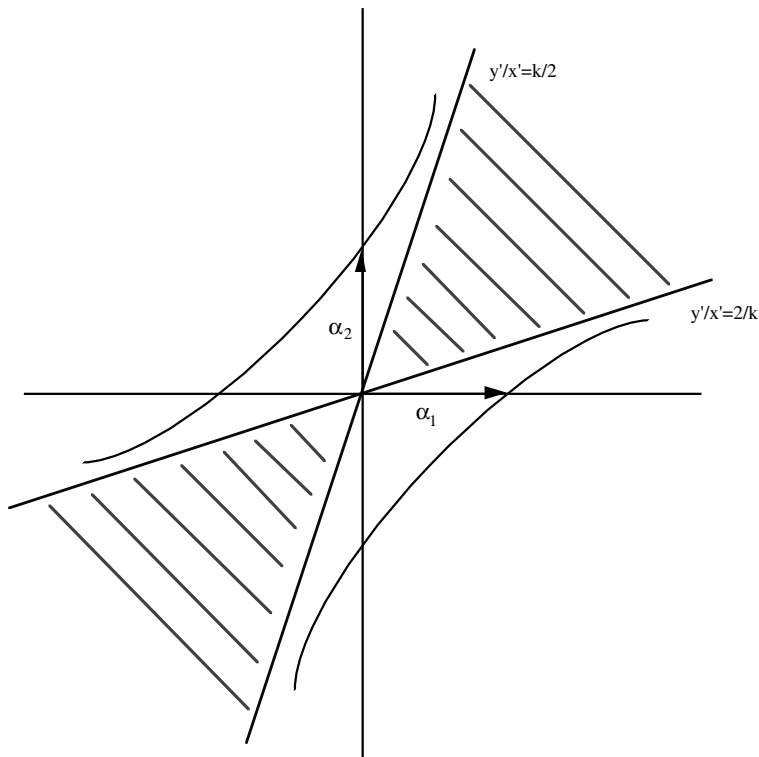
We are left with the question of describing $\mathcal{B}$ explicitly. We do this as follows: we let

$$\mathcal{B}_{\mathcal{I}} = \bigcap_{w \in W} wBw^{-1} \subset B,$$

$$T = \{(w_1 w_2)^n\}_{n \in \mathbb{Z}} \text{ (translations along } \mathcal{A}_0),$$

$$\mathcal{U} = \text{closure of } \mathcal{U}_0, \text{ where} \qquad\qquad (14.5)$$

$$\mathcal{U}_0 = \text{group generated by all } U_\alpha, \text{ with}$$

$$\alpha \in \Phi_1,$$

where

$$\Phi_1 := \{-\alpha_2, -w_2 \cdot \alpha_1, -w_2 w_1 \cdot \alpha_2 \cdots\} \cup \{\alpha_1, w_1 \cdot \alpha_2, w_1 w_2 \cdot \alpha_1 \cdots\}.$$

From the Appendix, we have Fig. A2, for $k > 2$:

We have that the Weyl roots are distributed along two branches of the hyperbola (pictured above) with α_1 and α_2 on two different branches. Since

$$w_1 \cdot \alpha_1 = -\alpha_1 , \quad w_2 \cdot \alpha_2 = -\alpha_2 ,$$

w_1 and w_2 must interchange the two branches of the hyperbola.

Hence the Weyl roots in

$$\Phi_1 = \{-\alpha_2, -w_2 \cdot \alpha_1, -w_2 w_1 \cdot \alpha_2 \cdots\} \cup \{\alpha_1, w_1 \cdot \alpha_2, w_1 w_2 \cdot \alpha_1 \cdots\} \qquad (14.6a)$$

are distributed along the lower branch of the hyperbola. The remaining Weyl roots, namely those in

$$\Phi_2 := \{-\alpha_1, \ -w_1\alpha_2, \ -w_1 w_2 \alpha_1, \ldots\} \cup \{\alpha_2, \ w_2\alpha_1, \ w_2 w_1 \alpha_2, \ldots\} \qquad (14.6b)$$

are distributed along the upper branch.

We have

$$\Phi = \Phi_1 \cup \Phi_2 , \quad (\text{disjoint union}) .$$

Moreover, the imaginary roots lie inside the imaginary cone which coincides with the shaded region between the asymptotes.

Since the sum of any two roots along the lower branch is not on either of the branches, nor in the shaded region between the asymptotes, such a sum cannot be a root.

Lemma 14.2. *Let α, β be Weyl roots. If $\alpha + \beta$ is not zero or a root, then $[U_\alpha, U_\beta] = 0$.*

Proof. Since $\alpha + \beta$ is not zero or a root, in the Kac–Moody algebra, the corresponding root vectors x_α and x_β commute. Hence any power or divided power of x_α commutes with any power or divided power of x_β. □

Hence the U_α with $\alpha \in \Phi_i$, $i = 1$ or $i = 2$, must commute among themselves, and the group $\mathcal{U}$ is commutative. We refer the reader to [20] for additional details on hyperbolic root systems, and [21] and [30] for more commutator relations in Kac–Moody groups.

The group T normalizes $\mathcal{U}$ and we have

Theorem 14.1. $\mathcal{B} = \mathcal{B}_{\mathcal{I}} \mathcal{U} T = \mathcal{B}_{\mathcal{I}} T \mathcal{U} = \mathcal{U} T \mathcal{B}_{\mathcal{I}} = \mathcal{U} \mathcal{B}_{\mathcal{I}} T$.

