EISENSTEIN SERIES ON RANK 2 HYPERBOLIC KAC–MOODY GROUPS OVER $\mathbb{R}$

LISA CARBONE, KYU–HWAN LEE, AND DONGWEN LIU

Abstract. We define Eisenstein series on rank 2 hyperbolic Kac–Moody groups over $\mathbb{R}$, induced from quasi–characters. We prove convergence of the constant term and hence the almost everywhere convergence of the Eisenstein series. Then we calculate other Fourier coefficients of the series. We also consider Eisenstein series induced from cusp forms and show that these are entire functions.

1. Introduction

After being developed by Langlands [La1, La2] in great generality, the theory of Eisenstein series has played a fundamental role in the formulation of the Langlands functoriality conjecture and in the study of L–functions by means of the Langlands–Shahidi method. Eisenstein series also appear in many other places throughout number theory and representation theory. The scope of applications is being extended to geometry and mathematical physics. On the other hand, since we have seen many successful generalizations of finite dimensional constructions to infinite dimensional Kac–Moody groups [K, Ku], it is a natural question to ask whether one can generalize the theory of Eisenstein series to Kac–Moody groups. Such an attempt is not merely for the sake of generalization. Even though it is hypothetical for the present, a satisfactory theory of Eisenstein series on Kac–Moody groups would have significant impact on some of the central problems in number theory [BFH, Sh]. It has recently come to light that these Eisenstein series may also play a role in certain supergravity theories [BCG].

In pioneering work, Garland developed a theory of Eisenstein series for the affine Kac–Moody groups over $\mathbb{R}$ in a series of papers [G99, G04, G06, GMS1, GMS2, GMS3, GMS4, G11], and he established absolute convergence and meromorphic continuation. The absolute convergence result has been generalized to the case of number fields by Liu [Li]. In a recent preprint [GMP], Garland, Miller and Patnaik showed that Eisenstein series induced from cups forms are entire functions. Garland's idea was extended to the function field case by Kapranov [Ka] through geometric methods and was systematically developed by Patnaik [P]. An algebraic approach to this case was made by Lee and Lombardo [LL]. Braverman and Kazhdan's recent preprint [BK] announces further results in the function field case.

The purpose of this paper is to construct and study Eisenstein series on rank 2 hyperbolic Kac–Moody groups over $\mathbb{R}$, generalizing Garland's work in the affine case. In [BCG], the authors defined Eisenstein series on higher rank hyperbolic Kac–Moody groups in analogy with the rank 2 case, studied in [CGGL] for Kac–Moody groups over finite fields.

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The rank 2 hyperbolic Kac–Moody groups form the first family beyond the affine case. However, in contrast to the affine case, our understanding of hyperbolic Kac–Moody groups (and algebras) is far from being complete. In particular, information regarding imaginary root multiplicities of hyperbolic Kac–Moody algebras is limited. A recent survey on this topic can be found in [CFL].

Nevertheless, we have the necessary information to construct Eisenstein series induced from quasi–characters on rank 2 hyperbolic Kac–Moody groups and to prove their almost everywhere convergence, thanks to the works of Lepowsky and Moody [LM], Feingold [Fein] and Kang and Melville [KM] on rank 2 hyperbolic root systems. We can also prove entirety of the Eisenstein series induced from cusp forms using the structure of rank 2 hyperbolic root systems and some of the ideas from [GMP]. Indeed, one of the benefits of working in the Kac–Moody group rather than its Kac–Moody algebra, is that the group is generated by root groups corresponding to only 'real' roots. The 'real' part of the Kac–Moody algebra is sufficiently well understood and carries many properties similar to finite dimensional simple Lie algebras [CG].

We assume that G is a rank 2 hyperbolic Kac–Moody group attached to a symmetric 2×2 generalized Cartan matrix, and we define Eisenstein series on the 'arithmetic' quotient $K(G_{\mathbb{R}})\backslash G_{\mathbb{R}}/G_{\mathbb{Z}}$, where $K = K(G_{\mathbb{R}})$ is the unitary form of G, an infinite dimensional analogue of a maximal compact subgroup. Our method is to choose a quasi–character ν on a Borel subgroup and then extend it to the whole of $G_{\mathbb{R}}$ via Iwasawa decomposition $G_{\mathbb{R}} = KA^+N$, which is given uniquely. We then average over an appropriate quotient of $G_{\mathbb{Z}}$ to obtain a $G_{\mathbb{Z}}$–invariant function $E_\nu(g)$ on $K\backslash G_{\mathbb{R}}/G_{\mathbb{Z}}$. Our first main result is:

Theorem 1.1. *Assume that ν satisfies Godement's criterion, and consider the cone*

$$A' = \{a \in A^+ : a^{\alpha_i} < 1, i = 1, 2\},$$

where α_1, α_2 are the two simple roots. Then for any compact subset A'_c of A', there is a measure zero subset N_0 of N such that $E_\nu(g)$ converges absolutely for $g \in KA'_cN'$, where $N' = N - N_0$.

Although the idea of the proof is similar to that of [G04], our proof heavily depends on a concrete description of root systems of rank 2 hyperbolic Kac–Moody algebras. We compute the constant term of the series $E_\nu(g)$ and show that the constant term is absolutely convergent, which implies almost everywhere convergence of the series. We conjecture that the Eisenstein series actually converges everywhere under a weaker condition than Godement's criterion (See Conjecture 5.3). As the argument in [G06] does not generalize to the hyperbolic case, the conjecture seems out of reach at the current time.

We also calculate other Fourier coefficients of the Eisenstein series in Section 6. Let ψ be a non-trivial character of $N/G_{\mathbb{Z}} \cap N$. Then we can write $\psi = \psi_1\psi_2$, where ψ_i corresponds to the simple root α_i for $i = 1, 2$. We call ψ *generic* if each ψ_i is non-trivial for $i = 1, 2$. We first show that the Fourier coefficients attached to generic characters vanish (Lemma 6.1). Then we consider characters of the form $\psi = \psi_i$ (*i.e.* either ψ_1 or ψ_2 is trivial) and compute the corresponding Fourier coefficients. The resulting formula is an infinite sum of products of the n-th Whittaker coefficient of the analytic Eisenstein series on SL_2 and quotients of the completed Riemann zeta function (Theorem 6.2).

The next main result is the entirety of the Eisenstein series $E_{s,f}(g)$ induced from a cusp form f. Our approach is similar to that of Garland, Miller and Patnaik in [GMP]; however, our method requires us to use information about the structure of the root system of G. We obtain:

Theorem 1.2. *Let f be an unramified cusp form on SL_2. For any compact subset A'_c of A', there is a measure zero subset N_0 of N such that $E_{s,f}(g)$ is an entire function of $s \in \mathbb{C}$ for $g \in KA'_cN'$, where $N' = N - N_0$.*

As mentioned earlier, rank 2 hyperbolic Kac–Moody algebras and groups form the first family beyond the affine case. It would be interesting to generalize the results of this paper to other hyperbolic Kac–Moody groups, for example, to the Kac–Moody group corresponding to the Feingold and Frenkel's rank 3 hyperbolic Kac–Moody algebra [FF]. Steve Miller has informed us that he can prove that for general Kac–Moody groups, the Eisenstein series $E_\nu(g)$ converges almost everywhere inside a cone in the region Re $\nu(h_{\alpha_i}) < -2$ for each simple coroot h_{α_i}, and that almost everywhere convergence can likely be extended to the full region Re $\nu(h_{\alpha_i}) < -2$. It will be very exciting to see further developments toward a satisfactory theory of Eisenstein series on Kac–Moody groups.

