CONGRUENCE SUBGROUPS OF LATTICES IN RANK 2 KAC–MOODY GROUPS OVER FINITE FIELDS

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Abstract. Let G be a rank 2 complete affine or hyperbolic simply–connected Kac–Moody group over a finite field k. Then G is locally compact and totally disconnected. Let $B^- = HU^-$ be the negative minimal parabolic subgroup of G, where H is the analog of a diagonal subgroup and U^- is generated by all negative real root groups. Let w_1 and w_2 be the generators of the Weyl group. Let $P_1^- = B^- \sqcup B^- w_1 B^-$ be the negative standard parabolic subgroup of G corresponding to w_1. It is known that the subgroups U^-, B^- and P_1^- are nonuniform lattice subgroups of G. Here we construct an infinite sequence of *congruence subgroups* of P_1^- as natural generalizations of the corresponding notions for lattices in Lie groups. We also show that the group U^- contains analogous congruence subgroups. Our technique involves determining graphs of groups presentations for U^-, B^- and P_1^- with the fundamental apartment of the Bruhat-Tits tree X a quotient graph for U^- and for B^- on X. When $k = \mathbb{F}_q$ and $q = 2^s$, the graph of groups for P_1^- has the the positive half of the fundamental apartment as quotient graph. We explicitly construct the graphs of groups for the principal (level 1) congruence subgroup of P_1^- and the analogous subgroups of U^- giving generalized amalgam presentations for them.

Dedicated to the memory of Eisa Abid whose short life touched many people.

1. Introduction

The principal congruence subgroup of the characteristic p modular group $SL_2(\mathbb{F}_q[t])$, namely the kernel of the reduction map $SL_2(\mathbb{F}_q[t]) \longrightarrow SL_2(\mathbb{F}_q)$ sending t to 0, plays an important role in the theory of modular forms in characteristic p. More generally, congruence subgroups of algebraic groups over local fields provide a source of arithmetic subgroups.

In this work, we construct the analog of congruence subgroups of complete affine and hyperbolic simply–connected Kac–Moody groups over finite fields. Complete Kac–Moody groups over finite fields are locally compact and totally disconnected groups and are 'infinite dimensional' analogs of algebraic groups. Even though these groups have no clear notion of algebraic or arithmetic structure, the twin BN-pair structure developed by Tits [Ti2] provides a framework which enables us to construct congruence subgroups in rank 2 by analogy with congruence subgroups of $SL_2(\mathbb{F}_q[t])$.

In general, recognizing whether a subgroup of the modular group is a congruence subgroup is a broad and difficult question. We give an explicit construction of congruence subgroups of affine and hyperbolic Kac–Moody groups in rank 2 as fundamental groups of certain graphs of groups. This gives rise to both a fundamental domain and a generalized amalgam presentation for the congruence subgroup.

To describe our results in more detail, we let $\mathfrak{g}$ be a rank 2 Kac–Moody algebra. In [Ti1] and [Ti2] Tits associated to $\mathfrak{g}$ a group functor on the category of commutative rings. In [CG] the authors gave an interpretation of Tits' group functor over arbitrary fields using representation theory of $\mathfrak{g}$. Carbone and Garland constructed a Kac–Moody group G that is a completion of Tits' 'minimal' group in [Ti1] and [Ti2].

Date: September 11, 2014.
The first author was supported in part by NSF grant #DMS-1101282.

Let G denote such an affine or hyperbolic group over a finite field $\mathbb{F}_q$. Let $B = HU$ be the completion of the subgroup B^+ of G (as in [CG] and Section 2) where H is a 'diagonal subgroup' and U is the closure of the group generated by all positive real root groups. Let $B^- = HU^-$ be the subgroup of G (as in [CG] and Section 2) where U^- is generated by all negative real root groups.

The group G has twin BN-pairs (G, B, N) and (G, B^-, N) ([CG], Sections 5 and 6) and admits a cocompact action on its corresponding Bruhat-Tits building, which in rank 2 is a homogeneous tree X of degree $q + 1$.

Let w_1 and w_2 be the generators of the Weyl group. Let $P_1^- = B^- \sqcup B^- w_1 B^-$ be the negative standard parabolic subgroup of G corresponding to w_1. It is known that the subgroups U^-, B^- and P_1^- are nonuniform lattice subgroups of G ([CG]). This result was also obtained independently by Rémy ([Re1]).

Here we construct the principal congruence subgroup of P_1^- the kernel of an appropriate reduction homomorphism. We also show that the group U^- contains analogous congruence subgroups. Our technique involves determining graphs of groups presentations for U^-, B^- and P_1^- with the fundamental apartment of the Bruhat-Tits tree X a quotient graph for U^- and for B^- on X. Similarly, the graph of groups for P_1^- has the the positive half of the fundamental apartment as quotient graph.

For some of our results, we restricted to the case where $q = 2^s$. This restriction simplified our methods, though is probably unnecessary.

Our work gives a 'Nagao' theorem for P_1^- (Section 3), *cf.* ([S], p 87) and ([BL], Ch 10). We explicitly construct the graphs of groups for the principal congruence subgroup of P_1^- and the relevant subgroups of U^- giving generalized amalgam presentations for them.

A theory of automorphic forms, particularly Eisenstein series, on 'arithmetic' quotients of Kac–Moody groups is desirable as an extension of some aspects of the Langlands program and is in its early days of development (see [BC], [CLL], [CGGL], [Ga]). In particular, in [CGGL], the authors study Eisenstein series on the quotient of the Tits building X of a rank 2 Kac- Moody group over a finite field by the nonuniform lattice P_1^-. Our construction of congruence subgroups will be an important ingredient in developing a full automorphic theory for Kac–Moody groups.

The authors would like to thank Mikhail Ershov and Howard Garland for helpful discussions. We are also grateful to the referee whose comments improved the exposition.

2. Rank 2 Kac–Moody group and Tits building

We shall consider the split form of a rank 2 affine or hyperbolic complete Kac–Moody group $G = G_A(k)$, as in [CG], over a field k with symmetric generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix},\ m \geq 2.$$

The 'symmetric' assumption is invoked in order to simplify the structure of the root system and root groups. Our results can also be extended to the case of rank 2 symmetrizable non-symmetric generalized Cartan matrices. Here, the real roots lie on 4 branches corresponding to 2 possible root lengths. The group U^- is a free product of metabelian groups in this case.

In this section, we define Tits' minimal form of G in terms of generators and relations, we describe a completion of the Tits functor by Carbone and Garland ([CG]) and we describe the Tits building of G which is a homogeneous tree of degree $|k| + 1$.

2.1. **Tits' presentation of minimal Kac–Moody groups.** Let $\mathfrak{g}$ be the Kac–Moody algebra with symmetric generalized Cartan matrix $A = \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, $m \geq 2$. When $m = 2$, $\mathfrak{g}$ is affine and when $m \geq 3$, $\mathfrak{g}$ is hyperbolic. Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$. Let Δ be the root system of $\mathfrak{g}$. Let $\Pi = \{\alpha_1, \alpha_2\}$ be a basis of simple roots for Π. For each simple root in $\Pi = \{\alpha_1, \alpha_2\}$, we define the simple root reflection $w_i(\alpha_j) := \alpha_j - \alpha_j(\alpha_i^\vee)\alpha_i$, where $\alpha_i^\vee$ is simple coroot. The w_i generate a subgroup $W = W(A) \subseteq Aut(\mathfrak{h}^*)$, called the *Weyl group* of A. The set $\Delta^{re} = W\Pi$ is a subset of Δ, called the set of *real roots.* The remaining roots $\Delta \backslash \Delta^{re}$ are called *imaginary roots.* We introduce an auxiliary group $W^* \subseteq Aut(\mathfrak{g})$, generated by elements $\{w_i^*\}_{i=1,2}$, where

$$w_i^* \;=\; exp(ad(e_i))exp(-ad(f_i))exp(ad(e_i)) \;=\; exp(-ad(f_i))exp(ad(e_i))exp(-ad(f_i)).$$

There is a surjective homomorphism $\epsilon : W^* \to W$ which sends w_i^* to w_i for each i. We define certain elements of $\mathfrak{g}$, denoted $\{e_\alpha\}_{\alpha \in \Delta^{re}}$, where Δ^{re} denotes the set of real roots of $\mathfrak{g}$. Given $\alpha \in \Delta^{re}$, write α in the form $w\alpha_j$ for $j = 1, 2$ and $w \in W$, choose $w^* \in W^*$ which maps onto w and set $e_\alpha = w^* e_{\alpha_j}$.

We define the group $G = G_A(k)$, called the *incomplete simply-connected Kac–Moody group* corresponding to A, to be the group generated by the set of symbols $\{\chi_\alpha(u) \mid \alpha \in \Delta^{re}, u \in k\}$ satisfying relations (R1)-(R7) below, where for $i, j = 1, 2$, u, v are elements of k and α, β are real roots ([Ti2] and [CER]).