Proof. $g \in \mathcal{B}$ implies that there exist $n, m \in \mathbb{Z}$ such that

$$g \cdot (w_2 w_1)^m (\mathcal{A}_0^+ \cup \sigma_B) = (w_2 w_1)^n (\mathcal{A}_0^+ \cup \sigma_B) .$$

We set $t = w_2 w_1$, and note that from the above equality, we have

$$t^{-n} g t^m (\mathcal{A}_0^+ \cup \sigma_B) = (\mathcal{A}_0^+ \cup \sigma_B) ,$$

and $t^{-n} g t^m$ fixes $\mathcal{A}_0^+ \cup \sigma_B$ pointwise. Consider

$$t^{-n} g t^m (\mathcal{A}_0^-) ;$$

then the argument of Lemma 14.1 shows that there exists $b \in \mathcal{U}$ which stabilizes $\mathcal{A}_0^+ \cup \sigma_B$ and satisfies

$$bt^{-n}gt^m(\mathcal{A}_0^-) = (\mathcal{A}_0^-)\,.$$

Then

$$bt^{-n}gt^m \text{ stabilizes } \mathcal{A}_0 \text{ pointwise.}$$

Therefore

$$bt^{-n}gt^m := b_{\mathcal{I}} \in \mathcal{B}_{\mathcal{I}}\,.$$

Then

$$g = t^n b^{-1} b_{\mathcal{I}} t^{-m} = (t^n b^{-1} t^{-n})\,(t^n b_{\mathcal{I}} t^{-n})\, t^{n-m}\,,$$

where

$$t^n b^{-1} t^{-n} \in \mathcal{U}\,,$$

$$t^n b_{\mathcal{I}} t^{-n} \in \mathcal{B}_{\mathcal{I}}\,,$$

$$t^{n-m} \in T\,,$$

which completes the proof. $\square$

15. Construction of Non-Uniform Lattices in Rank 2

In this section we use k to denote the finite field $\mathbb{F}_q$. In analogy with [18], we let $\mathcal{U}' \subseteq \mathcal{U}$ be the subgroup of $\mathcal{U}$ (as defined in (14.5)) defined as follows:

$$\mathcal{U}' = \text{closure of the group generated by all } U_\alpha, \text{ with } \alpha \in \Phi_1^+\,,$$

where

$$\Phi_1^+ := \{\alpha_1, w_1 \cdot \alpha_2, w_1 w_2 \cdot \alpha_1 \cdots\}\,.$$

Remark 15.1. The group $\mathcal{U}$ is topologically isomorphic to the additive group of $\mathbb{F}_q((t))$, the field of formal Laurent series in t over $\mathbb{F}_q$, the group $\mathcal{U}'$ is isomorphic to the additive group of the ring of formal power series $\mathbb{F}_q[[t]]$, $\mathcal{U}'$ is profinite, and equals $\mathcal{U} \cap B$. The group $\mathcal{U}$ (respectively $\mathcal{U}'$) plays the role of the (additive) subgroup of $SL_2(\mathbb{F}_q((t^{-1})))$ (respectively $SL_2(\mathbb{F}_q[[t^{-1}]])$) of upper triangular unipotent matrices in the proof of A. Lubotzky's Proposition 3.2, (see [18, p. 415]).

Our aim is to construct an infinite family of non-uniform lattices in our rank 2 Kac–Moody group G.

In analogy with [18], we let Δ denote the subgroup of G generated by all $\chi_\alpha(s)$, where $\alpha \in \Phi_1^-$ (as defined in Sec. 14) (there is an over-use of "Δ" here, but the notation is standard, for both the set of roots, and the subgroup Δ of [18]).

Lemma 15.1. *We have*

$$\Delta \cong \bigoplus_{i=1}^{\infty} (\mathbb{F}_q^+)_i\,,$$

where $\mathbb{F}_q^+$ *denotes the additive group of the finite field* $\mathbb{F}_q$.

Hence $\Delta \cong \bigoplus_{i=1}^{\infty} (\mathbb{Z}/p\mathbb{Z})_i$. It follows that the group Δ is isomorphic to the group Δ of ([18, Sec. 3]), and so Δ is residually finite.

Lemma 15.2. *The group Δ is a uniform lattice in $\mathcal{U}$.*

Proof. The group Δ is a subgroup of B^- (as in Sec. 6), which is discrete. We have $\mathcal{U} = \mathcal{U}' \cdot \Delta$, and $\mathcal{U}'$ is compact. □

To construct non-uniform lattices, our strategy, following Lubotzky [18], is to enlarge Δ to a non-uniform lattice in G. We have

$$G/\mathcal{B} = (\mathcal{B} \sqcup \mathcal{U}w\mathcal{B})/\mathcal{B}$$

$$\cong \{\infty\} \sqcup \mathcal{U}. \tag{15.1}$$

The decomposition $G/\mathcal{B} \cong \{\infty\} \sqcup \mathcal{U}$ is analogous to the decomposition of G/B in [18, p. 417], for a simple algebraic group G over a non-archimedean local field, and a minimal parabolic subgroup B. For example if Lubotzky's G is $= SL_2$, then the coset $\mathcal{B}/\mathcal{B}$ in (15.1) corresponds to $\{\infty\}$ in [18, p. 417], and the coset $\mathcal{U}w\mathcal{B}/\mathcal{B}$ in (15.1) corresponds to $U \cong \mathbb{F}_q((t^{-1}))$ in [18, p. 417], where U is the unipotent radical of B. In Lubotzky's setting, one may identify $\{\infty\} \sqcup \mathbb{F}_q((t^{-1}))$ with $\mathrm{Ends}(X)$ where X is the Bruhat–Tits tree of G, and

$$\mathrm{Ends}(X) = \partial X = \{\text{equivalence classes of semi-infinite paths in } X\},$$

where two such paths are equivalent if their intersection is infinite. Here we also identify $\{\infty\} \sqcup \mathcal{U}$ in (15.1) with $\mathrm{Ends}(X)$, where X is the Bruhat–Tits tree of our Kac–Moody group G.