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2. Rank 2 hyperbolic Kac–Moody algebras and $\mathbb{Z}$–forms

Let $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$ be the rank 2 hyperbolic Kac–Moody algebra associated with the symmetric generalized Cartan matrix

$$\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}, \qquad m \geq 3.$$

Let $\mathfrak{h} = \mathfrak{h}_{\mathbb{C}}$ be a Cartan subalgebra. Let Φ be the corresponding root system and let $\Phi_{\pm}$ denote the positive and negative roots respectively. Let

$$\mathfrak{g} \;=\; \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$$

be the triangular decomposition of $\mathfrak{g}$, where

$$\mathfrak{g}^- = \bigoplus_{\alpha \in \Phi_-} \mathfrak{g}_\alpha, \quad \mathfrak{g}^+ = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha.$$

Let $W = W(A)$ be the Weyl group of $\mathfrak{g}$. We have

$$W(A) = \langle r_1, r_2 \mid r_1^2 = 1, r_2^2 = 1 \rangle$$

which is the infinite dihedral group

$$W = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z} \rtimes \{\pm 1\},$$

where $\langle (r_1 r_2) \rangle \cong \mathbb{Z}$.

A root $\alpha \in \Phi$ is called a *real root* if there exists $w \in W$ such that $w\alpha$ is a simple root. A root α which is not real is called *imaginary*. We denote by Φ^{re} the set of real roots and Φ^{im} the set of imaginary roots.

Set $I = \{1, 2\}$. We let $\Lambda \subseteq \mathfrak{h}^*$ be the $\mathbb{Z}$–linear span of the simple roots α_i, for $i \in I$, and $\Lambda^\vee \subseteq \mathfrak{h}$ be the $\mathbb{Z}$–linear span of the simple coroots h_{α_i}, for $i \in I$. Let $e_i = e_{\alpha_i}$ and $f_i = f_{\alpha_i}$ be root vectors in $\mathfrak{g}$ corresponding to simple roots α_i, $i \in I$. Let $\mathcal{U}_{\mathbb{C}}$, $\mathcal{U}_{\mathbb{C}}^+$ and $\mathcal{U}_{\mathbb{C}}^-$ be the universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{g}^+$ and $\mathfrak{g}^-$ respectively. We define the following $\mathbb{Z}$-subalgebras: Let

(1) $\mathcal{U}_{\mathbb{Z}}^+ \subseteq \mathcal{U}_{\mathbb{C}}^+$ be the $\mathbb{Z}$-subalgebra generated by $\dfrac{e_i^n}{n!}$ for $i \in I$ and $n \geq 0$,

(2) $\mathcal{U}_{\mathbb{Z}}^- \subseteq \mathcal{U}_{\mathbb{C}}^-$ be the $\mathbb{Z}$-subalgebra generated by $\dfrac{f_i^n}{n!}$ for $i \in I$ and $n \geq 0$,

(3) $\mathcal{U}_{\mathbb{Z}}^0 \subseteq \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ be the $\mathbb{Z}$-subalgebra generated by $\begin{pmatrix} h \\ n \end{pmatrix}$, for $h \in \Lambda^\vee$ and $n \geq 0$, where

$$\begin{pmatrix} h \\ n \end{pmatrix} = \frac{h(h-1)\dots(h-n+1)}{n!},$$

(4) $\mathcal{U}_{\mathbb{Z}} \subseteq \mathcal{U}_{\mathbb{C}}$ be the $\mathbb{Z}$-subalgebra generated by $\dfrac{e_i^n}{n!}$, $\dfrac{f_i^n}{n!}$ for $i \in I$ and $\dbinom{h}{n}$, for $h \in \Lambda^\vee$ and $n \geq 0$.

It follows ([Ti1]) that $\mathcal{U}_{\mathbb{Z}}$ is a $\mathbb{Z}$–*form* of $\mathcal{U}_{\mathbb{C}}$, *i.e.* the canonical map

$$\mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathcal{U}_{\mathbb{C}}$$

is bijective.

Recall $\mathfrak{g}^+ = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$. Let V be a representation of $\mathfrak{g}$. Then V is called a *highest weight representation* with highest weight $\lambda \in \mathfrak{h}^*$ if there exists $0 \neq v_\lambda \in V$ such that

$$\mathfrak{g}^+(v_\lambda) = 0, \qquad h(v_\lambda) = \lambda(h) v_\lambda$$

for $h \in \mathfrak{h}$ and

$$V = \mathcal{U}_{\mathbb{C}} \cdot v_\lambda.$$

Since $\mathfrak{g}^+$ annihilates v_λ and $\mathfrak{h}$ acts as scalar multiplication on v_λ, we have

$$V = \mathcal{U}_{\mathbb{C}}^- \cdot v_\lambda.$$

We write $V = V^\lambda$ for the unique irreducible highest weight module with highest weight λ.

We shall construct a lattice $V_{\mathbb{Z}}$ in V by taking the orbit of a highest weight vector v_λ under $\mathcal{U}_{\mathbb{Z}}$. We have $\mathcal{U}_{\mathbb{Z}}^+ \cdot v_\lambda = \mathbb{Z} v_\lambda$ since all elements of $\mathcal{U}_{\mathbb{Z}}^+$ except for 1 annihilate v_λ. Also $\mathcal{U}_{\mathbb{Z}}^0$ acts as scalar multiplication on v_λ by a $\mathbb{Z}$–valued scalar, since $\dbinom{h}{n}$ for $h \in \Lambda^\vee$ and $n \geq 0$ acts on v_λ as

$$\binom{\lambda(h)}{n} = \frac{\lambda(h)(\lambda(h)-1)\dots(\lambda(h)-n+1)}{n!} \in \mathbb{Z}.$$

Thus

$$\mathcal{U}_{\mathbb{Z}}^0 \cdot v_\lambda = \mathbb{Z} v_\lambda, \qquad \mathcal{U}_{\mathbb{Z}} \cdot v_\lambda = \mathcal{U}_{\mathbb{Z}}^- \cdot (\mathbb{Z} v_\lambda) = \mathcal{U}_{\mathbb{Z}}^- \cdot v_\lambda.$$

Let α be any positive real root and let e_α and f_α be root vectors corresponding to α and $-\alpha$ respectively. Then

$$\frac{f_\alpha^n}{n!} v_\lambda \in V_{\lambda - n\alpha}.$$

For a weight $\mu < \lambda$ we have

$$\frac{e_\alpha^n}{n!} v_\mu \in V_{\mu + n\alpha}.$$

We set

$$V_{\mathbb{Z}} \;=\; \mathcal{U}_{\mathbb{Z}} \cdot v_\lambda \;=\; \mathcal{U}_{\mathbb{Z}}^- \cdot v_\lambda.$$

Then $V_{\mathbb{Z}}$ is a lattice in $V_{\mathbb{C}}$ and a $\mathcal{U}_{\mathbb{Z}}$-module.

For each weight μ of V, let V_μ be the corresponding weight space, and we set

$$V_{\mu,\mathbb{Z}} \;=\; V_\mu \cap V_{\mathbb{Z}}.$$

We have

$$V_{\mathbb{Z}} \;=\; \oplus_\mu V_{\mu,\mathbb{Z}},$$

where the sum is taken over the weights of V. Thus $V_{\mathbb{Z}}$ is a direct sum of its weight spaces. We set

$$V_{\mu,\mathbb{R}} \;=\; \mathbb{R} \otimes_{\mathbb{Z}} V_{\mu,\mathbb{Z}}$$

so that

$$V_{\mathbb{R}} \;:=\; \mathbb{R} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} \;=\; \oplus_\mu V_{\mu,\mathbb{R}}.$$

For each weight μ of V, we have $\mu = \lambda - (k_1\alpha_1 + k_2\alpha_2)$, where λ is the highest weight and $k_i \in \mathbb{Z}_{\geq 0}$. Define the *depth* of μ to be

$$depth(\mu) = k_1 + k_2.$$

A basis $\Psi = \{v_1, v_2, \dots\}$ of V is called *coherently ordered relative to depth* if

(1) Ψ consists of weight vectors;

(2) If $v_i \in V_\mu$, $v_j \in V_{\mu'}$ and $depth(\mu') > depth(\mu)$, then $j > i$;

(3) $\Psi \cap V_\mu$ consists of an interval $v_k, v_{k+1}, \dots, v_{k+m}$.