(R1) $\chi_\alpha(u + v) = \chi_\alpha(u)\chi_\alpha(v)$;

(R2) Let (α, β) be a *prenilpotent pair,* that is, there exist $w,\ w' \in W$ such that

$$w\alpha,\ w\beta \in \Delta_+^{re} \text{ and } w'\alpha,\ w'\beta \in \Delta_-^{re}.$$

Then

$$[\chi_\alpha(u), \chi_\beta(v)] = \prod_{m,n \geq 1} \chi_{m\alpha + n\beta}(C_{mn\alpha\beta} u^m v^n)$$

where the product on the right-hand side is taken over all real roots of the form $m\alpha + n\beta$, $m, n \geq 1$, in some fixed order and $C_{mn\alpha\beta}$ are integers independent of k. This product appearing on the right-hand side is finite.

For each $i = 1, 2$ set

$\chi_{\pm i}(u) = \chi_{\pm\alpha_i}(u)$, $u \in k$

$\widetilde{w}_i(u) = \chi_i(u)\chi_{-i}(-u^{-1})\chi_i(u)$, $u \in k^*$

$\widetilde{w}_i = \widetilde{w}_i(1)$ and $h_i(u) = \widetilde{w}_i(u)\widetilde{w}_i^{-1}$, $u \in k^*$.

The remaining relations are

(R3) $\widetilde{w}_i \chi_\alpha(u) \widetilde{w}_i^{-1} = \chi_{w_i\alpha}(\eta_{\alpha,i} u)$, where $\eta_{\alpha,i} \in \{\pm 1\}$,

(R4) $h_i(u)\chi_\alpha(v)h_i(u)^{-1} = \chi_\alpha(v u^{\langle \alpha, \alpha_i^\vee \rangle})$ for $u \in k^*$,

(R5) $\widetilde{w}_i h_j(u) \widetilde{w}_i^{-1} = h_j(u) h_i(u^{-a_{ji}})$, $u \in k^*$,

(R6) $h_i(uv) = h_i(u)h_i(v)$ for $u, v \in k^*$ and

(R7) $[h_i(u), h_j(v)] = 1$ for $u, v \in k^*$.

Let H be the subgroup of G generated by the $h_i(u)$, $i = 1, 2$, $u \in k^*$ Then $H \cong k^* \times k^*$.

An immediate consequence of relations (R3) is that $G_A(k)$ is generated by $\{\chi_{\pm i}(u)\}$. The elements $\widetilde{w}_i$ generate a group $\widetilde{W}$ which is isomorphic to the group W^* above.

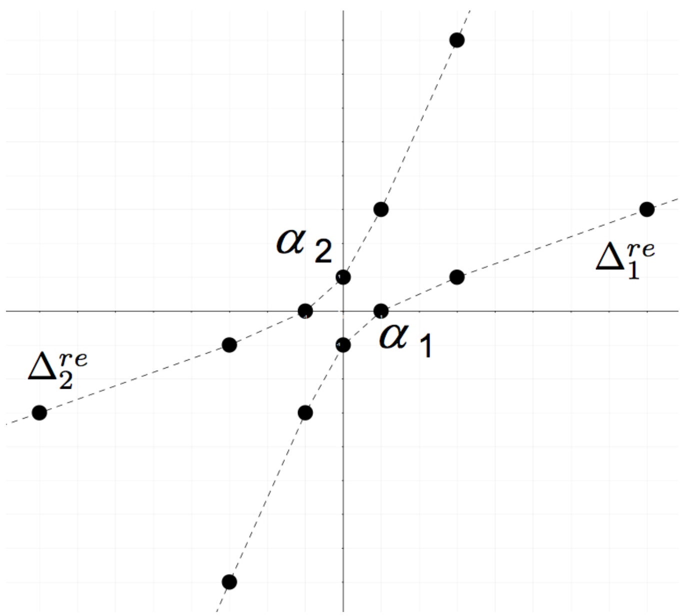
2.2. **Commutation relations in rank 2 Kac–Moody groups.** In this subsection, we describe the commutation relations (R2) of subsection 2.1 over a field k in Tits' minimal Kac–Moody group $G = G_A(k)$ associated to the generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}, \ m \geq 2.$$

Analogous relations appear in the Steinberg presentation for finite dimensional Chevalley groups. The real roots are partitioned into 2 disjoint sets ([CG])

$$\Delta_1^{re} := \{-\alpha_2, \ -w_2\alpha_1, \ -w_2w_1\alpha_2, \dots\} \cup \{\alpha_1, \ w_1\alpha_2, \ w_1w_2\alpha_1, \dots\}$$

$$\Delta_2^{re} := \{-\alpha_1, \ -w_1\alpha_2, \ -w_1w_2\alpha_1, \dots\} \cup \{\alpha_2, \ w_2\alpha_1, \ w_2w_1\alpha_2, \dots\}$$



In this example, prenilpotence is an equivalence relation on real roots and Δ_1^{re} and Δ_2^{re} are the equivalence classes under this equivalence relation. Since

$$w_1 \cdot \alpha_1 = -\alpha_1, \quad w_2 \cdot \alpha_2 = -\alpha_2,$$

w_1 and w_2 interchange Δ_1^{re} and Δ_2^{re}.

If α and β both belong to Δ_1^{re}, or if α and β both belong to Δ_2^{re} then $\alpha + \beta$ cannot be zero or a root ([CG]), thus

$$[\chi_\alpha(s), \chi_\beta(t)] = 1, \ s, t \in k.$$

This completes the commutation relations necessary to define the group G.

2.3. **Completion of Tits' minimal Kac–Moody group.** The following interpretation of Kac–Moody groups was suggested by Tits ([Ti1]) and given by Carbone and Garland ([CG]). This construction generalizes that of simply–connected Chevalley groups. Let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{g}$. Let $\Lambda \subseteq \mathfrak{h}^*$ be the linear span of α_i, for $i = 1, 2$ and $\Lambda^\vee \subseteq \mathfrak{h}$ be the linear span of $\alpha_i^\vee$, for $i = 1, 2$. Let $\mathcal{U}_\mathbb{Z} \subseteq \mathcal{U}$ be the $\mathbb{Z}$-subalgebra generated by $e_i^m/m!$, $f_i^m/m!$ and $\binom{h}{m}$, for $i = 1, 2$, $h \in \Lambda^\vee$ and $m \geq 0$. Then $\mathcal{U}_\mathbb{Z}$ is a $\mathbb{Z}$-form of $\mathcal{U}$, that is $\mathcal{U}_\mathbb{Z}$ is a subring and the canonical map $\mathcal{U}_\mathbb{Z} \otimes \mathbb{Q} \longrightarrow \mathcal{U}$ is bijective. For a field k, let $\mathcal{U}_k = \mathcal{U}_\mathbb{Z} \otimes k$ and $\mathfrak{g}_k = \mathfrak{g}_\mathbb{Z} \otimes k$.

Now let $\lambda \in \mathfrak{h}^*$ be a regular dominant integral weight, that is, $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{>0}$ for each $i = 1, 2$. Let V^λ be the corresponding irreducible highest weight module. Choose a highest-weight vector $v_\lambda \in V^\lambda$ and let $V^\lambda_{\mathbb{Z}} \subset V^\lambda$ be the orbit of v_λ under the action of $\mathcal{U}_{\mathbb{Z}}$. Then $V^\lambda_{\mathbb{Z}}$ is a $\mathbb{Z}$-form of V_λ as well as a $\mathcal{U}_{\mathbb{Z}}$-module. Similarly, $V^\lambda_k := k \otimes_{\mathbb{Z}} V^\lambda_{\mathbb{Z}}$ is a $\mathcal{U}_k$-module. Then there is a (unique) homomorphism $\pi_\lambda : G \to Aut(V^\lambda_k)$ such that

$$\pi_\lambda(\chi_{\alpha_i}(u)) = \sum_{m=0}^{\infty} u^m \frac{e_i^m}{m!} \quad \text{for } i = 1, 2 \text{ and } u \in k,$$
$$\pi_\lambda(\chi_{-\alpha_i}(u)) = \sum_{m=0}^{\infty} u^m \frac{f_i^m}{m!} \quad \text{for } i = 1, 2 \text{ and } u \in k.$$

The expressions on the right-hand side are well defined automorphisms of V^λ_k since e_i and f_i are locally nilpotent on V^λ_k. Let $G^\lambda = \pi_\lambda(G)$.

We define the *weight topology* on G^λ by taking stabilizers of elements of V^λ_k as a subbase of neighborhoods of the identity. The completion of G^λ in this topology was given in [CG]. For a field k, the G^λ are all isomorphic for different dominant integral weights λ ([Ti2]).

2.4. **The BN-pair of a complete Kac–Moody group.** Now take $k = \mathbb{F}_q$ and let $G = G_A(\mathbb{F}_q)$ denote the completion of the Tits' functor associated to A and to k ([CG] and Section 2.3). Then G has subgroups $B^\pm \subseteq G$, $N \subseteq G$ and Weyl group $W = N/H$, where $H = N \cap B^\pm$ is a normal subgroup of N. We have $B^\pm = HU^\pm$ where U^+ is the closure of the group generated by $\chi_\alpha(u)$, $\alpha \in \Delta_+^{re}, u \in k$, U^- is generated by $\chi_\alpha(u), \alpha \in \Delta_-^{re}, u \in k$. The subgroup B^+ is compact, in fact a profinite neighborhood of the identity in G ([CER]) and B^- is discrete ([CG]). The group W coincides with the Weyl group of the previous subsection, $H = \langle h_i(u) \mid i = 1, 2,\ u \in k^* \rangle$ and $N = \langle \widetilde{W}, H \rangle$, where $\widetilde{W}$ is as defined in the previous subsection.