We remark that

$$\Delta \subseteq \mathcal{U} \subseteq \mathcal{B} = \mathrm{Stab}_G(\infty),$$

where $\mathcal{B}$ is the stabilizer of the end ∞ represented by the positive half ray (denoted by $\mathcal{A}_{0,\infty}^+$) of the standard apartment $\mathcal{A}_0$ (as in Sec. 9). Hence Δ fixes ∞, but no other end of X. Let $\mathcal{A}_{0,\infty}^+$ have vertex sequence $(x_0, x_1, x_2, \ldots)$ corresponding to the cosets $(P_1, P_2, w_2 P_1, \ldots)$ as in Fig. 15.1.

We let $F \subseteq X$ consist of the union of all bi-infinite lines containing x_0 (notation of Fig. 15.1) and having ∞ as one end (F is shown in Fig. 15.1, in the case $|k| = 2$). Then any semi-infinite ray in F with origin x_0 and end different from ∞ can be "collapsed" to the standard ray (with end $-\infty$ determined by $\mathcal{A}_0^-$) by an element of $\mathcal{U}' \subseteq \mathrm{Stab}_G(x_0)$, and conversely, the orbit of the end $-\infty$ under $\mathcal{U}'$ corresponds to semi-infinite rays in F with origin x_0, and end different from ∞.

We will now prove

$$\Delta F = X. \tag{15.2}$$

Let x be any vertex in X, and let $\mathcal{A} \subseteq X$ be an apartment such that:

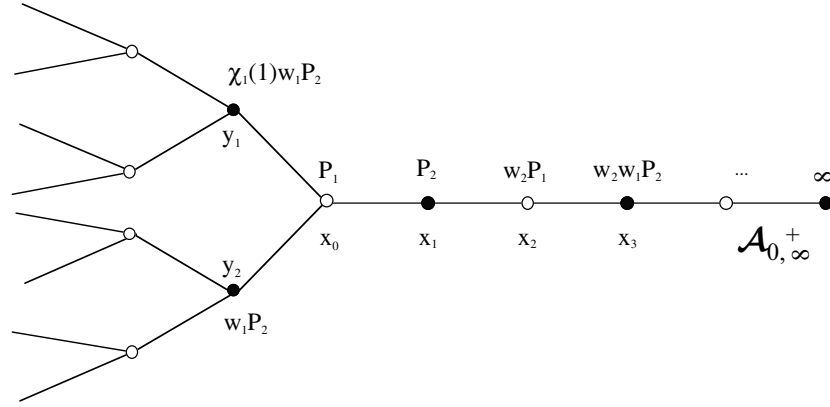$$x \text{ is a vertex of } \mathcal{A}. \tag{15.3a}$$

Fig. 15.1.   For $|k| = 2$, the diagram of F, a fundamental domain of Δ on X.

and

$$\mathcal{A} \cap \mathcal{A}_0 \text{ contains all simplices of } \mathcal{A}_0 \text{ to the right of some fixed } x_i \qquad (15.3b)$$

(see Fig. 15.1).

Then $\mathcal{A}$ determines two ends, ξ, ∞, where ξ corresponds to some element of $\mathcal{U}$. We can find $\delta \in \Delta$ such that $\delta \cdot \xi \in \mathcal{U}'$, as remarked in the proof of Lemma 15.2. But δ leaves ∞ fixed, and so $\delta \cdot \mathcal{A} \subseteq F$. In particular $\delta x \in F$, which proves (15.1).

We quote [18, Proposition 1.10], which we shall use to prove Theorem 15.1.

Proposition 15.1 ([18, (1.10)]). *Let $\Gamma_1, \ldots, \Gamma_s$ be discrete subgroups of $\mathrm{Aut}(X)$ and $x_0 \in VX$ such that x_0 is fixed by no non-trivial element of any of the $\Gamma_i's$. Let D_i be a fundamental domain for Γ_i containing x_0, and*

$$E_i = E(D_i) := \{x \in D_i | (\Gamma_i)_x = \{1\}\} \,.$$

Assume $E_i \supseteq X - E_j$ for every pair $i \neq j \in \{1, \ldots, s\}$ and that $\gamma(E_i) \cap E_i = \varnothing$, for every $1 \neq \gamma \in \Gamma_i$. Let Γ be the group generated by $\Gamma_1, \ldots, \Gamma_s$. Then

  (i) Γ *is discrete.*
 (ii) Γ *is the free product of $\Gamma_1, \ldots, \Gamma_s$.*
(iii) $D_0 = \bigcap_{i=1}^{s} D(\Gamma_i, x_0)$ *contains the Dirichlet fundamental domain $D(\Gamma, x_0)$ for Γ where*

$$D(\Gamma, x_0) := \{x \in VX | d(x, x_0) \leq d(\gamma x, x_0) \quad for \; every \; \gamma \in \Gamma\} \,.$$