Theorem 2.1 ([CG, G78]). *The lattice $V_\mathbb{Z}$ has a coherently ordered $\mathbb{Z}$–basis $\{v_1, v_2, \dots\}$ where $v_i \in V_\mathbb{Z}$, $v_i = \xi_i v_\lambda$, for some $\xi_i \in \mathcal{U}_\mathbb{Z}$. Let $w_i = k_i \otimes v_i$, $k_i \in \mathbb{R} \setminus \{0\}$. Then the set $\{w_1, w_2, \dots\}$ is a coherently ordered basis for $V_\mathbb{R}$. Any vector in $V_\mathbb{Z}$ has an integer valued norm relative to a Hermitian inner product $\langle , \rangle$ on V.*

3. The Kac–Moody group G and Iwasawa decomposition

Our next step is to construct our Kac–Moody group G over $\mathbb{R}$. The construction below can be used to construct G over any field F [CG]. As before, let V be a highest weight module for $\mathfrak{g}_\mathbb{C}$. Then the simple root vectors e_i and f_i are locally nilpotent on V.

We let $V_\mathbb{Z}$ be a $\mathbb{Z}$–form of V as in Section 2. Since $V_\mathbb{Z}$ is a $\mathcal{U}_\mathbb{Z}$-module, we have

$$\frac{e_i^n}{n!}(V_\mathbb{Z}) \subseteq V_\mathbb{Z} \quad \text{and} \quad \frac{f_i^n}{n!}(V_\mathbb{Z}) \subseteq V_\mathbb{Z} \quad \text{for} \quad n \in \mathbb{N},\ i \in I.$$

Let $V_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} V_\mathbb{Z}$. For $s, t \in \mathbb{R}$ and $i \in I$, set

$$\chi_{\alpha_i}(s) \ = \ \sum_{n=0}^{\infty} s^n \frac{e_i^n}{n!} \ = \exp(se_i), \qquad \chi_{-\alpha_i}(t) \ = \ \sum_{n=0}^{\infty} t^n \frac{f_i^n}{n!} \ = \exp(tf_i).$$

Then $\chi_{\alpha_i}(s)$, $\chi_{-\alpha_i}(t)$ define elements in $Aut(V_\mathbb{R})$, thanks to the local nilpotence of e_i, f_i. More generally, for a real root α, we choose a root vector $x_\alpha \in \mathfrak{g}_\alpha$ and define

$$\chi_\alpha(s) = \exp(sx_\alpha) \in Aut(V_\mathbb{R}), \qquad s \in \mathbb{R}.$$

For $t \in \mathbb{R}^\times$, we set

$$w_{\alpha_i}(t) = \chi_{\alpha_i}(t)\chi_{-\alpha_i}(-t^{-1})\chi_{\alpha_i}(t) \quad \text{for } i \in I,$$

and define

$$h_{\alpha_i}(t) = w_{\alpha_i}(t)w_{\alpha_i}(1)^{-1}.$$

We let $G_\mathbb{R}$ be the subgroup of $Aut(V_\mathbb{R})$ generated by the linear automorphisms $\chi_{\alpha_i}(s)$ and $\chi_{-\alpha_i}(t)$ of $V_\mathbb{R}$, for $s, t \in \mathbb{R}$, $i \in I$. That is, we define

$$G_\mathbb{R} = \langle \exp(se_i),\ \exp(tf_i) : s, t \in \mathbb{R},\ i \in I\rangle.$$

One can see that $\chi_\alpha(s) \in G_\mathbb{R}$ for real roots α. By replacing $\mathbb{R}$ with F in the above construction, we obtain the group G_F for any field F.

We define the following subgroups of $G_\mathbb{R}$:

(1) $K = \{k \in G_\mathbb{R} : k$ preserves the inner product $\langle , \rangle$ on $V_\mathbb{R}\}$,
(2) $A = \langle h_{\alpha_i}(s) : s \in \mathbb{R}^\times, i \in I\rangle$, $\qquad A^+ = \langle h_{\alpha_i}(s) : s \in \mathbb{R}_+, i \in I\rangle$,
(3) $N = \langle \chi_\alpha(s) : \alpha \in \Phi_+^{re}, s \in \mathbb{R}\rangle$, where Φ_+^{re} is the set of positive real roots.

Theorem 3.1 ([CG, DGH]). *We have the Iwasawa decomposition:*

$$G_{\mathbb{R}} = KA^+N \tag{3.1}$$

with uniqueness of expression.

As in [CG], we now define the '$\mathbb{Z}$–form' $G_{\mathbb{Z}}$ of $G_{\mathbb{R}}$ in the following way. We set

$$G_{\mathbb{Z}} = G_{\mathbb{R}} \cap Aut(V_{\mathbb{Z}}).$$

Then

$$G_{\mathbb{Z}} = \{\gamma \in G_{\mathbb{R}} : \gamma \cdot V_{\mathbb{Z}} \subseteq V_{\mathbb{Z}}\}.$$

Remark 3.2. For a discussion on dependence on the choice of V and $V_{\mathbb{Z}}$, we refer the reader to [CG]. In this paper, we work with fixed V and $V_{\mathbb{Z}}$.

4. Eisenstein series on rank 2 hyperbolic Kac–Moody groups

Let $g = k_g a_g n_g \in G_{\mathbb{R}}$ be the Iwasawa decomposition according to (3.1). Let $\nu : A^+ \to \mathbb{C}^\times$ be a quasi–character and define

$$\Phi_\nu : G_{\mathbb{R}} \to \mathbb{C}^\times$$

to be the function

$$\Phi_\nu(g) = \nu(a_g)$$

Then Φ_ν is well defined since the Iwasawa decomposition is unique and Φ_ν is left K–invariant and right N–invariant. For convenience, we write $\Gamma = G_{\mathbb{Z}}$.

Let B denote the minimal parabolic subgroup of $G_{\mathbb{R}}$. Relative to a coherently ordered basis Ψ for $V^\lambda_{\mathbb{Z}}$, the group Γ has a representation in terms of infinite matrices with integral entries. Define the Eisenstein series on $G_{\mathbb{R}}$ to be the infinite formal sum

$$E_\nu(g) \quad := \sum_{\gamma \in \Gamma/\Gamma\cap B} \Phi_\nu(g\gamma).$$

Recall that $\mathfrak{h}$ is the Lie algebra of A and that h_{α_i}, $i \in I$, are the simple coroots. We say that ν satisfies *Godement's criterion* if

$$\text{Re } \nu(h_{\alpha_i}) < -2, \quad i \in I.$$

We do not expect that the Eisenstein series will be convergent over the whole space $K\backslash G_{\mathbb{R}}/\Gamma$ but rather a subspace, where if $G_{\mathbb{R}} = KA^+N$ is decomposed in terms of the Iwasawa decomposition, the A^+–component is replaced by the 'group' corresponding to the Tits cone to obtain $G'_{\mathbb{R}} = KA'N$. Godement's criterion places the sum in a cone in the region Re $\nu(h_{\alpha_i}) < -2$ for each i. In the next section, we deduce almost everywhere convergence of the Eisenstein series from convergence of the constant term.

We will not comment on the difficult question of meromorphic continuation to the whole complex plane here. However, we will prove in §7 that cuspidal Eisenstein series are entire.

5. Convergence of the constant term

In this section we prove convergence of the constant term and thus almost everywhere convergence of the Eisenstein series itself. Assume first that $\nu : A^+ \to \mathbb{C}^\times$ is real valued and positive. Then we may interpret the infinite sum $E_\nu(g)$ as a function taking values in $\mathbb{R}_+ \cup \{\infty\}$. The function E_ν may be regarded as a function on

$$K\backslash G_{\mathbb{R}}/\Gamma \cap N \cong A^+ \times N/\Gamma \cap N.$$

Under the identification $\mathbb{R}_+^2 \cong A^+$:

$$(x_1, x_2) \mapsto h_{\alpha_1}(x_1)h_{\alpha_2}(x_2),$$

we have the measure da on A^+, corresponding to the measure

$$\frac{dx_1}{x_1}\frac{dx_2}{x_2}$$

on $\mathbb{R}_+^2$. As in [G04] we know that $N/\Gamma \cap N$ is the projective limit of a projective family of finite–dimensional compact nil–manifolds and thus admits a projective limit measure dn, which is a left N–invariant probability measure. We define for all $g \in G_{\mathbb{R}}$ the constant term

$$E_\nu^\sharp(g) = \int_{N/\Gamma\cap N} E_\nu(gn)dn$$

which is left K–invariant and right N–invariant. In particular $E_\nu^\sharp(g)$ is determined by the A^+–component of g in the Iwasawa decomposition. Let $\rho \in \mathfrak{h}^*$ satisfying $\rho(h_{\alpha_i}) = 1$, $i \in I$. Then

$$\rho = \frac{\alpha_1 + \alpha_2}{2 - m}.$$

Applying the Gindikin–Karpelevich formula, a formal calculation as in [G04] yields that for $a \in A^+$

$$E_\nu^\sharp(a) = \sum_{w\in W} a^{w(\nu+\rho)-\rho}c(\nu, w),$$

where

$$c(\nu, w) = \prod_{\alpha\in\Phi_+\cap w^{-1}\Phi_-} \frac{\xi(-(\nu + \rho)(h_\alpha))}{\xi(1 - (\nu + \rho)(h_\alpha))},$$

and $\xi(s)$ is the completed Riemann zeta function

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\prod_p \frac{1}{1 - p^{-s}}.$$