Tits ([Ti1]) and Carbone and Garland ([CG]) showed that (G, B^+, N) and (G, B^-, N) are BN-pairs and

$$G = B^+NB^- = B^-NB^+.$$

It follows that G has Bruhat decomposition

$$G = \sqcup_{w \in W} B^\pm w B^\pm.$$

Let $S = \{w_1, w_2\}$ be the standard generating set for the Weyl group W consisting of simple root reflections. The *standard parabolic subgroups* are

$$P_i^\pm = B^\pm \sqcup B^\pm w_i B^\pm.$$

From now on, we often drop the '+' and refer to P_i^+ as P_i and B^+ as B.

2.5. **The Tits building of G, a tree.** The Tits building of G is a simplicial complex X of dimension $dim(X) = |S| - 1 = 1$. Since W is infinite, by the Solomon-Tits theorem X is contractible and so X is a tree. In fact we may associate to G a twin building $X^\pm$ where X^+ and X^- come from the twin BN-pairs (G, B^+, N) and (G, B^-, N) respectively. The buildings X^+ and X^- are isomorphic as chamber complexes and have constant thickness $q+1$. However, we shall not use the twin structure of the building and hence we drop the $\pm$ and simply write 'X' for X^+.

The Tits building of G is constructed as follows

$$VX \cong G/P_1 \sqcup G/P_2$$
$$EX \cong G/B \sqcup \overline{G/B},$$

where $\overline{G/B}$ is a copy of the set G/B, giving an orientation to EX. If Q_1 and Q_2 are vertices, then there is an edge connecting Q_1 and Q_2 if and only if $Q_1 \cap Q_2$ contains a conjugate of B.

There is a *standard simplex* corresponding to the identity coset of B. The group G acts by left multiplication on cosets. There are natural projections on cosets induced by the inclusion of B in P_1 and P_2:

$$\pi : G/B \longrightarrow G/P_i,\ i = 1, 2.$$

If $v_i \in G/P_i$ is a vertex and $St^X(v_i) = \pi^{-1}(v_i)$ is the set of edges with origin v_i, then we may index $St^X(v_i)$ by $P_i/B \subseteq G/B$, $i = 1, 2$. The Tits building X is a *homogeneous*, bipartite tree of degree

$$[P_1 : B] = [P_2 : B] = q + 1.$$

The following describes how the cosets Bw_1B and Bw_2B are indexed modulo B ([CG]):

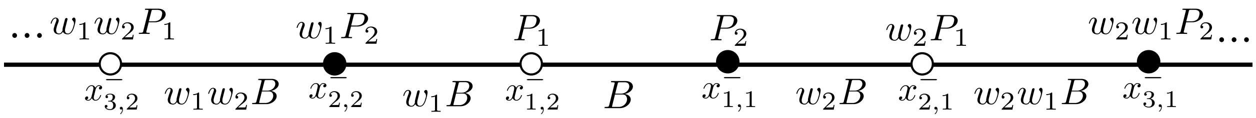
$$Bw_1B/B = \{\chi_{\alpha_1}(s)w_1B/B \mid s \in k\},$$
$$Bw_2B/B = \{\chi_{\alpha_2}(s)w_2B/B \mid s \in k\},$$

where α_1 and α_2 are the simple roots corresponding to w_1 and w_2 respectively.

It follows that the edges emanating from P_1 and P_2 may be indexed as follows:

$$\begin{array}{rcl} St^X(P_1) & = & \{B\} \sqcup \{\chi_{\alpha_1}(s)w_1B/B \mid s \in k\}, \\ St^X(P_2) & = & \{B\} \sqcup \{\chi_{\alpha_2}(s))w_2B/B \mid s \in k\}, \end{array}$$

where B denotes the identity coset and the stars of other vertices are obtained by translating these. Apartments in X are infinite lines. The *fundamental apartment*, denoted by $\mathcal{A}_0$, in X consists of all Weyl group translates of the standard simplex.



2.6. **Structure theorems for lattice subgroups.** For a general nonuniform lattice $\Gamma \in G$, we will assume that $\Gamma\backslash X$ has the structure of a finite core graph union finitely many cusps which are semi-infinite rays. We conjecture this to be true for any nonuniform lattice $\Gamma \leq G$ and all our known examples are of this type (see [C] for further discussion).

2.7. **Graph of groups presentations for subgroups of G.** We will use the fundamental theory of Bass and Serre for reconstructing group actions on trees to obtain a graph of groups presentations for various subgroups of G. Our references for the Bass–Serre theory are ([B], [S]). We refer the reader to ([S], Section 5), for the construction of a quotient graph of groups $\Gamma\backslash\backslash X$ and universal covering tree $\widetilde{\Gamma\backslash\backslash X}$ corresponding to a group Γ acting on X on a tree X.

Theorem 2.1. *([B], [S]) Let X be a tree and Γ a group acting on X without inversions and with quotient graph of groups $\Gamma\backslash\backslash X$. Then $\Gamma \cong \pi_1(\Gamma\backslash\backslash X)$ is an isomorphism of groups and $X \cong \widetilde{\Gamma\backslash\backslash X}$ is an isomorphism of trees.*

Here we take $\Gamma \leq G$, where G is the rank 2 simply–connected Kac–Moody group. Then $\pi_1(\Gamma\backslash\backslash X)$ gives a presentation for Γ as a generalized amalgam of groups which are conjugate to vertex stabilizers.

If $\Gamma\backslash\backslash X$ is an infinite graph of finite groups with finite volume, then Γ is discrete and has finite covolume in G with respect to some (hence any) Haar measure on G, hence is a nonuniform lattice in G ([BL]).

3. Graphs of groups presentations for U^-, B^- and P_1^-

Let $\alpha \in \Delta^{re}$ be a real root. Then α has an expression $\alpha = w\alpha_i$, for α_i simple and $w \in W$. Define

$$\ell(\alpha) \quad := \quad \ell(w) + 1,$$

where $\ell(\cdot)$ is the standard length function on W, so that for α_i simple

$$\ell(\alpha_i) \quad = \quad 1,$$

and for α_i simple and w_j a simple root reflection

$$\ell(w_j\alpha_i) \quad = \quad 2, \quad i \neq j.$$

For $n \geq 1$ and $i = 1, 2$, we define (finite) *cuspidal root subgroups* of $U^- \subseteq B^-$ and $U^+ \subseteq B^+$:

$$U^-_{n,i} \quad = \quad \langle U_\alpha \mid \alpha \in \Delta^{re}_{i,-},\ 1 \leq \ell(\alpha) \leq n \rangle \quad = \{ \prod_{\substack{\alpha \in \Delta^{re}_{i,-} \\ 1 \leq \ell(\alpha) \leq n}} \chi_\alpha(u_\alpha) \mid u_\alpha \in \mathbb{F}_q \},$$

$$U^+_{n,i} \quad = \quad \langle U_\alpha \mid \alpha \in \Delta^{re}_{i,+},\ 1 \leq \ell(\alpha) \leq n \rangle \quad = \{ \prod_{\substack{\alpha \in \Delta^{re}_{i,+} \\ 1 \leq \ell(\alpha) \leq n}} \chi_\alpha(u_\alpha) \mid u_\alpha \in \mathbb{F}_q \},$$

respectively. Where, for $i = 1, 2$,

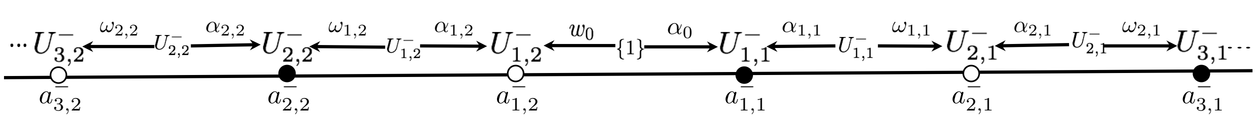
$$\Delta^{re}_{i,+} \quad = \quad \{\alpha_i,\ w_i\alpha_{3-i},\ w_iw_{3-i}\alpha_i, \dots\}$$
$$\Delta^{re}_{i,-} \quad = \quad \{-\alpha_{3-i},\ -w_{3-i}\alpha_i,\ -w_{3-i}w_i\alpha_{3-i}, \dots\}.$$

Then for $i = 1, 2$, $U^{\pm}_{n,i}$ are finite subgroups of $U^{\pm}$ of order $|U^{\pm}_{n,i}| = q^n$. Such subgroups were also introduced in [Ro], Section 6 and [KP], Section 3.

3.1. **Graph of groups for U^-.** Our aim in this subsection is to construct a graph of groups whose fundamental group is the group U^-.

It is known from [CG] and [Re1] that the group B^- is a nonuniform lattice subgroup of the affine or hyperbolic rank 2 Kac–Moody group G over any finite field $k = \mathbb{F}_q$. Since U^- is a finite index subgroup of B^-, it follows that U^- is also a nonuniform lattice in G.