Theorem 15.1. *Using the notation in Fig.* 15.1, *let $y_1, \ldots, y_q$ be the neighbours of x_0 other than x_1, where $q = |k|$. Choose $g_0 = id$, $g_1, \ldots, g_q$ in $P_1/B = \mathrm{Stab}_G(x_0)/\mathrm{Stab}_G(x_0, x_1)$ subject to:*

$$g_0 = id \,, \quad g_1(x_1) = y_1 \,, \quad g_2(x_1) = y_2, \ldots, g_q(x_1) = y_q \,,$$

and set $\Delta_i = g_i \Delta g_i^{-1}$, $i = 0, \ldots, q$. *Let* Γ *be the group generated by* $\Delta_0, \ldots, \Delta_q$. *Then*

(1) Γ *is discrete,*
(2) Γ *equals the free product* $\Delta_0 * \cdots * \Delta_q$, *and*
(3) Γ *is a non-uniform lattice in* G.

Proof. We apply [18, Proposition 1.10] (cf. [18, Proposition 3.2]). First we show that

$$\Delta_0 \cap P_1 = \{1\} . \qquad (15.4)$$

It then follows that the vertex x_0 is fixed by no non-trivial element of Δ_i, $i = 0, \ldots, q$. To prove (15.4), we show:

$$w_1 \Delta_0 w_1^{-1} \subseteq U^- \qquad (15.5a)$$

and

$$P_1 \cap U^- = U_{-\alpha_1} . \qquad (15.5b)$$

Assertion (15.5a) follows at once from the definition of Δ_0. For (15.5b), if $\xi \in P_1$ is in U^-, it is lower triangular unipotent with respect to a coherently ordered basis. Let

$$v_\lambda = \text{a highest weight vector,}$$

$$L_1 = \text{the subgroup generated by } \chi_{\pm\alpha_1}(s), \ s \in k.$$

Let W be the L_1-submodule of V_k^λ generated by v_λ; then W is spanned by vectors

$$v_\lambda, v_{\lambda-\alpha_1}, \ldots, v_{\lambda-m\alpha_1} = w_1 v_\lambda ,$$

where $v_{\lambda-q\alpha_1} \in V_{\lambda-q\alpha_1}^\lambda$, $q = 0, 1, \ldots, m$.

Then W is P_1-invariant, and moreover $P_1 = L_1 U_1$,

$$U_1 = U \cap w_1 U w_1^{-1} ,$$

and U_1 acts trivially on W, as follows from the equality $U_{0,w_i} = U'$ of Sec. 6. Then we have

$$\xi = \xi_L \xi_U , \quad \xi_L \in L_1, \ \xi_U \in U_1 ,$$

and

$$\xi_L|_W = \xi|_W , \quad \text{so } \xi_L \text{ is lower triangular, unipotent.}$$

Hence $\xi_L \in U_{-\alpha_1}$, and thus

$$\xi_U = \xi_L^{-1} \xi ,$$

where the right-hand side is lower triangular unipotent, and the left-hand side is upper triangular unipotent. Thus $\xi_U = 1$, and so $\xi = \xi_L \in U_{-\alpha_1}$, and we have proven (15.5b).

It follows from (15.5a) and (15.5b) that

$$w_1 \Delta_0 \cap P_1 w_1^{-1} = w_1 \Delta_0 w_1^{-1} \cap P_1 \subseteq U^- \cap P_1 \subseteq U_{-\alpha_1},$$

and so

$$\Delta_0 \cap P_1 \subseteq U_{\alpha_1}.$$

But

$$\Delta_0 \cap U_{\alpha_1} = \{1\},$$

since U_{α_1} is upper triangular with 1's on the diagonal, and Δ_0 is lower triangular with 1's on the diagonal. Thus $\Delta_0 \cap P_1 = \{1\}$ and we have proven (15.4).

Since Δ fixes the unique end ∞ of X but not x_0, we have

$$E_0 = E_0(\Delta) := \{x \in F | \Delta_x = \{1\}\}$$

$$= \{x \in F | d(x, x_0) \le d(x, x_1)\}.$$

It follows that

$$\gamma E_0 \cap E_0 = \varnothing, \quad 1 \ne \gamma \in \Delta.$$

For $i = 1, \ldots, q$, we set

$$E_i = E_i(\Delta) := \{x \in F | (\Delta_i)_x = \{1\}\}.$$

Then each subtreee $E_i \subseteq X$ is a rotation in X of the subtree E_0 around the vertex x_0. It follows that

$$\gamma E_i \cap E_i = \varnothing, \quad 1 \ne \gamma \in \Delta_i, \ i = 1, \ldots, q.$$

It is also easy to verify that

$$X - E_j \subseteq E_i, \quad i \ne j,$$

$i = 1, \ldots, q$.