Before proving the convergence of the constant term, let us first give some preliminaries for the structure of the root system of $\mathfrak{g}$, following [KM]. Let

$$\gamma = \frac{m + \sqrt{m^2 - 4}}{2}, \tag{5.1}$$

which is a root of the polynomial $x^2 - mx + 1$. Let r_1, r_2 be the simple reflections corresponding to the simple roots α_1, α_2. Then the Weyl group W is generated by r_1, r_2 subject to the relations $r_1^2 = r_2^2 = 1$, and has an explicit description

$$W = \{1, r_1(r_2r_1)^n, r_2(r_1r_2)^n, (r_1r_2)^{n+1}, (r_2r_1)^{n+1} : n \geq 0\}.$$

We introduce a sequence $\{A_n\}$ defined by

$$A_0 = 0, \quad A_1 = 1, \quad A_{n+2} = aA_{n+1} - A_n + 1, \quad n \geq 0.$$

Then we have the explicit formula

$$A_n = \frac{\gamma^{2n+1} - \gamma^n(1 + \gamma) + 1}{\gamma^{n-1}(\gamma + 1)(\gamma - 1)^2} = \frac{\gamma^{n+2}}{(\gamma + 1)(\gamma - 1)^2} + O(1), \quad n \geq 0. \tag{5.2}$$

We also need another sequence

$$B_n = A_n - A_{n-1} = \frac{\gamma^{2n} - 1}{\gamma^{n-1}(\gamma^2 - 1)} = \frac{\gamma^{n+1}}{\gamma^2 - 1} + o(1), \quad n \geq 0. \tag{5.3}$$

Then we have the following formulas for the actions of $r_1(r_2r_1)^n$ and $(r_1r_2)^{n+1}$ on simple roots:

$$\begin{cases} r_1(r_2r_1)^n\alpha_1 = -B_{2n+1}\alpha_1 - B_{2n}\alpha_2, \\ r_1(r_2r_1)^n\alpha_2 = B_{2n+2}\alpha_1 + B_{2n+1}\alpha_2. \end{cases} \tag{5.4}$$

$$\begin{cases} (r_1r_2)^{n+1}\alpha_1 = B_{2n+3}\alpha_1 + B_{2n+2}\alpha_2, \\ (r_1r_2)^{n+1}\alpha_2 = -B_{2n+2}\alpha_1 - B_{2n+1}\alpha_2. \end{cases} \tag{5.5}$$

Switching α_1 and α_2, we may also obtain the similar actions of $r_2(r_1r_2)^n$ and $(r_2r_1)^{n+1}$. Regarding $w\rho - \rho$ we have

$$\begin{cases} -r_1(r_2r_1)^n\rho - \rho = -A_{2n+1}\alpha_1 - A_{2n}\alpha_2, \\ r_2(r_1r_2)^n\rho - \rho = -A_{2n}\alpha_1 - A_{2n+1}\alpha_2, \\ (r_1r_2)^{n+1}\rho - \rho = -A_{2n+2}\alpha_1 - A_{2n+1}\alpha_2, \\ (r_2r_1)^{n+1}\rho - \rho = -A_{2n+1}\alpha_1 - A_{2n+2}\alpha_2. \end{cases} \tag{5.6}$$

Theorem 5.1. *Assume that ν satisfies Godement's criterion. Then the constant term $E^\sharp_\nu(g)$ converges absolutely for $g \in KA'N$, where A' is the cone*

$$A' = \{a \in A^+ : a^{\alpha_i} < 1, i \in I\}.$$

Proof. We may without loss of generality assume that $\nu(h_{\alpha_i})$ has real values, $i \in I$. Then we may write

$$\nu = s_1\alpha_1 + s_2\alpha_2$$

where $s_1, s_2 \in \mathbb{R}$. Godement's criterion then reads

$$\nu(h_{\alpha_1}) = 2s_1 - ms_2 < -2, \quad \nu(h_{\alpha_2}) = 2s_2 - ms_1 < -2. \tag{5.7}$$

In particular we have $s_1, s_2 > 0$. Let us consider a typical term

$$a^{w(\nu+\rho)-\rho}c(\nu, w)$$

in $E^\sharp_\nu(a)$, where $a \in A'$. By symmetry we only need to consider $w = r_1(r_2r_1)^n$ and $w = (r_1r_2)^{n+1}$, $n \geq 0$.

For $w = r_1(r_2r_1)^n$, by (5.4) and (5.6) we have

$$w(\nu + \rho) - \rho = (-s_1B_{2n+1} + s_2B_{2n+2} - A_{2n+1})\alpha_1 + (-s_1B_{2n} + s_2B_{2n+1} - A_{2n})\alpha_2.$$

By (5.2) and (5.3) we have

$$\begin{aligned} & -s_1B_{2n+1} + s_2B_{2n+2} - A_{2n+1} \\ = \;& (\gamma s_2 - s_1)\frac{\gamma^{2n+2}}{\gamma^2 - 1} - \frac{\gamma^{2n+3}}{(\gamma+1)(\gamma-1)^2} + O(1) \\ = \;& (\gamma s_2 - s_1 - \frac{\gamma}{\gamma - 1})\frac{\gamma^{2n+2}}{\gamma^2 - 1} + O(1). \end{aligned}$$

From (5.7) it follows that

$$\begin{aligned} \gamma s_2 - s_1 &= \frac{m\gamma - 2}{4 - m^2}(2s_1 - ms_2) + \frac{2\gamma - m}{4 - m^2}(2s_2 - ms_1) \\ &> \frac{2(m\gamma - 2 + 2\gamma - m)}{m^2 - 4} \\ &= \frac{2(\gamma - 1)}{m - 2} = \frac{2\gamma}{\gamma - 1}, \end{aligned}$$

where the last equation follows from $\gamma^2 - m\gamma + 1 = 0$. If we introduce a constant

$$C_\nu = (\gamma s_2 - s_1 - \frac{\gamma}{\gamma - 1})\frac{1}{\gamma^2 - 1} > 0,$$

then

$$r_1(r_2 r_1)^n(\nu + \rho) - \rho = (C_\nu \gamma^{2n+2} + O(1))\alpha_1 + (C_\nu \gamma^{2n+1} + O(1))\alpha_2.$$

Similarly, we have

$$(r_1 r_2)^{n+1}(\nu + \rho) - \rho = (D_\nu \gamma^{2n+3} + O(1))\alpha_1 + (D_\nu \gamma^{2n+2} + O(1))\alpha_2,$$

where

$$D_\nu = (\gamma s_1 - s_2 - \frac{\gamma}{\gamma - 1})\frac{1}{\gamma^2 - 1} > 0.$$

By Godement's criterion we have

$$-\text{Re }(\nu + \rho)(h_{\alpha_i}) > 1$$

for $i \in I$. Then there exists $\varepsilon > 0$ such that

$$-\text{Re }(\nu + \rho)(h_\alpha) > 1 + \varepsilon$$

for any real root α. Using properties of the Riemann zeta function we can find a constant $C_\varepsilon > 0$ depending on ε such that

$$|c(\nu, w)| < C_\varepsilon^{\ell(w)}$$

where $\ell(w)$ is the length of w.