Theorem 3.1. *The group U^- is isomorphic to the fundamental group of the following graph of groups:*



where

$$\omega_0 : \{1\} \hookrightarrow U^-_{1,2}$$
$$\alpha_0 : \{1\} \hookrightarrow U^-_{1,1}$$

are inclusion maps and for $i = 1, 2$, $n \geq 1$

$$\alpha_{n,i} : U^-_{n,i} \hookrightarrow U^-_{n,i}$$

are identity maps and for $i = 1, 2$, $n \geq 1$

$$\omega_{n,i} : U^-_{n,i} \hookrightarrow U^-_{n+1,i}$$

are inclusion maps.

Let

$$U^-_{\infty,1} = \bigcup_{n\geq 1} U^-_{n,1}$$
$$U^-_{\infty,2} = \bigcup_{n\geq 1} U^-_{n,2}$$

be the ascending unions of finite root groups.

Since for $i = 1, 2$, $n \geq 1$, $\omega_{n,i} : U^-_{n,i} \hookrightarrow U^-_{n+1,i}$ and $U^-_{\infty,2} \cap U^-_{\infty,1} = \{1\}$, it follows that

$$U^- = U^-_{\infty,2} * U^-_{\infty,1}.$$

This was also proven in [KP], Proposition 3.5 (c) in a different setting.

To prove Theorem 3.1, we will make use of the following lemma which follows from a simple inductive argument.

Lemma 3.2. *For $i = 1, 2$, $n \geq 1$,*

$$U^-_{n,i} = u_n U^+_{n,j} u_n^{-1}$$

where $j = i$ if n is even, $j = 3 - i$ if n is odd and $u_n \in W$ is the unique element of the set Ω_i with $\ell(u_n) = n$, where

$$\Omega_1 = \{1, w_2, w_2w_1, w_2w_1w_2, \dots\},$$
$$\Omega_2 = \{1, w_1, w_1w_2, w_1w_2w_1, \dots\}.$$

Note that for $n \geq 1$, $i = 1, 2$ $u_n \in \Omega_i$ with $\ell(u_n) = n$, we have

$$u_{n-1} P_j u_{n-1}^{-1} = Stab_G(x^-_{n,i})$$

where $j = i$ for n even and $j = 3 - i$ for n odd, with $u_0 = 1$.

Lemma 3.3. *([CG], Section 15) Let P_1, P_2 and U^- be as before, then we have,*

$$P_1 \cap U^- = U_{-\alpha_1} \leq U^-_{1,2}$$

$$P_2 \cap U^- = U_{-\alpha_2} \leq U^-_{1,1}.$$

Theorem 3.1 follows immediately from the following.

Proposition 3.4. *Let G be a rank 2 affine or hyperbolic simply–connected Kac–Moody group over a finite field. Let U^- be the subgroup of G generated by all negative real root subgroups. For $i = 1, 2$ and $n \geq 1$, $U^-_{n,i}$ be cuspidal root subgroups. Then we have:*

(1) For $i = 1, 2$, $n \geq 1$ the group $U^-_{n,i}$ is the stabilizer of the vertex $x^-_{n,i}$ modulo U^-.

(2) For $i = 1, 2$, $n \geq 1$ the group $U^-_{n,i}$ leaves the edge $(x^-_{n,i}, x^-_{n+1,i})$ fixed and acts transitively on the set of edges with origin $x^-_{n,i}$ distinct from $(x^-_{n,i}, x^-_{n+1,i})$, sending all such edges to $u_{n-1}B$, where $u_{n-1} \in \Omega_i$ with $\ell(u_{n-1}) = n - 1$, setting $u_0 = 1$.

(3) For $i = 1, 2$, $n \geq 1$ the vertices $x^-_{n,i}$ are not equivalent under the action of U^-.

(4) $Stab_{U^-}(x^-_{1,2}, x^-_{1,1}) = \{1\}$.

Proof. For (1), $i = 1, 2$, $n \geq 1$ we have,

$$U^+_{n,i} \quad = \{ \prod_{\substack{\alpha \in \Delta^{re}_{i,+} \\ 1 \leq \ell(\alpha) \leq n}} \chi_\alpha(u_\alpha) \mid u_\alpha \in k\}$$

since $\chi_\alpha(u_\alpha)$ commute among themselves for $\alpha \in \Delta^+_{n,i}$ ([CG], Section 14), where

$$\Delta^+_{n,i} = \{\alpha \in \Delta^{re}_{i.+} \mid l(\alpha) \leq n\}.$$

From Lemma 3.2, for $i = 1, 2$, $n \geq 1$ we have

$$U^-_{n,i} \quad = \quad u_n U^+_{n,j} u_n{}^{-1} \quad \subseteq \quad u_n U u_n{}^{-1} \quad \subseteq \quad u_n B u_n{}^{-1}$$

where $j = i$ for even n, $j = 3 - i$ for odd n and $u_n \in W$ is the unique element of the set Ω_i with $\ell(u_n) = n$, setting $u_0 = 1$.

Thus for each $n \geq 1$, $i \;=\; 1, 2$

$$U^-_{n,i} \quad \subseteq \quad u_n B u_n{}^{-1} \quad = \quad u_{n-1} P_j u^{-1}_{n-1} \cap u_n P_{3-j} u_n^{-1} \quad \subseteq \quad u_{n-1} P_j u^{-1}_{n-1}$$

where $j = i$ for even n, $j = 3 - i$ for odd n and $u_n \in \Omega_i$ with $\ell(u_n) = n$, having set $u_0 = 1$.

Hence

$$U^-_{n,i} \quad \subseteq \quad u_{n-1} P_j u^{-1}_{n-1} \quad = \quad Stab_G(x^-_{n,i})$$

so

$$\begin{aligned} U^-_{n,i} \quad \subseteq u_{n-1} P_j u^{-1}_{n-1} \cap U^- \quad &= \quad Stab_G(x^-_{n,i}) \cap U^- \\ &= \quad Stab_{U^-}(x^-_{n,i}). \end{aligned}$$

Conversely, for $n \geq 1$ and $i = 1, 2$

$$\begin{aligned} u_{n-1} P_j u^{-1}_{n-1} \cap U^- \quad &= \quad u_{n-1}(P_j \cap U^-) u^{-1}_{n-1} \\ &= \quad u_{n-1} U_{-\alpha_j} u^{-1}_{n-1} \quad \textit{by Lemma } 3.3 \\ &= \quad U_{-u_{n-1}\alpha_j} \\ &\subseteq \quad U^-_{n,i} \end{aligned}$$

where $j = i$ for even n and $j = 3 - i$ for odd n, $u_{n-1} \in \Omega_i$ with $\ell(u_{n-1}) = n - 1$ setting $u_0 = 1$. So

$$U^-_{n,i} = Stab_{U^-}(x^-_{n,i}).$$

For (2), the inclusion $U^-_{n,i} \subseteq U^-_{n+1,i}$, for $i = 1, 2$, $n \geq 1$, shows that the group $U^-_{n,i}$ fixes the edge $(x^-_{n,i}, x^-_{n+1,i})$. We observe that for a vertex on the fundamental apartment, denoted by wP_i with $i = 1, 2$, $w \in W$, we may identify $E^X_0(wP_i)$ the set of edges with the origin wP_i with

$$E^X_0(wP_i) \quad = \quad \{wB\} \cup \{w\chi_{\alpha_i}(s) w_i B \mid s \in \mathbb{F}_q\}.$$

The vertex $x^-_{n,i}$, $n \geq 1$, $i = 1, 2$ corresponds to the coset $u_{n-1} P_j$ where $u_{n-1} \in W$ is the unique element of the set Ω_i with $\ell(u_{n-1}) = n - 1$ and $j = i$ if n is even, $j = 3 - i$ if n is odd having set $u_0 = 1$. The edge $(x^-_{n,i}, x^-_{n+1,i})$ then corresponds to the coset $u_{n-1} w_j B$, that is the element of the set $\{u_{n-1}B\} \cup \{u_{n-1}\chi_{\alpha_j}(s) w_j B \mid s \in \mathbb{F}_q\}$ corresponding to $s = 0$.

We are left to consider $\{u_{n-1}B\} \cup \{u_{n-1}\chi_{\alpha_j}(s) w_j B \mid s \in \mathbb{F}_q^\times\}$ which is the set of edges with origin $x^-_{n,i}$ distinct from $(x^-_{n,i}, x^-_{n+1,i})$. We claim that if $u \cdot [u_{n-1}\chi_{\alpha_j}(s) w_j B] = u_{n-1}B$, for some $s \in \mathbb{F}_q$ and for some $u \in G$, then $u \in U^-_{n,i}$. In fact, the element u is equal to $u^{-1}_{n-1} w^{-1}_j \chi_{\alpha_j}(-s) u^{-1}_{n-1}$, and

$$\begin{aligned} u_{n-1} w^{-1}_j \chi_{\alpha_j}(-s) u^{-1}_{n-1} &\in u_{n-1} U_{-\alpha_j} u^{-1}_{n-1} \\ &= U_{-u_{n-1}\alpha_j} \\ &\leq U^-_{n,i} \end{aligned}$$

for $i = 1, 2$, $n \geq 1$ and $j = i$ if n is even, $j = 3 - i$ if n is odd and $u_{n-1} \in \Omega_i$ with $\ell(u_{n-1}) = n - 1$ setting $u_0 = 1$, the proof of (2) is complete.