It follows from [18, Proposition 1.10] that Γ is discrete, Γ equals the free product $\Delta_0 * \cdots * \Delta_q$, and that a fundamental domain Y for Γ on X is

$$Y = \bigcap_{i=0}^{q} g_i(F) = \bigcup_{i=0}^{q} g_i(\mathcal{A}_{0,\infty}^+),$$
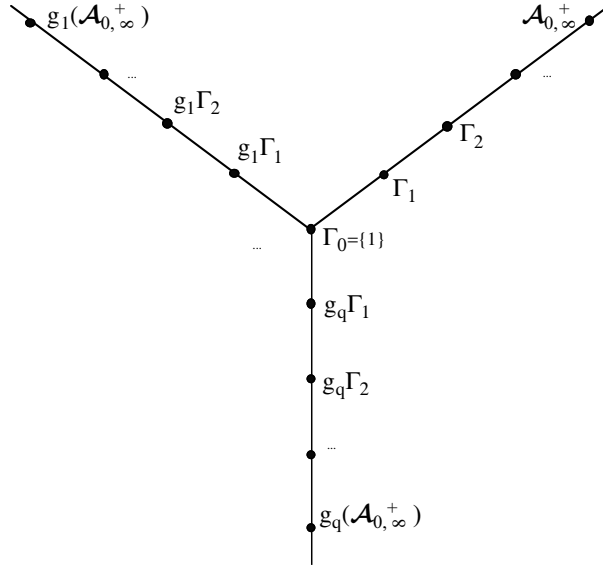
depicted in Fig. 15.2:

Fig. 15.2.

To compute the isotropy groups of vertices and edges in Y, we recall (Sec. 9) that

$$E\mathcal{A}_{0,\infty}^+ = \{B, w_2 B, w_2 w_1 B, \ldots\}.$$

For $w \in W$, let Γ_w denote the isotropy group of the coset wB:

$$\Gamma_w = \{\gamma \in \Gamma | \gamma wB = wB\}$$

$$= \Gamma \cap wBw^{-1}.$$

Then Γ_w is a discrete subgroup of the compact group wBw^{-1} and hence is finite.

We have

$$|\Gamma \cap wBw^{-1}| \geq |\Delta_0 \cap wBw^{-1}| \geq q^{l(w)} \qquad (15.6)$$

(the last inequality following from the arguments of Lemma 8.3), where $l(\cdot)$ is the length function on W.

For $n \geq 0$, let Γ_n be the isotropy group of the vertex x_n of the positive half ray $\mathcal{A}_{0,\infty}^+ = (x_0, x_1, x_2, \ldots)$ (using notation as in Figs. 15.1 and 15.2). Then for $n \geq 1$, $i = 1, \ldots, q$

$$\Gamma_n \cong g_i(\Gamma_n).$$

Applying (15.6) we obtain

$$\Gamma_0 = \{1\},$$

and for $n \geq 1$, $i = 1, \ldots, q$:

$$|\Gamma_n| = |g_i(\Gamma_n)| \geq q^n \,.$$

Thus the covolume of Γ, suitably normalized, is at most:

$$(q+1) \sum_{i=0}^{\infty} \frac{1}{q^i} \,.$$

So Γ is a non-uniform lattice in G. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 15.2. (1) It follows from Lemma 15.1 that the group Γ constructed in Theorem 15.1 is isomorphic to Γ of [18, Proposition 3.2].

(2) The group Γ constructed in Theorem 15.1 *cannot* be finitely generated [3, (0.4)].

16. Deformations of Non-Uniform Lattices in Rank 2

In this section, we describe how the construction of non-uniform lattices in Sec. 15 is flexible enough to admit an uncountably infinite family of non-uniform lattices. We describe a compact parameter space for this uncountably infinite family. It remains to determine whether non-uniform lattices form uncountably many conjugacy classes in the Kac–Moody group. We conjecture that this is the case (Conjecture 16.1).

16.1. *Cuspidal deformations*

Using the notation of Sec. 15, and Theorem 15.1, we have a non-uniform lattice Γ which is the free product

$$\Gamma = \Delta_0 * \cdots * \Delta_q \,.$$

By varying the g_i over the open set $B/(B \cap N_G(\Delta))$, where $N_G(\Delta)$ denotes the normalizer in G of Δ, for $i = 1, \ldots q$, we have an uncountably infinite family of non-uniform lattices here, parametrized by an open set of the form:

$$\mathcal{P}(\Gamma, G) = B/(B \cap N_G(\Delta)) \times \cdots \times B/(B \cap N_G(\Delta))$$

($q = |k|$ factors).

Lemma 16.1. *We have* $N_G(\Delta) \subseteq \mathcal{B}$.

Proof. Recall (Sec. 15) that Δ fixes ∞ but no other end of X. We claim that if $g\Delta g^{-1} \subseteq \Delta \subseteq \mathcal{B}$, then $g \in \mathcal{B}$. Suppose, conversely that $g \notin \mathcal{B}$. If $g \notin \mathrm{Stab}_G(\infty)$, then for every $1 \neq \delta \in \Delta$ one has $g\delta g^{-1} \notin \mathcal{B}$. Otherwise $g\delta g^{-1} \in \mathcal{B}$, and has ∞ as a fixed point. However, since δ has ∞ as a fixed point, it is the case that $g\delta g^{-1}$ has $g \cdot \infty$ as a fixed point. Since $g \notin \mathrm{Stab}_G(\infty)$, $g \cdot \infty \neq \infty$, and so $g\delta g^{-1}$ has two fixed

points on the boundary. Hence δ has two fixed points on the boundary, which is a contradiction. Hence $g \in \mathcal{B}$ as asserted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The question arises as to whether we have uncountably many distinct conjugacy classes of non-uniform lattices in G. We have natural maps:

$$B/(B \cap N_G(\Delta)) \times \cdots \times B/(B \cap N_G(\Delta))$$
$$\downarrow$$
$$B/(B \cap \mathcal{B}) \times \cdots \times B/(B \cap \mathcal{B})$$
$$\downarrow$$
$$G/\mathcal{B} \times \cdots \times G/\mathcal{B},$$
$$\cong$$
$$(\mathcal{U} \sqcup \infty) \times \cdots \times (\mathcal{U} \sqcup \infty)$$
$$\cong$$
$$\partial X \times \cdots \partial X.$$