Since $a \in A'$, we have $a^{\alpha_i} < 1$, $i \in I$. Combining above estimates we see that the series defining $E_\nu^\sharp(a)$ converges absolutely. □

Corollary 5.2. *Assume that ν satisfies Godement's criterion. Then for any compact subset A'_c of A', there is a measure zero subset N_0 of N such that $E_\nu(g)$ converges absolutely for $g \in KA'_cN'$, where $N' = N - N_0$.*

We propose the following conjecture, which weakens Godement's criterion and asserts everywhere convergence instead of almost everywhere convergence.

Conjecture 5.3. *$E_\nu(g)$ converges absolutely for $g \in KA'N$ and ν satisfying* Re $\nu(h_{\alpha_i}) < -1$, *$i \in I$.*

6. Fourier coefficients

In this section we shall define and calculate other Fourier coefficients of $E_\nu(g)$. To facilitate the computation, we work with adelic groups. Let $\mathbb{A} = \mathbb{R} \times \prod'_p \mathbb{Q}_p$ and $\mathbb{I} = \mathbb{A}^\times$ be the adele ring and idele group of the rational number field $\mathbb{Q}$, respectively. According to Section 3, for any prime p we have the group $G_{\mathbb{Q}_p} \subset Aut(V_{\mathbb{Q}_p})$, and we let $K_p \subset G_{\mathbb{Q}_p}$ be the subgroup $K_p = \{g \in G_{\mathbb{Q}_p} : g \cdot V_{\mathbb{Z}_p} = V_{\mathbb{Z}_p}\}$. Let $G_\mathbb{A} = G_\mathbb{R} \times \prod'_p G_{\mathbb{Q}_p}$ and $G_{\mathbb{A}_f} = \prod'_p G_{\mathbb{Q}_p}$ (the adele and finite adele groups respectively) be the restricted products with respect to the family of subgroups K_p. Note that we have the diagonal embedding $\iota : G_\mathbb{Q} \hookrightarrow G_\mathbb{R} \times \prod_p G_{\mathbb{Q}_p}$. Set $\Gamma_\mathbb{Q} = \iota^{-1}(G_\mathbb{A})$ and $K_\mathbb{A} = K \times \prod_p K_p$.

We shall extend the definition of $E_\nu(g)$ to $g \in G_\mathbb{A}$. For each prime p we have an Iwasawa decomposition [DGH]

$$G_{\mathbb{Q}_p} = K_p A_{\mathbb{Q}_p} N_{\mathbb{Q}_p},$$

where $A_{\mathbb{Q}_p}$ is generated by $h_{\alpha_i}(s)$, $i = 1, 2$, $s \in \mathbb{Q}_p^\times$, and $N_{\mathbb{Q}_p}$ is generated by $\chi_\alpha(s)$, $\alpha \in \Phi_+^{re}$, $s \in \mathbb{Q}_p$. From the local Iwasawa decompositions we have

$$G_{\mathbb{A}} = K_{\mathbb{A}} A_{\mathbb{A}} N_{\mathbb{A}}.$$

If $\iota = (\iota_\infty, \iota_p) \in \mathbb{I}$ is an idele, define the usual norm $|\iota|$ of ι by

$$|\iota| = |\iota_\infty| \prod_p |\iota_p|_p.$$

An element $a \in A_{\mathbb{A}}$ can be decomposed as $a = h_{\alpha_1}(s_1) h_{\alpha_2}(s_2)$, $s_1, s_2 \in \mathbb{I}$. We let $|a| \in A^+$ be the element $h_{\alpha_1}(|s_1|) h_{\alpha_2}(|s_2|)$. Let $g \in G_{\mathbb{A}}$ be decomposed as

$$g = k_g a_g n_g, \quad k_g \in K_{\mathbb{A}},\ a_g \in A_{\mathbb{A}},\ n_g \in N_{\mathbb{A}}.$$

Note that $|a_g|$ is uniquely determined by g, although above decomposition is not unique. Then we may define

$$\Phi_\nu(g) = |a_g|^\nu.$$

The Eisenstein series is defined by

$$E_\nu(g) = \sum_{\gamma \in \Gamma_{\mathbb{Q}} / \Gamma_{\mathbb{Q}} \cap B_{\mathbb{Q}}} \Phi_\nu(g\gamma), \quad g \in \mathbb{A}.$$

When $g \in G_{\mathbb{R}}$ this coincides with our previous definition, since $\Gamma_{\mathbb{Q}} / \Gamma_{\mathbb{Q}} \cap B_{\mathbb{Q}} \cong \Gamma / \Gamma \cap B$.

For a positive real root α, let U_α be the root subgroup $\{\chi_\alpha(u) : u \in \mathbb{R}\}$, where $\chi_\alpha(u) = \exp(u x_\alpha)$ for a root vector $x_\alpha \in \mathfrak{g}$ corresponding to α.

Let ψ be a non-trivial character of $N/\Gamma \cap N$. Then we have $\psi = \psi_1 \psi_2$, where ψ_i is a character of $U_{\alpha_i}/\Gamma \cap U_{\alpha_i}$. This follows from the fact that $N/[N, N] \cong U_{\alpha_1} \times U_{\alpha_2}$. We extend ψ to a character of $N_{\mathbb{A}}/N_{\mathbb{Q}}$, where $N_{\mathbb{Q}} := \Gamma_{\mathbb{Q}} \cap N_{\mathbb{A}}$ and define the ψ-th Fourier coefficient of $E_\nu(g)$ along B by

$$E_{\nu,\psi}(g) = \int_{N/\Gamma \cap N} E_\nu(gn) \bar{\psi}(n) dn = \int_{N_{\mathbb{A}}/N_{\mathbb{Q}}} E_\nu(gn) \bar{\psi}(n) dn.$$

Then $E_{\nu,\psi}(g)$ is a Whittaker function on G, that is, a function W satisfying the relation

$$W(gn) = \psi(n) W(g),$$

for each $n \in N$.

We call ψ *generic* if each ψ_i is non-trivial for $i = 1, 2$. Then we have the following vanishing result for generic characters, which in fact holds generally for infinite—dimensional Kac–Moody groups (cf. [Li]).

Lemma 6.1. *If ψ is generic, then $E_{\nu,\psi}(g) = 0$.*

Proof. Recall that we have the Bruhat decomposition

$$G = BWB = \bigsqcup_{w \in W} N_w w B,$$

where $N_w = \prod_{\alpha \in \Phi_+ \cap w\Phi_-} U_\alpha$. Then we have

$$E_\nu(g) = \sum_{\gamma \in \Gamma/\Gamma \cap B} \Phi_\nu(g\gamma) = \sum_{w \in W} \sum_{\gamma \in N_{w,\mathbb{Q}}} \Phi_\nu(g\gamma w).$$

We introduce $N'_w = \prod_{\alpha\in\Phi_+\cap w\Phi_+} U_\alpha$. Then $N = N_w N'_w$ and it follows that

$$\begin{aligned} E_{\nu,\psi}(g) &= \sum_{w\in W}\int_{N_{\mathbb{A}}/N_{\mathbb{Q}}}\sum_{\gamma\in N_{w,\mathbb{Q}}}\Phi_\nu(gn\gamma w)\bar{\psi}(n)dn \\ &= \sum_{w\in W}\int_{N_{\mathbb{A}}/N'_{w,\mathbb{Q}}}\Phi_\nu(gnw)\bar{\psi}(n)dn \\ &= \sum_{w\in W}\int_{N_{w,\mathbb{A}}}\bar{\psi}(n_w)\int_{N'_{w,\mathbb{A}}/N'_{w,\mathbb{Q}}}\Phi_\nu(gn_w n'_w w)\bar{\psi}(n'_w)dn'_w dn_w. \end{aligned}$$

For each $w \in W$, at least one of the two roots $w^{-1}\alpha_i$, $i = 1, 2$, is positive. Since Φ_ν is right N–invariant, the inner integral of the last equation involves a factor

$$\int_{U_{\alpha_i,\mathbb{A}}/U_{\alpha_i,\mathbb{Q}}}\bar{\psi}_i(u)du = \int_{U_{\alpha_i}/\Gamma\cap U_{\alpha_i}}\bar{\psi}_i(u)du$$

for some i, which is zero by the assumption that ψ is generic. □

Using this lemma, we may assume that $\psi = \psi_1$ or ψ_2, and is non-trivial. Any character of U_{α_i}, which is trivial on $\Gamma \cap U_{\alpha_i}$, is of the form

$$\psi_{i,n} : \chi_{\alpha_i}(u) \mapsto e^{2\pi i n u}, \quad u \in \mathbb{R}$$

for some $n \in \mathbb{Z}$.