For (3), suppose that for some $u^- \in U^-$, $u^- x^-_{n,i} = x^-_{n+m,i}$ with $m \neq 0$. Assume that $m > 0$. Then for $i = 1, 2$, some $w, w^{'} \in W$ with $\ell(w^{'}) = n + m \geq \ell(w) = n$ we have

$$u^-(wP_i) = w^{'} P_i \text{ if and only if } (w^{'})^{-1} u^- w \in P_i = B \sqcup B w_i B$$

which is impossible since $U^- \cap B = \{1\}$, unless $w^{'} = w$ and thus vertices wP_i and $w^{'}P_i$ coincide. For (4),we have

$$\begin{aligned} Stab_{U^-}(x^-_{1,2}, x^-_{1,1}) &= Stab_G(x^-_{1,2}, x^-_{1,1}) \cap U^- \\ &= B \cap U^- \\ &= HU \cap U^- \\ &= \{1\} \end{aligned}$$

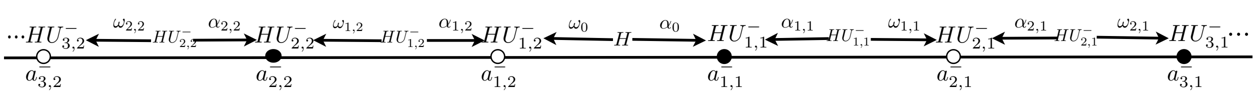
as follows from axiom RD1-RD5 of [Ti2]. $\square$

3.2. **Graph of groups for B^-.** We can now obtain the graph of groups for B^-. We have

$$H \quad \cong \quad \mathbb{F}_q^\times \times \mathbb{F}_q^\times$$

from Section 2. Thus $|H| = (q-1)^2$ and for each $i = 1, 2$, $n \geq 1$ the group $HU^-_{n,i}$ is a finite subgroup of B^- of order $q^n(q-1)^2$. The following theorem is an easy generalization of Theorem 3.1 using the additional property that H fixes the fundamental apartment point wise.

Theorem 3.5. *The group $B^- = HU^-$ is isomorphic to the fundamental group of the following graph of groups:*



where

$$\omega_0 : H \hookrightarrow HU^-_{1,2}$$

$$\alpha_0 : H \hookrightarrow HU^-_{1,1}$$

are inclusion maps and for $i = 1, 2$, $n \geq 1$

$$\alpha_{n,i} : HU^-_{n,i} \hookrightarrow HU^-_{n,i}$$

are identity maps and for $i = 1, 2$, $n \geq 1$

$$\omega_{n,i} : HU^-_{n,i} \hookrightarrow HU^-_{n+1,i}$$

are inclusion maps. Moreover B^- has the expression

$$B^- \quad = \quad HU^- \quad = \quad HU^-_{\infty,2} \ *_H \ HU^-_{\infty,1}.$$

The volume of the graph of groups for B^- is a scalar multiple of $2+\sum_{n\geq 1}\dfrac{1}{q^n}$ (*cf.* [CG], Lemma 8.1). Theorem 3.5 also proves that the group B^- is a nonuniform lattice subgroup of G (see ([CG], Section 8) for an alternate proof).

This also shows that the fundamental apartment $\mathcal{A}_0$ is a fundamental domain for U^- and for B^- on X, as was observed in [Ti2] and [CG].

3.3. **Graph of groups for** P_1^-**.** The following gives a 'Nagao theorem' for P_1^- in analogy with $GL_2(\mathbb{F}_q[t])$ and $SL_2(\mathbb{F}_q[t])$ as in [S]. However, our method of proof restricts us to constructing a graph of groups for P_1^- only when $k=\mathbb{F}_q$ and $q=2^s$. We note though, that it was shown in [CG] that the semi–infinite ray is a fundamental domain for P_1^- on $X=X_{q+1}$ with no restriction on q.

We will need the following lemma.

Lemma 3.6. *Let G be an affine or hyperbolic complete simply–connected Kac–Moody group over the finite field $k=\mathbb{F}_q$. Let P_1^- be the negative standard parabolic subgroup of G. Then P_1^- contains a subgroup of order $q+1$.*

Proof: As in [CG], there is a homomorphism

$$\phi_1 : SL_2(\mathbb{F}_q)\to G.$$

Since G is of simply connected type, ϕ_1 is injective (For the adjoint form of G, ϕ_1 has kernel the cyclic group of order 2, but we consider only the simply connected form here). Moreover, $\phi_1(SL_2(\mathbb{F}_q))$ is generated by $\chi_{\alpha_1}(s)$, $\chi_{-\alpha_1}(t)$, $s,t\in\mathbb{F}_q$ as in Subsection 2.1. Since the Weyl group element w_1 flips α_1 to $-\alpha_1$, $\phi_1(SL_2(\mathbb{F}_q))$ is contained in the union of Bruhat cells $B^-\sqcup B^-w_1B^-$. But this is precisely the parabolic subgroup P_1^-. So $\phi_1(SL_2(\mathbb{F}_q))$ is contained in P_1^-. Thus there is a subgroup $T=\phi_1(\mathcal{T})$ of P_1^-, where $\mathcal{T}$ is the non–split torus of $SL_2(\mathbb{F}_q)$ a subgroup of order $q+1$. (We refer the reader to [CC], Theorem 3, (i) and [Lu], Lemma 3.5 for further details about the non–split torus of $SL_2(\mathbb{F}_q)$.$\square$

Remarks.

(1) When $q=2$, T is generated by $\chi_{\alpha_1}(1)\chi_{-\alpha_1}(1)$ which has order 3 ([CC]).

(2) Lemma 3.6 can also be proven using the fact that P_1^- has a Levi factor which is a finite group of Lie type.

Proposition 3.7. *Let G be an affine or hyperbolic complete simply–connected Kac–Moody group over the finite field $k=\mathbb{F}_q$ and assume that $q=2^s$. Let $X=X_{q+1}$ be the Tits building of G. Let P_1^- be the negative standard parabolic subgroup of G. Then*

$$\{P_1\}\cup\mathcal{A}_0^+=(P_1,\ P_2,\ w_2P_1,\ w_2w_1P_2,\ w_2w_1w_2P_1,\dots)$$

is a fundamental domain for P_1^- on X where $\mathcal{A}_0^+=(P_2,\ w_2P_1,\ w_2w_1P_2,\ w_2w_1w_2P_1,\dots)$ is the positive half of the fundamental apartment $\mathcal{A}_0$ and

$$P_1^-\cong B_{0,1}^-\ *_H \cup_{n\geq 1}B_{n,1}^-,$$

$$B_{0,1}^-=\langle H,T\rangle,\ |B_{0,1}^-|=(q-1)^2(q+1)$$

where T is the subgroup of P_1^- of order $q+1$ as in Lemma 3.6, and for $n\geq 1$,

$$B_{n,1}^-=\langle H,U_{n,1}^-\rangle,\quad |U_{n,1}^-|=q^n,\ U_{n,1}^-\cong\{\begin{pmatrix}1 & tf(t)\\ 0 & 1\end{pmatrix},\ tf(t)\in t\mathbb{F}_q[t],\ deg(tf(t))\leq n\},$$

$$H=\langle h_{\alpha_1}(u)\mid u\in\mathbb{F}_q^\times\rangle\ *\ \langle h_{\alpha_2}(u)\mid u\in\mathbb{F}_q^\times\rangle\ \cong\ \mathbb{F}_q^\times\ \times\mathbb{F}_q^\times,$$

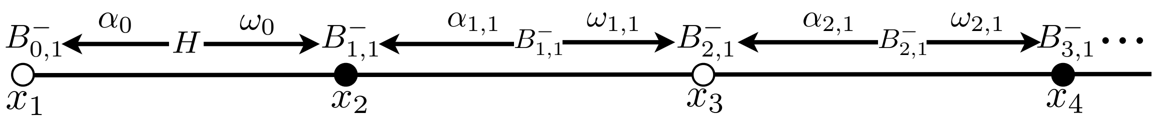
and $|H| = (q-1)^2$ *so that* $|B^-_{n,1}| = (q-1)^2 q^n$.

Proposition 3.7 follows immediately from Proposition 3.10.

We will use the following Lemma, proven in [CC], Theorem 3, (i) and in [Lu], Lemma 3.5.

Lemma 3.8. *For* $k = \mathbb{F}_q$ *and* $q = 2^s$, *the group* T *of order* $q+1$ *acts transitively on* $P_1/B \cong \mathbb{P}^1(\mathbb{F}_q)$.

Theorem 3.9. *When* $k = \mathbb{F}_q$ *and* $q = 2^s$, *the group* P_1^- *is isomorphic to the fundamental group of the following graph of groups:*



where

$$\omega_0 : H \hookrightarrow B^-_{1,1}$$

$$\alpha_0 : H \hookrightarrow B^-_{0,1}$$

are inclusion maps and for $n \geq 1$

$$\alpha_{n,i} : B^-_{n,1} \hookrightarrow B^-_{n,1}$$

are identity maps and for $n \geq 1$

$$\omega_{n,i} : B^-_{n,1} \hookrightarrow B^-_{n+1,1}$$

are inclusion maps.