$(q = |k|$ factors$)$. We have the following (cf. [7, Conjecture 2.12]):

Conjecture 16.1. *The diagonal action of G on*

$$G/\mathcal{B} \times \cdots \times G/\mathcal{B}$$

$(|k|$ *factors*$)$, *has the property that every orbit is nowhere dense.*

An affirmative answer to Conjecture 16.1 would imply the existence of uncountably many conjugacy classes of non-uniform lattices in G. Suppose that A is affine, that is,

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Then G is a central extension, denoted $\widehat{SL_2}$

$$1 \longrightarrow k^\times \longrightarrow \widehat{SL_2} \longrightarrow SL_2 \longrightarrow 1$$

of SL_2 over a Laurent series field $k((t))$ over the finite field k, by the (one dimensional) multiplicative group $k^\times$. In this case, our uncountably infinite family of non-uniform lattices essentially coincides with Lubotzky's construction of uncountably many conjugacy classes of non-uniform lattices in $SL_2(k((t)))$ (see [18, p. 415]), where Conjecture 16.1 can be easily verified by dimension counting for $q \geq 4$, where $q = |k|$. Thus Conjecture 16.1 is true for $m = 2$, and $q \geq 4$.

To give a geometric interpretation of the cuspidal deformations of Γ in G for $m = 2$, and $q \geq 4$, we observe that the above discussion implies that we can vary the ends of q of the $q+1$ cusps of Γ, each in an open neighborhood of the boundary of the Bruhat–Tits tree X. This gives a fundamental domain isomorphic to that of Γ, and isomorphisms of isotropy groups.

16.2. *Conjugacy in* $\mathrm{Aut}(X)$

Let $\mathbb{A} = (A, \mathcal{A})$ and $\mathbb{A}' = (A', \mathcal{A}')$ be graphs of groups. We say that $\mathbb{A}$ and $\mathbb{A}'$ are *isomorphic*, if there is an isomorphism

$$\phi : A \longrightarrow A'$$

of graphs, and for each $v \in VA$, and each $e \in EA$, there are group isomorphisms

$$\phi_v : \mathcal{A}_v \longrightarrow \mathcal{A}'_{\phi(v)}$$

$$\phi_e : \mathcal{A}_e \longrightarrow \mathcal{A}'_{\phi(e)} \, .$$

The cuspidal deformations of non-uniform lattices that we described above in Sec. 16.1 give rise to isomorphic graphs of groups. The following proposition indicates that the non-uniform lattices in the uncountably infinite families in Sec. 16.1 are all conjugate in $\mathrm{Aut}(X)$.

Proposition 16.1. *Let $\Phi : \mathbb{A} \longrightarrow \mathbb{A}'$ be an isomorphism of graphs of groups. Let $X = \widetilde{(\mathbb{A}, a)}$ and identify X with $\widetilde{(\mathbb{A}', a')}$ via Φ. Then $\pi_1(\mathbb{A}, a)$ and $\pi_1(\mathbb{A}', a')$ are conjugate in $\mathrm{Aut}(X)$.*

Proof. The isomorphism $\Phi : \mathbb{A} \longrightarrow \mathbb{A}'$ is a morphism of graphs of groups in the sense of [1, Sec. 2]. Identifying $X = \widetilde{(\mathbb{A}, a)}$ with $\widetilde{(\mathbb{A}', a')}$ via Φ, we get an induced automorphism of covering trees [1, Proposition 2.4]

$$
\begin{array}{ccc}
X & \xrightarrow{\;g \in \mathrm{Aut}(X)\;} & X \\
\downarrow & & \downarrow \\
(\mathbb{A}, a) & \xrightarrow{\;\cong\;} & (\mathbb{A}', a')
\end{array}
$$

with the property that

$$g \cdot \pi_1(\mathbb{A}, a) = \pi_1(\mathbb{A}', a') \cdot g \, ,$$

that is, $\pi_1(\mathbb{A}', a') = g \pi_1(\mathbb{A}, a) g^{-1}$, for some $g \in \mathrm{Aut}(X)$. $\square$

Remark 16.1. If we take $\mathbb{A}$ and $\mathbb{A}'$ to be graphs of *trivial* groups, then we obtain the result of Proposition 12.1.

17. Failure of Property T in Rank 2

If a property T group H acts on a tree, then the group H must fix a vertex [33]. Therefore, if a non-compact group G acts on a tree with compact vertex stabilizers, G cannot have property T. This is the case for the Kac–Moody group G in the case $l = 2$, and thus we have the following.

Proposition 17.1. *When $l = 2$, the group G does not have Kazhdan's property T.*

For a locally compact group H containing lattices, H has property T if and only if lattices of H have property T. Thus we obtain the following:

Corollary 17.1. *When $l = 2$, lattices of G do not have Kazhdan's property T.*

Corollary 17.1 also follows from the fact that non-uniform lattices in G cannot be finitely generated when $l = 2$ (cf. Remark 15.1 (2)). Proposition 17.1 and Corollary 17.1 are in contrast to the higher rank case (see Sec. 18).