Before we state and prove the main result of this section, let us first recall some Fourier coefficients for SL_2. For $F = \mathbb{R}$ or $\mathbb{Q}_p$, one has the Iwasawa decomposition $SL_2(F) = KAN$, where $K = SO(2,\mathbb{R})$ or $SL_2(\mathbb{Z}_p)$ is a maximal compact subgroup of $SL_2(F)$,

$$A = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} : a \in F^\times\right\}, \quad N = \left\{\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} : x \in F\right\}.$$

Let $g \in SL_2(F)$ be decomposed as

$$g = k\begin{pmatrix} a_g & 0 \\ 0 & a_g^{-1}\end{pmatrix} n.$$

For $s \in \mathbb{C}$, define a function $\Phi_s(g)$ on $SL_2(F)$ by $\Phi_s(g) = |a_g|^{-s}$. Clearly Φ_s is well–defined. Given a character ψ of F, we shall consider the Fourier coefficient

$$\int_F \Phi_s\begin{pmatrix} 1 & 0 \\ x & 1\end{pmatrix}\bar{\psi}(x)dx, \tag{6.1}$$

which is convergent for Re $s > 0$. If we write $\begin{pmatrix} a_x & 0 \\ 0 & a_x^{-1}\end{pmatrix}$ for the A–component of $\begin{pmatrix} 1 & 0 \\ x & 0\end{pmatrix}$ in the Iwasawa decomposition, then $|a_x| = \sqrt{1+x^2}$ for $F = \mathbb{R}$ and $|a_x| = \max(1, |x|_p)$ for $F = \mathbb{Q}_p$.

The character $\psi(u) = e^{2\pi i u}$ of $\mathbb{R}/\mathbb{Z}$ corresponds to the character $\psi_\infty \prod_p \psi_p$ of $\prod_p \mathbb{Z}_p\backslash\mathbb{A}/\mathbb{Q}$, where

$$\begin{aligned} &\psi_\infty(x) = e^{2\pi i x}, \quad x \in \mathbb{R} \\ &\psi_p(x) = e^{-2\pi i(\text{fractional part of } x)}, \quad x \in \mathbb{Q}_p. \end{aligned}$$

Fix $y \in \mathbb{R}_+$ and associate an idele $(y_v)_v \in \mathbb{I}$ by $y_\infty = y$, $y_p = 1$. For $n \in \mathbb{Z}$, $n \neq 0$, we twist the nth power of ψ by y, *i.e.* consider the characters $\psi_\infty(nyx)$ of $\mathbb{R}$ and $\psi_p(nx)$ of $\mathbb{Q}_p$. Then the Fourier coefficients (6.1) are given by the following functions. For $F=\mathbb{R}$, Re $s>1$, we have (cf. [Bu, pp. 66–67])

$$\begin{aligned} W_n^\infty(y,s) &= \int_{-\infty}^{\infty}(1+x^2)^{-\frac{s}{2}}\bar{\psi}_\infty(nyx)dx \\ &= 2\pi^{s/2}\Gamma(s/2)^{-1}|ny|^{\frac{s-1}{2}}K_{\frac{s-1}{2}}(2\pi|n|y), \end{aligned}$$

where $K_s(y)$ is the *K–Bessel function*, also known as the *Macdonald Bessel function*, defined by

$$K_s(y)=\frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2}\ t^s\frac{dt}{t}, \quad \text{Re } s>0.$$

Write $|n|=\prod_p p^{n_p}$ into the primary decomposition. Then for $p<\infty$, Re $s>1$, we have

$$\begin{aligned} W_n^p(s) &= 1+\sum_{i=1}^{\infty}p^{-is}\int_{p^{-i}\mathbb{Z}_p^\times}\bar{\psi}_p(nx)dx \\ &= \frac{(1-p^{-s})(1-p^{(n_p+1)(1-s)})}{1-p^{1-s}}. \end{aligned}$$

In the above computations we have made use of the Iwasawa decomposition for SL_2. Now we form a product

$$W_n(y,s)=W_n^\infty(y,s)\prod_p W_n^p(s)=2\sigma_{1-s}(|n|)|ny|^{\frac{s-1}{2}}K_{\frac{s-1}{2}}(2\pi|n|y)\frac{1}{\xi(s)}, \quad \text{Re } s>1, \tag{6.2}$$

where σ_s is the divisor power sum function defined by $\sigma_s(n)=\sum_{d|n}d^s$ for $n\in\mathbb{N}$.

Theorem 6.2. *Assume that ν satisfies Godement's criterion. Then for $a\in A'$, $i\in I$, $n\in\mathbb{Z}$, $n\neq 0$, one has the $\psi_{i,n}$-th Fourier coefficient*

$$E_{\nu,\psi_{i,n}}(a)=\sum_{w\in W,\ w^{-1}\alpha_i<0}a^{w(\nu+\rho)-\rho}c_{\psi_{i,n}}(\nu,w)(a),$$

where

$$c_{\psi_{i,n}}(\nu,w)(a)=W_n(a^{-\alpha_i},1+w(\nu+\rho)(h_{\alpha_i}))\prod_{\substack{\alpha\in\Phi_+\cap w^{-1}\Phi_- \\ \alpha\neq -w^{-1}\alpha_i}}\frac{\xi(-(\nu+\rho)(h_\alpha))}{\xi(1-(\nu+\rho)(h_\alpha))}$$

with $W_n(y,s)$ being defined by (6.2).

Proof. As in the proof of Lemma 6.1, we see that

$$E_{\nu,\psi}(a)=\sum_{w\in W,\ w^{-1}\alpha_i<0}\int_{N_{w,\mathbb{A}}}\Phi_\nu(an_ww)\bar{\psi}(n_w)dn_w.$$

We follow the computation in [G04] and [Li, 4.4]. Let $w^{-1}=r_{k_1}\cdots r_{k_\ell}$ be the reduced expression of w^{-1}, where $\ell=\ell(w)$ and $k_j=1$ or 2 for $j=1,\dots,\ell$. Let

$$\Phi_{w^{-1}}=\Phi_+\cap w^{-1}\Phi_-=\{\beta_1,\dots,\beta_\ell\},$$

where $\beta_j=r_{k_1}\cdots r_{k_{j-1}}\alpha_{k_j}$. Then

$$\Phi_w=\Phi_+\cap w\Phi_-=\{\gamma_1,\dots,\gamma_\ell\},$$

where $\gamma_j=-w\beta_j=r_{k_\ell}\cdots r_{k_{j+1}}\alpha_{k_j}$. Note that

$$\beta_1+\cdots+\beta_\ell=\rho-w^{-1}\rho, \quad \gamma_1+\cdots+\gamma_\ell=\rho-w\rho.$$

From these formulas it is clear that if $w^{-1}\alpha_i < 0$, then $\gamma_\ell = \alpha_i = -w\beta_\ell$. By decomposing N_w into a product of root subgroups, we get

$$\begin{aligned}
&\int_{N_{w,\mathbb{A}}} \Phi_\nu(an_w w)\bar{\psi}(n_w)dn_w \\
=\ &\int_{\mathbb{A}^\ell} \Phi_\nu\big(a\chi_{\gamma_\ell}(u_\ell)\cdots\chi_{\gamma_1}(u_1)w\big)\bar{\psi}_{i,n}(u_\ell)du_\ell\cdots du_1 \\
=\ &\int_{\mathbb{A}^\ell} \Phi_\nu\big(\chi_{\gamma_\ell}(a^{\gamma_\ell}u_\ell)\cdots\chi_{\gamma_1}(a^{\gamma_1}u_1)aw\big)\bar{\psi}_{i,n}(u_\ell)du_\ell\cdots du_1 \\
=\ &\int_{\mathbb{A}^\ell} a^{w\nu-\gamma_1-\cdots-\gamma_\ell}\Phi_\nu\big(\chi_{\gamma_\ell}(u_\ell)\cdots\chi_{\gamma_1}(u_1)w\big)\bar{\psi}_{i,n}(a^{-\alpha_i}u_\ell)du_\ell\cdots du_1 \\
=\ &a^{w(\nu+\rho)-\rho}\int_{\mathbb{A}^\ell} \Phi_\nu\big(\chi_{-\beta_\ell}(u_\ell)\cdots\chi_{-\beta_1}(u_1)\big)\bar{\psi}_{i,n}(a^{-\alpha_i}u_\ell)du_\ell\cdots du_1.
\end{aligned}$$