Theorem 3.9 follows immediately from the following.

Proposition 3.10. *Let* G *be an affine or hyperbolic complete simply–connected Kac–Moody group over the finite field* $k = \mathbb{F}_q$ *and assume that* $q = 2^s$. *Let* $X = X_{q+1}$ *be the Tits building of* G. *Let* P_1^- *be the negative standard parabolic subgroup of* G *and let* $P_1 = P_1^+$ *be the positive standard parabolic subgroup of* G. *Let* $(x_1,\ x_2,\ x_3,\ x_4,\ x_5, \dots)$ *be an indexing of vertices*

$$\{P_1\} \cup \mathcal{A}_0^+ = (P_1,\ P_2,\ w_2P_1,\ w_2w_1P_2,\ w_2w_1w_2P_1, \dots)$$

of X, *where* $\mathcal{A}_0^+ = (P_2,\ w_2P_1,\ w_2w_1P_2,\ w_2w_1w_2P_1, \dots)$ *is the positive half of the fundamental apartment* $\mathcal{A}_0$. *Then*

(1) $B^-_{0,1}$ *acts transitively on* P_1/B *sending all edges to* B *and fixing the coset* P_1.

(2) $P_1^- \cap P_1^+ = B^-_{0,1}$.

(3) For each $n \geq 1$, *the group* $B^-_{n,1}$ *leaves the edge* (x_{n+1}, x_{n+2}) *fixed and acts transitively on the set of edges with origin* x_{n+1}, *distinct from* (x_{n+1}, x_{n+2}) *sending all the edges to* $u_{n-1}B$, *where* $u_{n-1} \in \Omega_1$ *with* $\ell(u_{n-1}) = n-1$, *setting* $u_0 = 1$.

(4) For $n \geq 1$, *the vertices* x_n *in the set*

$$\{P_1\} \cup \mathcal{A}_0^+ = (P_1,\ P_2,\ w_2P_1,\ w_2w_1P_2,\ w_2w_1w_2P_1, \dots)$$

are not equivalent modulo P_1^-.

Proof. For $k = \mathbb{F}_q$ and $q = 2^s$, the group T acts transitively on P_1/B (Lemma 3.8). The group H is contained in B and hence P_1 and so H fixes P_1. Hence $B_{0,1}^- = \langle H, T\rangle$ acts transitively on P_1/B sending all edges to B and fixing the coset P_1.

For (2), by (1) we have

$$P_1^- \cap P_1^+ = Stab_{P_1^-}(P_1^+) = B_{0,1}^-.$$

For (3), since $B_{n,1}^- \subseteq B_{n+1,1}^-$ for $n \geq 1$, $B_{n,1}^-$ fixes the edge (x_{n+1}, x_{n+2}). As in Proposition 3.4, we have if $b^- \in P_1^-$ acts transitively on the set of edges with origin x_{n+1} for $n \geq 1$, distinct from (x_{n+1}, x_{n+2}) sending all the edges to $u_{n-1}B$, then $b^- \in U_{n,1}^-$. Also from the equation

$$b^-[u_{n-1}\chi_{\alpha_t}(s)w_t B] = u_{n-1}B$$

where $s \in \mathbb{F}_q$, $t = 1$ if n is even and $t = 2$ if n is odd, we obtain

$$u_{n-1}^{-1} b^- u_{n-1}\chi_{\alpha_t}(s)w_t \in B$$

also

$$u_{n-1}^{-1} b^- u_{n-1}\chi_{\alpha_t}(s)w_t \in P_1^-.$$

Hence

$$u_{n-1}^{-1} b^- u_{n-1}\chi_{\alpha_t}(s)w_t \in P_1^- \cap B = H,$$

so $b^- \in H$. Therefore $b^- \in B_{n,1}^- = \langle H, U_{n,1}^-\rangle$ and the proof of (3) is complete.

Part (4) follows easily from Proposition 3.4(3). □

4. Congruence subgroups of Kac–Moody groups

Let $\Gamma = SL_2(\mathbb{F}_q[t]) \leq H = SL_2(\mathbb{F}_q((t^{-1})))$. Then Γ is a nonuniform (arithmetic) lattice subgroup of H ([S]). Let

$$\Gamma_0 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q[t]) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} mod\ t\}$$

be the principal congruence subgroup of Γ. Then

$$\Gamma_0 = Ker\{\rho : SL_2(\mathbb{F}_q[t]) \longrightarrow SL_2(\mathbb{F}_q) \mid \ t \mapsto 0\}.$$

By [S] we have:

$$\Gamma_0 = *_{s\in S} \quad s\Lambda s^{-1},$$

where

$$\Lambda = \begin{pmatrix} 1 & t\mathbb{F}_q[t] \\ 0 & 1 \end{pmatrix}$$

and S is a set of coset representatives for $SL_2(\mathbb{F}_q)/B(SL_2(\mathbb{F}_q))$, where $B(SL_2(\mathbb{F}_q))$ denotes the Borel subgroup of $SL_2(\mathbb{F}_q)$.

For $r > 1$, we also have the family of congruence subgroups:

$$\Gamma_0(r) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q[t]) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} mod\ t^r\}.$$

Our aim is to show that the subgroup P_1^- of our Kac–Moody group G contains a congruence subgroup analogous to Γ_0. In Section 5 we comment on the existence of congruence subgroups in Kac–Moody groups analogous to the family $\Gamma_0(r)$, $r > 1$.

We first define *principal congruence subgroups* of U^- as follows. We set

$$K_{1,U^-} \quad = \quad Ker\{\psi_{1,U^-} : U^- \to U_{-\alpha_2} \mid U_\alpha \mapsto 1 \text{ if } \ell(\alpha) \geq 1, \alpha \in \Delta_-^{re} \setminus \{-\alpha_2\}\}$$

$$K_{2,U^-} \quad = \quad Ker\{\psi_{2,U^-} : U^- \to U_{-\alpha_1} \mid U_\alpha \mapsto 1 \text{ if } \ell(\alpha) \geq 1, \alpha \in \Delta_-^{re} \setminus \{-\alpha_1\}\}.$$

The groups K_{i,U^-} are the subgroups of U^- generated up to conjugacy by all negative root groups except for $U_{\alpha_{3-i}}$. Here we explicitly construct the graphs of groups for K_{i,U^-} giving generalized amalgam presentations for them.

For each $i = 1, 2$, the set $\mathcal{S} = \infty \sqcup \{\chi_{\alpha_i}(s)w_i \mid s \in \mathbb{F}_q\}$ may be identified with $\mathbb{P}^1(\mathbb{F}_q)$ whose elements are coset representatives of P_i/B.

For $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$, $s_0 = 0$, we set

$$\begin{aligned} S_1 &= \{g_0 = w_1, g_1 = \chi_{\alpha_1}(s_1)w_1, g_2 = \chi_{\alpha_1}(s_2)w_1, \dots, g_{q-1} = \chi_{\alpha_1}(s_{q-1})w_1\} \\ S_2 &= \{h_0 = w_2, h_1 = \chi_{\alpha_2}(s_1)w_2, h_2 = \chi_{\alpha_2}(s_2)w_2, \dots, h_{q-1} = \chi_{\alpha_2}(s_{q-1})w_2\}. \end{aligned}$$

For each $j = 0, 1, 2, \dots, q-1$, let $S_{1,j}\mathcal{A}_0^+$ and $S_{2,j}\mathcal{A}_0^-$ denote translates of the positive half $\mathcal{A}_0^+ = (P_2,\ w_2P_1,\ w_2w_1P_2,\ w_2w_1w_2P_1, \dots)$ of the fundamental apartment $\mathcal{A}_0$ by the elements g_j of S_1 and of the negative half $\mathcal{A}_0^- = (P_1,\ w_1P_2,\ w_1w_2P_1,\ w_1w_2w_1P_2, \dots)$ of the fundamental apartment $\mathcal{A}_0$ by the elements h_j of of S_2 respectively. Then $S_{1,j}\mathcal{A}_0^+$ has vertex sequence

$$\{x_{0,j}^{1,-} = P_2, x_{1,j}^{1,-} = \chi_{\alpha_1}(s_j)w_1P_2, x_{2,j}^{1,-} = \chi_{\alpha_1}(s_j)w_1w_2P_1, ... \mid s_j \in \mathbb{F}_q, j = 0, 1, 2, \dots, q-1, s_0 = 0\}$$

and $S_{2,j}\mathcal{A}_0^-$ has vertex sequence

$$\{x_{0,j}^{2,-} = P_1, x_{1,j}^{2,-} = \chi_{\alpha_2}(s_j)w_2P_1, x_{2,j}^{2,-} = \chi_{\alpha_2}(s_j)w_2w_1P_2, ... \mid s_j \in \mathbb{F}_q, j = 0, 1, 2, \dots, q-1, s_0 = 0\}.$$

The following theorem gives the graph of groups for K_{i,U^-} $i = 1, 2$.