18. Cohomology

For $l \geq 3$, the methods of Garland in [12] yield results for various cohomologies on the Bruhat–Tits building X associated with the BN-pair (G, B, N), of our Kac–Moody group G, and on discrete subgroups $\Gamma \subseteq G$ (see also Ballmann–Swiatkowski [5], Pansu [24] and Zuk [35]). For example, we have the following:

Theorem 18.1. *Let $\Gamma \subseteq G$ be a uniform lattice of G, and let*

$$\rho : \Gamma \longrightarrow \mathrm{Aut}(V)$$

be a unitary representation Γ in the complex Hilbert Space V (not necessarily finite dimensional). If $l \geq 3$, and if $q = |k|$ is sufficiently large, then

$$H^i(\Gamma, \rho) = 0, \quad 0 < i < l - 1.$$

By Theorem 18.1 and [33] we have the following:

Corollary 18.1. *Given G and q as in the theorem, if G contains a uniform lattice, then G has property T.*

Applying the result of Valette and de la Harpe [33] in this way has already been utilized in the works of Ballmann–Swiatkowski, Pansu and Zuk cited above.

If G is affine, then by Borel–Harder [4], we may deduce that G contains uniform lattices when $l \geq 3$. However, if G is not affine, we do not know if G contains uniform lattices when $l \geq 3$.

Dymara and Januszkiewicz [10] have also used the results in [12], to obtain vanishing theorems, and to prove that G has property T when $l \geq 3$ and G is of compact hyperbolic type. In [11] Dymara and Januszkiewicz give criteria for lattices in G to have property T in some non-compact cases.

It then follows (see, in contrast, Corollary 17.1) that in these cases the lattices we construct here (uniform and non-uniform) have property T. In particular these lattices are finitely generated and have finite commutator quotients. This is the case for the subgroups Γ as in Theorem 8.1.

Appendix. Explicit Computations for Symmetric, 2×2, Generalized Cartan Matrices

We will consider matrices

$$A = \begin{pmatrix} 2 & -k \\ -k & 2 \end{pmatrix}, \quad k \in \mathbb{Z}_{\geq 3}.$$

We have

$$\det(A - \lambda I) = \det \begin{pmatrix} 2 - \lambda & -k \\ -k & 2 - \lambda \end{pmatrix} = \lambda^2 - 4\lambda + (4 - k^2).$$

Then the characteristic roots are given by:

$$\lambda = 2 \pm k.$$

Indeed:

$$\begin{pmatrix} 2 & -k \\ -k & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 - k \\ 2 - k \end{pmatrix} = (2 - k) \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 2 & -k \\ -k & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 + k \\ -(2 + k) \end{pmatrix} = (2 + k) \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so

$$f_1 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad f_2 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are eigenvectors corresponding to eigenvalues $(2 - k)$, $(2 + k)$, respectively. We have that

$$e_1 := \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad e_2 := \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

constitute an *orthonormal* basis of eigenvectors. Relative to the basis e_1, e_2, the matrix A then has the form

$$A' = \begin{pmatrix} 2 - k & 0 \\ 0 & 2 + k \end{pmatrix},$$

and the corresponding quadratic form is

$$(x, y)A' \begin{pmatrix} x \\ y \end{pmatrix} = x^2(2 - k) + y^2(2 + k)$$

$$= (2 - k)(2 + k) \left(\frac{x^2}{2 + k} - \frac{y^2}{k - 2} \right), \quad k - 2 > 0.$$

We consider the hyperbola

$$\frac{x^2}{(2 + k)} - \frac{y^2}{(k - 2)} = 2.$$

We recall that for a hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = c,$$

the asymptotes are given by

$$\frac{x^2}{a^2} = \frac{y^2}{b^2},$$

or

$$y = \pm \frac{b}{a} x \,.$$

In our case,

$$a^2 = 2 + k$$
$$b^2 = k - 2 \,,$$

and

$$\frac{b^2}{a^2} = \frac{k-2}{k+2} \longrightarrow 1 \,,$$

as $k \to \infty$. But also
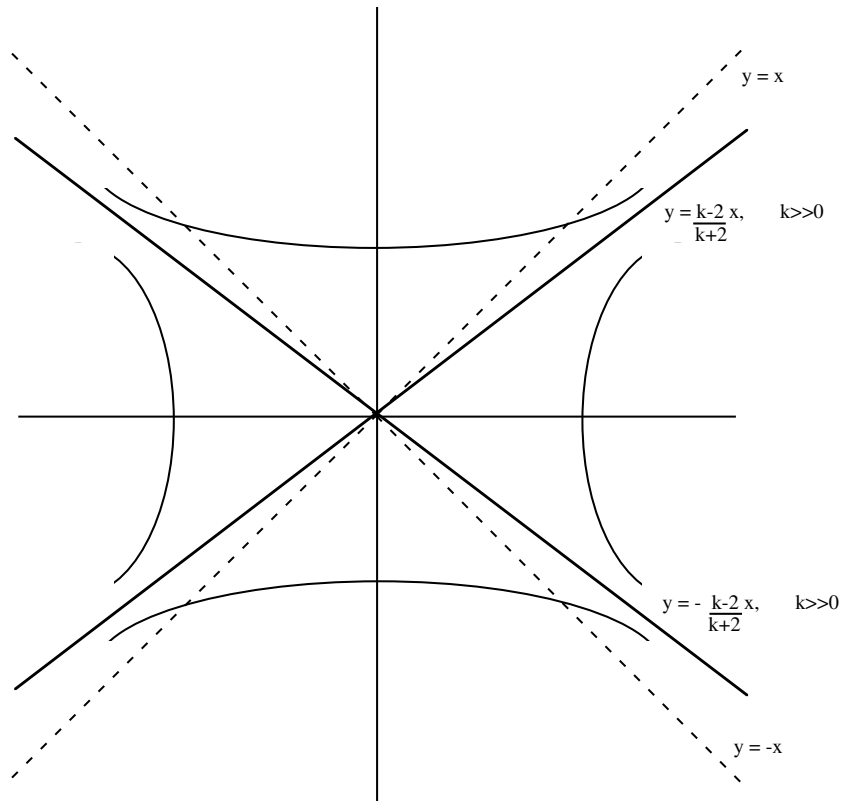
$$\frac{k-2}{k+2} < 1 \,.$$