Let $\chi_{-\beta_\ell}(u) = k(u)a(u)n(u)$ be the Iwasawa decomposition, with $k(u)\in K$, $n(u)\in U_{\beta_\ell}$, $a(u)\in A$. Put $w'^{-1} = r_{k_1}\cdots r_{k_{\ell-1}}$, then $\{\beta_1,\ldots,\beta_{\ell-1}\} = \Phi_+\cap w'^{-1}\Phi_-$. Consider the decomposition $N = N_{w'}N'_{w'}$ (see the proof of Lemma 6.1). Then we have

$$U_{-\beta_{\ell-1}}\cdots U_{-\beta_1} = w'^{-1}N_{w'}w'.$$

Let us define the projection

$$\pi : w'^{-1}Nw' \to w'^{-1}N_{w'}w'.$$

Since $U_{-\beta_1},\ldots,U_{-\beta_{\ell-1}}$, $U_{\beta_\ell}\subset w'^{-1}Nw'$, the following map

$$\pi\circ\mathrm{Ad}(n(u)) : w'^{-1}N_{w'}w' \to w'^{-1}N_{w'}w',$$

is well–defined and unimodular. From this fact, and noting that Φ_ν is right invariant under $w'^{-1}N'_{w'}w'\subset N$, it follows

$$\begin{aligned}
&\int_{\mathbb{A}^\ell} \Phi_\nu\big(\chi_{-\beta_\ell}(u_\ell)\cdots\chi_{-\beta_1}(u_1)\big)\bar{\psi}_{i,n}(a^{-\alpha_i}u_\ell)du_\ell\cdots du_1 \\
=\ &\int_{\mathbb{A}} a(u_\ell)^{\nu+\beta_1+\cdots\beta_{\ell-1}}\bar{\psi}_{i,n}(a^{-\alpha_i}u_\ell)du_\ell\int_{\mathbb{A}^{\ell-1}}\Phi_\nu\big(\chi_{-\beta_{\ell-1}}(u_{\ell-1})\cdots\chi_{-\beta_1}(u_1)\big)du_{\ell-1}\cdots du_1 \\
=\ &\int_{\mathbb{A}} a(u_\ell)^{\nu+\rho-w'^{-1}\rho}\bar{\psi}_{i,n}(a^{-\alpha_i}u_\ell)du_\ell\int_{\mathbb{A}^{\ell-1}}\Phi_\nu\big(\chi_{-\beta_{\ell-1}}(u_{\ell-1})\cdots\chi_{-\beta_1}(u_1)\big)du_{\ell-1}\cdots du_1.
\end{aligned}$$

Note that $w'^{-1}\rho(h_{\beta_\ell}) = \rho(w'h_{\beta_\ell}) = \rho(h_{\alpha_i}) = 1$. Then the first integral in the last equation equals $W_n(a^{-\alpha_i}, 1-(\nu+\rho)(h_{\beta_\ell})) = W_n(a^{-\alpha_i}, 1+w(\nu+\rho)(h_{\alpha_i}))$, and the second one, by Gindikin–Karpelevich formula, equals

$$\prod_{j=1}^{\ell-1}\frac{\xi\big(-(\nu+\rho)(h_{\beta_j})\big)}{\xi\big(1-(\nu+\rho)(h_{\beta_j})\big)}.$$

Hence our assertion follows. □

7. Entirety of cuspidal Eisenstein series

In analogy with [BK, Theorem 5.2] and [GMP], we shall prove that the Eisenstein series on G induced from cusp forms on SL_2 are entire functions. Let P be the maximal parabolic subgroup of G generated by B and the simple reflection r_1, and $P = MU$ be the Levi decomposition, where M is the Levi subgroup and U is the pro–unipotent radical of P. Let $L\cong SL_2$ be the subgroup generated by $\chi_{\pm\alpha_1}(t)$, $t \in \mathbb{R}$, and let $A_1 = \langle h_{\alpha_1}(t) : t \in \mathbb{R}^\times \rangle$, $H = \{a \in A : a^{\alpha_1} = \pm 1\}$. Then we have an almost direct product $A = A_1 \times H$, and $M = LH$. We introduce

$$H^+ = \{h_{\alpha_1}(t^m)h_{\alpha_2}(t^2) : t \in \mathbb{R}_+\} \cong K \cap H\backslash H,$$
$$A_1^+ = \{h_{\alpha_1}(t) : t \in \mathbb{R}_+\} \cong K \cap A_1\backslash A_1.$$

From the Iwasawa decomposition $G = KP = KMU$, the following maps

$$\mathrm{Iw}_L : G \to K \cap L\backslash L, \quad \mathrm{Iw}_{H^+} : G \to H^+$$

are well–defined. Similarly, using the Iwasawa decomposition for L we may define the map

$$\mathrm{Iw}_{A_1^+} : K \cap L\backslash L \to A_1^+.$$

For convenience we also denote by Iw_{A^+} the map $G \to A^+$, $g = k_g a_g n_g \mapsto a_g$ which we used previously. Then it is clear that

$$\mathrm{Iw}_{A^+} = (\mathrm{Iw}_{A_1^+} \circ \mathrm{Iw}_L) \times \mathrm{Iw}_{H^+} : G \to A^+ \cong A_1^+ \times H^+. \tag{7.1}$$

Let ϖ_2 be the fundamental weight corresponding to α_2, that is

$$\varpi_2 = \frac{m\alpha_1 + 2\alpha_2}{4 - m^2}.$$

Note that ϖ_2 is trivial on $A_1 = L \cap A$, hence (7.1) implies that

$$\mathrm{Iw}_{A^+}(\cdot)^{\varpi_2} = \mathrm{Iw}_{H^+}(\cdot)^{\varpi_2}. \tag{7.2}$$

Similarly, since α_1 is trivial on H we also have

$$\mathrm{Iw}_{A^+}(\cdot)^{\alpha_1} = \mathrm{Iw}_{A_1^+} \circ \mathrm{Iw}_L(\cdot)^{\alpha_1}. \tag{7.3}$$

We may regard ϖ_2 as an algebraic character of M. For $s \in \mathbb{C}$, define the Eisenstein series

$$E_s(g) = \sum_{\gamma \in \Gamma/\Gamma \cap P} \mathrm{Iw}_{H^+}(g\gamma)^{s\varpi_2}.$$

Moreover, for an unramified cusp form f on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, that is, a $SO(2)$–invariant cusp form, we define

$$E_{s,f}(g) = \sum_{\gamma \in \Gamma/\Gamma \cap P} \mathrm{Iw}_{H^+}(g\gamma)^{s\varpi_2} f\big(\mathrm{Iw}_L(g\gamma)\big).$$

Theorem 7.1. *Assume that* Re $s < -2$. *Then for any compact subset A'_c of A', there is a measure zero subset N_0 of N such that $E_s(g)$ converges absolutely for $g \in KA'_cN'$, where $N' = N - N_0$.*

Proof. Our proof is similar to previous sections where we induce Eisenstein series from characters on Borel subgroups. We have the Bruhat decomposition

$$G = \bigsqcup_{w \in W_1} BwP = \bigsqcup_{w \in W_1} N_w wP,$$

where $W_1 = \{w \in W : w\alpha_1 > 0\}$ is a set of representatives of minimal length for the quotient $W/\langle r_1 \rangle$. Formal calculations show that the constant term of E_s for $a \in A'$ is

$$E_s^\sharp(a) = \sum_{w \in W_1} a^{w(s\varpi_2+\rho)-\rho} c(s\varpi_2, w).$$

Then we only need to prove the absolute convergence of $E_s^\sharp(a)$ under the conditions of the theorem. We may assume that s is real. There are two cases for $w \in W_1$: $w = (r_1r_2)^n$ or $r_2(r_1r_2)^n$. We shall only deal with the first case, and the second case can be treated similarly.