Theorem 4.1. *Let G be an affine or hyperbolic complete simply–connected Kac–Moody group over the finite field $k = \mathbb{F}_q$. Let U^- be subgroup of G generated by all negative real root subgroups. Let K_{i,U^-} be the principal congruence subgroup of U^- for $i = 1, 2$. Then the group*

$$K_{i,U^-} \quad = \quad Ker\{\psi_{i,U^-} : U^- \to U_{-\alpha_{3-i}} \mid U_\alpha \mapsto 1 \textit{ if } \ell(\alpha) \geq 1, \alpha \in \Delta_{i,-}^{re} \setminus \{\alpha_{3-i}\}\}$$

is isomorphic to the fundamental group of the following graph of groups:

where the total number of rays is $q+1$ and is indexed over the set $\mathcal{S}$,

$$\alpha_n^{i,j} : \chi_{\alpha_i}(s_j)w_iU_{n,i}^-w_i^{-1}\chi_{\alpha_i}(-s_j) \to \chi_{\alpha_i}(s_j)w_iU_{n,i}^-w_i^{-1}\chi_{\alpha_i}(-s_j)$$

are identity maps, for $i = 1, 2$, $n \geq 0$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$ setting $s_0 = 0$ and $U_{0,i}^- = \{1\}$, and

$$\omega_n^{i,j} : \chi_{\alpha_i}(s_j)w_iU_{n,i}^-w_i^{-1}\chi_{\alpha_i}(-s_j) \hookrightarrow \chi_{\alpha_i}(s_j)w_iU_{n+1,i}^-w_i^{-1}\chi_{\alpha_i}(-s_j)$$

are inclusion maps, for $i = 1, 2$, $n \geq 0$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$ setting $s_0 = 0$ and $U_{0,i}^- = \{1\}$.

Lemmas 4.2 and 4.3 show that the vertex groups of the graph of groups in Theorem 4.1 are subgroups of K_{i,U^-} for $i = 1, 2$.

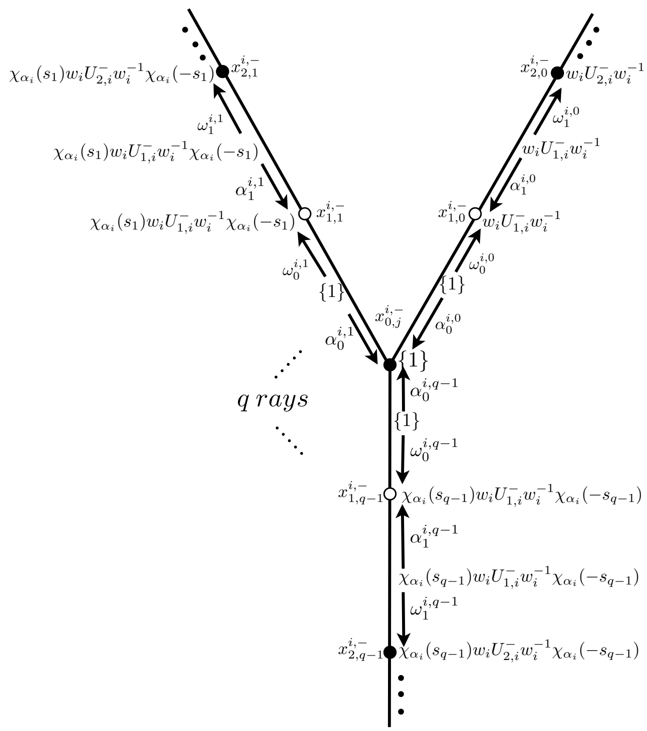
Lemma 4.2. *For $s \in \mathbb{F}_q$, $i = 1, 2$*

$$\psi_{i,U^-}[\chi_{\alpha_i}(s)] = \psi_{i,U^-}[\chi_{w_i(-\alpha_i)}(s)] = 1.$$

Proof. This is obvious from the definition of ψ_{i,U^-}. □

Lemma 4.3. *For each $i = 1, 2$, $s_j \in \mathbb{F}_q$, $j = 0,\ 1,\ 2, \dots,\ q-1$, with $s_0 = 0$ we have*

$$\chi_{\alpha_i}(s_j)w_iU_{n,i}^-w_i^{-1}\chi_{\alpha_j}(-s_j) \subseteq K_{i,U^-}.$$



Proof. For each $i = 1, 2,\ n \geq 1,\ s_k \in \mathbb{F}_q,\ k = 0, 1, 2, \ldots, n$ we have,

$$\begin{aligned} w_iU^-_{n,i}w_i^{-1} &= \{w_i\chi_{-\alpha_{3-i}}(s_1)\chi_{-w_{3-i}\alpha_i}(s_2)\ldots\chi_{\alpha_k}(s_n)w_i^{-1}\} \\ &= \{(w_i\chi_{-\alpha_{3-i}}(s_1)w_i^{-1})(w_i\chi_{-w_{3-i}\alpha_i}(s_2)w_i^{-1})\ldots(w_i\chi_{\alpha_k}(s_n)w_i^{-1})\} \\ &= \{\chi_{-w_i\alpha_{3-i}}(s_1)\chi_{-w_iw_{3-i}\alpha_i}(s_2)\ldots\chi_{w_i\alpha_k}(s_n)\} \end{aligned}$$

where for each k, $\alpha_k \in \Delta^{re}_{i,-}$ with $l(\alpha_k) = n$. Hence for each $i = 1, 2,\ s_j \in \mathbb{F}_q,\ j = 0,\ 1,\ 2, \ldots,\ q-1$, with $s_0 = 0$

$$\begin{aligned} \psi_{i,U^-}[\chi_{\alpha_i}(s_j)w_iU^-_{n,i}w_i^-\chi_{\alpha_i}(-s_j)] &= \psi_{i,U^-}[\chi_{\alpha_i}(s_j)\chi_{-w_i\alpha_{3-i}}(s_1)\chi_{-w_iw_{3-i}\alpha_i}(s_2)\ldots\chi_{w_i\alpha_k}(s_n)\chi_{\alpha_i}(-s_j)] \\ &= \{1\}, \end{aligned}$$

by the definition of ψ_{i,U^-}. $\square$

Theorem 4.4. *Let G be a rank 2 affine or hyperbolic complete simply–connected Kac–Moody group over the finite field $k = \mathbb{F}_q$. Let U^- be subgroup of G generated by all negative real root subgroups. Let K_{i,U^-} be principal congruence subgroup of U^- for $i = 1, 2$, then we have,*

(1) for $i = 1, 2$, $K_{i,U^-} \cap P_i = \{1\}$.

(2) For $i = 1, 2$, $n \geq 1$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \ldots, q-1$, with $s_0 = 0$,

$$Stab_{K_{i,U^-}}(x^{i,-}_{n,j}) = \chi_{\alpha_i}(s_j)w_iU^-_{n,i}w_i^{-1}\chi_{\alpha_i}(-s_j).$$

(3) For $i = 1, 2$, $n \geq 1$, $j = 0, 1, 2, \ldots, q-1$, the vertices $x^{i,-}_{n,j}$ are not equivalent mod K_{i,U^-}.

(4) For $i = 1, 2$, $n \geq 1$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \ldots, q-1$, with $s_0 = 0$ and Ω_i be as before the coset $\chi_{\alpha_i}(s_j) w_i U^-_{n,i} w_i^{-1} \chi_{\alpha_i}(-s_j)$ fixes the edge $(x^{i,-}_{n,j}, x^{i,-}_{n+1,j})$ and acts transitively on the set of edges with origin $x^{i,-}_{n,j}$ distinct from $(x^{i,-}_{n,j}, x^{i,-}_{n+1,j})$, sending all edges to $\chi_{\alpha_i}(s_j) w_i u_{n-1} B$, where $u_{n-1} \in \Omega_i$ with $\ell(u_{n-1}) = n-1$, setting $u_0 = 1$.

Proof. For (1), we recall that

$$K_{i,U^-} \cap P_i \subseteq (U^- \cap P_i) = U_{-\alpha_i}, \text{ by Lemma 3.3.}$$

But

$$K_{i,U^-} \cap U_{-\alpha_i} = \{1\}$$

by definition of K_{i,U^-}, so $K_{i,U^-} \cap P_i = \{1\}$.