So we have:

Fig. A.1.

for the corresponding "system of hyperbolas". What are the asymptotes in the old coordinates? We have for the "old" canonical basis f_1, f_2

$$e_1 = \frac{1}{\sqrt{2}} f_1 + \frac{1}{\sqrt{2}} f_2$$

$$e_2 = \frac{1}{\sqrt{2}} f_1 - \frac{1}{\sqrt{2}} f_2 \,.$$

Hence

$$x e_1 + y e_2 = \frac{1}{\sqrt{2}} x f_1 + \frac{1}{\sqrt{2}} x f_2 + \frac{1}{\sqrt{2}} y f_1 - \frac{1}{\sqrt{2}} y f_2$$

$$= \frac{1}{\sqrt{2}} (x + y) f_1 + \frac{1}{\sqrt{2}} (x - y) f_2 \,,$$

and the old coordinates x', y' (relative to (f_1, f_2)) are given in terms of the new coordinates (x, y) (relative to (e_1, e_2)) by

$$x' = \frac{1}{\sqrt{2}} (x + y)$$

$$y' = \frac{1}{\sqrt{2}} (x - y) \,.$$

The transform of the line

$$y = \pm \frac{k - 2}{k + 2} x$$

is

$$x' = \frac{1}{\sqrt{2}} \left(x \pm \frac{k - 2}{k + 2} x \right)$$

$$y' = \frac{1}{\sqrt{2}} \left(x \mp \frac{k - 2}{k + 2} x \right) ,$$

or

$$x' = \frac{1}{\sqrt{2}} \left(\frac{(k + 2) \pm (k - 2)}{k + 2} x \right)$$

$$y' = \frac{1}{\sqrt{2}} \left(\frac{(k + 2) \mp (k - 2)}{k + 2} x \right) .$$

This is the pair of lines:

$$x' = \frac{1}{\sqrt{2}} \left(\frac{2k}{k + 2} x \right)$$

$$y' = \frac{1}{\sqrt{2}} \left(\frac{4}{k + 2} x \right) ,$$

and

$$x' = \frac{1}{\sqrt{2}}\left(\frac{4}{k+2}x\right)$$

$$y' = \frac{1}{\sqrt{2}}\left(\frac{2k}{k+2}x\right),$$

or

$$\frac{y'}{x'} = \frac{\frac{4}{k+2}}{\frac{2k}{k+2}} = \frac{4}{2k} = \frac{2}{k},$$

and

$$\frac{y'}{x'} = \frac{k}{2}.$$

Since A is symmetric, the squared lengths of the simple roots α_1 and α_2 are the same and equal to 2, and hence all the Weyl roots are of equal length. The Weyl roots hence lie on the hyperbola $\frac{x^2}{(2+k)} - \frac{y^2}{(k-2)} = 2$ with asymptotes $y'/x' = k/2$ and $y'/x' = 2/k$ (see (14.9) and [20] for more details on hyperbolic root systems).
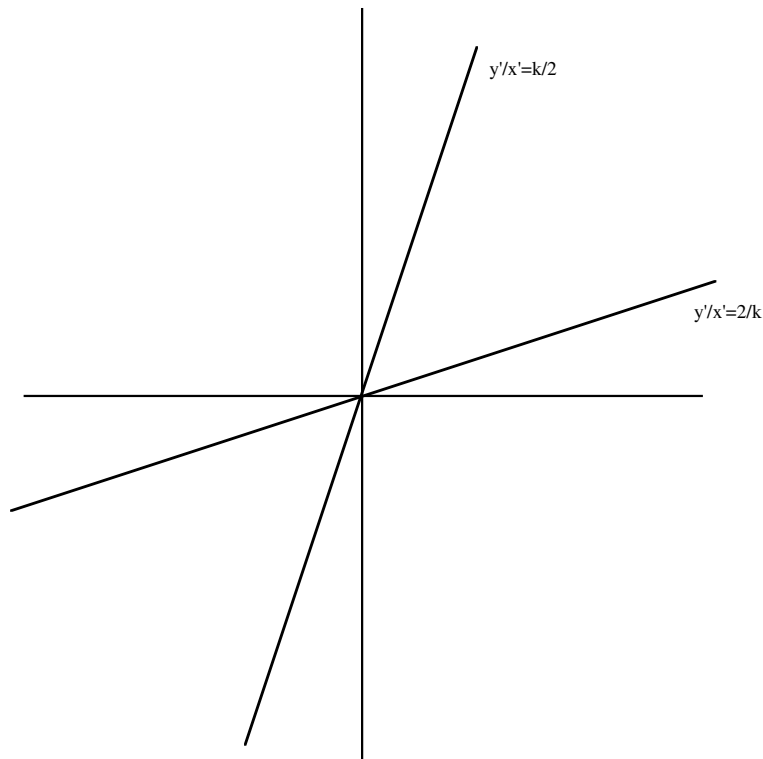

Fig. A.2.

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Acknowledgment

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