Assume $w = (r_1 r_2)^n$. Then using (5.5) and (5.6), for $a \in A'$ there exists a positive constant M only depending on a such that

$$a^{w(s\varpi_2+\rho)-\rho} \leq M a^{\gamma C_n \alpha_1 + C_n \alpha_2},$$

where

$$C_n = \frac{\gamma^{2n}}{(m^2-4)(\gamma+1)(\gamma-1)^2}\big[-s(m\gamma-2)(\gamma-1)-(m^2-4)\gamma\big].$$

It follows that $C_n \to \infty$ if and only if

$$s < -\frac{(m^2-4)\gamma}{(m\gamma-2)(\gamma-1)} = -1-\gamma^{-1}. \tag{7.4}$$

We also need to consider the factor $c(s\varpi_2, w)$. One can show that

$$\Phi_+ \cap w^{-1}\Phi_- = \{B_i\alpha_1 + B_{i+1}\alpha_2 : i = 0, \ldots, 2n-1\}.$$

For $\alpha = B_i\alpha_1 + B_{i+1}\alpha_2$ we have

$$-(s\varpi_2+\rho)(h_\alpha) = (-s-1)B_{i+1} - B_i > B_{i+1} - B_i \geq 1$$

when $s < -2$. This implies that $c(s\varpi_2, w) \leq C^{\ell(w)}$ for some constant C depending on s.

Combining above analysis it is easy to see the convergence of $E_s^\sharp(a)$ for $a \in A'$. □

We remark that, from $B_{i+1} > \gamma B_i$ it follows that

$$(-s-1)B_{i+1} - B_i \to \infty \quad \text{as} \quad i \to \infty$$

when $s < -1-\gamma^{-1}$. Together with (7.4), this suggests the following

Conjecture 7.2. *$E_s(g)$ converges absolutely for $g \in KA'N$ and* Re $s < -1-\gamma^{-1}$.

Now let us state the main result of this section.

Theorem 7.3. *Let f be an unramified cusp form on SL_2. For any compact subset A'_c of A', there is a measure zero subset N_0 of N such that $E_{s,f}(g)$ is an entire function of $s \in \mathbb{C}$ for $g \in KA'_cN'$, where $N' = N - N_0$.*

We shall follow the strategy in [GMP] to prove Theorem 7.3. The following lemma is in analogy with [GMP, Lemma 3.2], where we set $x^y := yxy^{-1}$ for $x, y \in G$.

Lemma 7.4. *If $\gamma \in \Gamma \cap BwB$, then*

$$\mathrm{Iw}_{A^+}(g\gamma) = \big(\mathrm{Iw}_{A^+} g\big)^{w^{-1}} \cdot \mathrm{Iw}_{A^+}(n_w w)$$

for some $n_w \in N_{w,\mathbb{A}}$ depending on γ and g.

Recall from [GMS2, Lemma 6.1] that for $n_w \in N_{w,\mathbb{A}}$,

$$\ln\big(\mathrm{Iw}_{A^+}(n_w w)\big) = \sum_{\alpha\in\Phi_w} c_\alpha h_\alpha \quad \text{with} \quad c_\alpha \geq 0, \tag{7.5}$$

where $\Phi_w = \Phi_+ \cap w\Phi_-$. Now we can establish the Iwasawa inequalities:

Lemma 7.5. *There exists a constant $D > 0$ such that*

$$\mathrm{Iw}_{A^+}(g\gamma)^{\alpha_1} \geq \mathrm{Iw}_{A^+}(g\gamma)^{D\varpi_2}$$

for any $g \in KA'N$, $w \in W_1$ and $\gamma \in \Gamma \cap BwB$.

Proof. Put $a = \mathrm{Iw}_{A^+} g \in A'$. From Lemma 7.4 and (7.5), it suffices to find a constant D such that the following two inequalities

$$a^{w^{-1}\alpha_1} \geq a^{Dw^{-1}\varpi_2}, \tag{7.6}$$

$$\alpha_1(h_\alpha) \geq D\varpi_2(h_\alpha) \tag{7.7}$$

hold for any $w \in W_1$ and $\alpha \in \Phi_w$. Again we only consider the case $w = (r_1r_2)^n$, and similar arguments apply to the other case $w = r_2(r_1r_2)^n$. Put $w = (r_1r_2)^n$, then $\Phi_w = \{B_{i+1}\alpha_1 + B_i\alpha_2, i = 0, \dots, 2n-1\}$. For any α of the form $B_{i+1}\alpha_1 + B_i\alpha_2$ one has

$$\alpha_1(h_\alpha) = 2B_{i+1} - mB_i \geq (2\gamma - m)B_i = (2\gamma - m)\varpi_2(h_\alpha).$$

This proves (7.7) with $D = 2\gamma - m > 0$. Applying (5.5) and (5.6), and noting that $a \in A'$, it is not hard to prove (7.6) for suitable D by comparing the coefficients of simple roots in $w^{-1}\alpha_1$ and $w^{-1}\varpi_2$. In particular we can choose the constant D to be independent of $a \in A'$. □

We also need the decay estimate [GMP, Theorem 4.6] for cusp forms on finite dimensional real Chevalley groups. In our case, if f is an $SO(2)$–finite cusp form on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, then there exists a constant $C > 0$ depending on f such that for any natural number $n \geq 1$, we have

$$|f(g)| \leq (Cn)^{Cn} \mathrm{Iw}_{A_1^+}(g)^{-n\alpha_1}. \tag{7.8}$$

Proof of Theorem 7.3. Since cusp forms are bounded, the assertion follows from Theorem 7.1 when Re $s < -2$. Assume that Re $s \geq -2$. Choose $s_0 \in \mathbb{R}$ with $s_0 < -2$. Choose a real number $d >$ Re $s - s_0 > 0$ such that $\frac{d}{D} \in \mathbb{N}$, where D is given in Lemma 7.5. Then Re $s - d < s_0 < -2$ and there exists a subset N' of N with measure zero complement such that $E_{\mathrm{Re}\ s-d}(g)$ converges for any $g \in KA'_cN'$.

Put $n = \frac{d}{D} \in \mathbb{N}$ as above. From (7.2), (7.3), Lemma 7.5 and (7.8), we obtain that for any $\gamma \in \Gamma/\Gamma \cap P$,

$$\begin{aligned} & \left|\mathrm{Iw}_{H^+}(g\gamma)^{s\varpi_2} f\big(\mathrm{Iw}_L(g\gamma)\big)\right| \\ \leq\ & (Cn)^{Cn}\mathrm{Iw}_{H^+}(g\gamma)^{(\mathrm{Re}\ s)\varpi_2}\mathrm{Iw}_{A_1^+} \circ \mathrm{Iw}_L(g\gamma)^{-n\alpha_1} \\ \leq\ & (Cn)^{Cn}\mathrm{Iw}_{H^+}(g\gamma)^{(\mathrm{Re}\ s)\varpi_2}\mathrm{Iw}_{H^+}(g\gamma)^{-nD\varpi_2} \\ =\ & (Cn)^{Cn}\mathrm{Iw}_{H^+}(g\gamma)^{(\mathrm{Re}\ s-d)\varpi_2}. \end{aligned}$$

Taking the summation over γ, it follows that $E_{s,f}(g)$ is absolutely convergent. □

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Department of Mathematics, Rutgers University, Piscataway, NJ 08854–8019, U.S.A.

E-mail address: `carbonel@math.rutgers.edu`

Department of Mathematics, University of Connecticut, Storrs, CT 06269, U.S.A.

E-mail address: `khlee@math.uconn.edu`

Department of Mathematics, University of Connecticut, Storrs, CT 06269, U.S.A.

E-mail address: `dongwen.liu@uconn.edu`