For (2), from Proposition 3.4 for each $n \geq 1$, $i = 1, 2$, we have

$$U^-_{n,i} \quad \leq \quad u_n B u_{n-1}^{-1} \quad = \quad u_{n-1} P_t u_{n-1}^{-1} \cap u_n P_{3-t} u_n^{-1},$$

where $u_n \in W$ is the unique element of the Ω_i, where

$$\Omega_1 = \{1, w_2, w_2 w_1, w_2 w_1 w_2, ...\}, \quad \Omega_2 = \{1, w_1, w_1 w_2, w_1 w_2 w_1, ...\},$$

with $\ell(u_n) = n$, with setting $u_0 = 1$, and $t = i$ if n is even and $t = 3 - i$ if n is odd. Then for each $n \geq 1$, $i = 1, 2$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \ldots, q-1$ with $s_0 = 0$ we have

$$\begin{aligned}
\chi_{\alpha_i}(s_j) w_i U^-_{n,i} w_i^{-1} \chi_{\alpha_i}(-s_j) &\subseteq \chi_{\alpha_i}(s_j) w_i u_n B u_n^{-1} w_i^{-1} \chi_{\alpha_i}(-s_j) \\
&= \chi_{\alpha_i}(s_j) w_i u_{n-1} P_t u_{n-1}^{-1} w_i^{-1} \chi_{\alpha_i}(-s_j) \cap \chi_{\alpha_i}(s_j) w_i u_n P_{3-t} u_n^{-1} w_i^{-1} \chi_{\alpha_i}(-s_j) \\
&\subseteq \chi_{\alpha_i}(s_j) w_i u_{n-1} P_t u_{n-1}^{-1} w_i^{-1} \chi_{\alpha_i}(-s_j).
\end{aligned}$$

where $t = i$ if n is even and $t = 3 - i$ if n is odd. Hence

$$\begin{aligned}
\chi_{\alpha_i}(s_j) w_i U^-_{n,i} w_i^{-1} \chi_{\alpha_i}(-s_j) &\subseteq \chi_{\alpha_i}(s_j) w_i u_{n-1} P_t u_{n-1}^{-1} w_i^{-1} \chi_{\alpha_i}(-s_j) \cap K_{i,U^-} \\
&= Stab_G(x^{i,-}_{n,j}) \cap K_{i,U^-} \\
&= Stab_{K_{i,U^-}}(x^{i,-}_{n,j})
\end{aligned}$$

for $i = 1, 2$, $n \geq 1$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \ldots, q-1$, with $s_0 = 0$. Conversely,

$$\begin{aligned}
Stab_{K_{i,U^-}}(x^{i,-}_{n,j}) &= \chi_{\alpha_i}(s_j) w_i u_{n-1} P_t u_{n-1}^{-1} w_i^{-1} \chi_{\alpha_i}(-s_j) \cap K_{i,U^-} \\
&= \chi_{\alpha_i}(s_j) w_i (u_{n-1} P_t u_{n-1}^{-1} \cap K_{i,U^-}) w_i^{-1} \chi_{\alpha_i}(-s_j) \\
&\subseteq \chi_{\alpha_i}(s_j) w_i (u_{n-1} P_t u_{n-1}^{-1} \cap U^-) w_i^{-1} \chi_{\alpha_i}(-s_j) \\
&= \chi_{\alpha_i}(s_j) w_i Stab_{U^-}(x^{i,-}_{n,j}) w_i^{-1} \chi_{\alpha_i}(-s_j) \\
&= \chi_{\alpha_i}(s_j) w_i U^-_{n,i} w_i^{-1} \chi_{\alpha_i}(-s_j).
\end{aligned}$$

So, for $i = 1, 2$, $n \geq 1$,

$$\chi_{\alpha_i}(s_j) w_i U^-_{n,i} w_i^{-1} \chi_{\alpha_i}(-s_j) = Stab_{K_{i,U^-}}(x^{i,-}_{n,j}),$$

for each $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \ldots, q-1$, with $s_0 = 0$.

Since $K_{i,U^-} \leq U^-$, (3) of the theorem follows easily from Proposition 3.4(3).

For (4), the chain of inclusions

$$U_{1,i}^{-} \subseteq U_{2,i}^{-} \subseteq U_{3,i}^{-} \subseteq \dots,$$

for $i = 1, 2$, yields

$$\chi_{\alpha_i}(s_j) w_i U_{1,i}^{-} w_i^{-1} \chi_{\alpha_i}(-s_j) \quad \subseteq \quad \chi_{\alpha_i}(s_j) w_i U_{2,i}^{-} w_i^{-1} \chi_{\alpha_i}(-s_j) \quad \subseteq \quad \chi_{\alpha_i}(s_j) w_i U_{3,i}^{-} w_i^{-1} \chi_{\alpha_i}(-s_j) \dots$$

which shows that for $n \geq 1$, $i = 1, 2$ and $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$, with $s_0 = 0$, the coset $\chi_{\alpha_i}(s_j) w_i U_{n,i}^{-} w_i^{-1} \chi_{\alpha_i}(-s_j)$ fixes the edge $(x_{n,j}^{i,-}, x_{n+1,j}^{i,-})$.

For $n \geq 1$, $i = 1, 2$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$ with $s_0 = 0$ the edge $(x_{n,j}^{i,-}, x_{n+1,j}^{i,-})$ corresponds to the coset $\chi_{\alpha_i}(s_j) w_i u_{n-1} w_t B$ where u_{n-1} is the unique element of Ω_i with $\ell(u_{n-1}) = n-1$ having set $u_0 = 1$, and $t = i$ if n is even and $t = 3 - i$ if n is odd. For each $n \geq 1$, $i = 1, 2$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$ with $s_0 = 0$, the coset $\chi_{\alpha_i}(s_j) w_i u_{n-1} w_t B$ belongs to the set

$$E_0^X(x_{n,j}^{i,-}) = \{\chi_{\alpha_i}(s_j) w_i u_{n-1} B\} \cup \{\chi_{\alpha_i}(s_j) w_i u_{n-1} \chi_{\alpha_t}(s) w_t B \mid s_j \in \mathbb{F}_q, s \in \mathbb{F}_q\}.$$

We are left to consider

$$\{\chi_{\alpha_i}(s_j) w_i u_{n-1} B\} \cup \{\chi_{\alpha_i}(s_j) w_i u_{n-1} \chi_{\alpha_t}(s) w_t B \mid s_j \in \mathbb{F}_q, s \in \mathbb{F}_q^{\times}\}$$

which are the edges with origin $x_{n,j}^{i,-}$ distinct from $(x_{n,j}^{i,-}, x_{n+1,j}^{i,-})$, for $n \geq 1$, $i = 1, 2$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$ with $s_0 = 0$. We claim that if for some $u_i \in G$

$$u_i \cdot [\chi_{\alpha_i}(s_j) w_i u_{n-1} \chi_{\alpha_t}(s) w_t B] = \chi_{\alpha_i}(s_j) w_i u_{n-1} B = (x_{n-1,j}^{i,-}, x_{n,j}^{i,-})$$

then

$$u_i \in \chi_{\alpha_i}(s_j) w_i U_{n,i}^{-} w_i^{-1} \chi_{\alpha_i}(-s_j).$$

In fact for $n \geq 1$, $i = 1, 2$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$ with $s_0 = 0$

$$\begin{aligned} u_i &= \chi_{\alpha_i}(s_j) w_i u_{n-1} w_t^{-1} \chi_{\alpha_t}(-s) u_{n-1}^{-1} w_i^{-1} \chi_{\alpha_i}(-s_j) \\ &\in \chi_{\alpha_i}(s_j) w_i u_{n-1} U_{-\alpha_t} u_{n-1}^{-1} w_i^{-1} \chi_{\alpha_i}(-s_j) \\ &= \chi_{\alpha_i}(s_j) w_i U_{-u_{n-1}\alpha_t} w_i^{-1} \chi_{\alpha_i}(-s_j) \\ &\subseteq \chi_{\alpha_i}(s_j) w_i U_{n,i}^{-} w_i^{-1} \chi_{\alpha_i}(-s_j). \end{aligned}$$

This completes the proof of (4). □

The following corollary is immediate.

Corollary 4.5. *For $i = 1, 2$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$ with $s_0 = 0$, we have*

$$K_{i,U^-} \quad = \quad *_{s_j \in \mathbb{F}_q}[\bigcup_{n \geq 1} \chi_{\alpha_i}(s_j) w_i U_{n,i}^{-} w_i^{-1} \chi_{\alpha_i}(-s_j)].$$

We note that for $i = 1, 2$, the group K_{i,U^-} is a nonuniform lattice subgroup of G.

4.1. **Principal congruence subgroup of P_1^-.** For $k = \mathbb{F}_q$ and $q = 2^s$, we define the principal congruence subgroup $K_{P_1^-}$ of P_1^- as follows:

$$K_{P_1^-} = Ker\{\psi_{P_1^-} : P_1^- \to P_0 \mid U_\alpha \mapsto \{1\} \text{ if } \ell(\alpha) \geq 1, \alpha \in \Delta_-^{re} \setminus \{-\alpha_1\}\}.$$

Where $P_0 = \phi_1(SL_2(\mathbb{F}_q))$ is generated by $\chi_{\alpha_1}(s)$, $\chi_{-\alpha_1}(t)$, $s, t \in \mathbb{F}_q$ and ϕ_1 is the map defined in Lemma 3.6.

Lemma 4.6 and Lemma 4.7 justify the definition of $K_{P_1^-}$.

Lemma 4.6. *For $k = \mathbb{F}_q$ and $q = 2^s$*

$$\psi_{P_1^-}(U^-) = U_{-\alpha_1}.$$

The proof of Lemma 4.6 follows from the definition of $\psi_{P_1^-}$.

Lemma 4.7. *For $s \in \mathbb{F}_q^\times$ and $q = 2^s$, we have*

$$\psi_{P_1^-}[h_{\alpha_1}(s)] = \chi_{-\alpha_1}(-s^{-1})\chi_{-\alpha_1}(1^{-1}).$$

Proof. For $s \in \mathbb{F}_q^\times$ and $q = 2^s$

$$\begin{aligned} h_{\alpha_1}(s) &= \widetilde{w}_1(s)\widetilde{w}_1(1)^{-1} \\ &= \chi_{\alpha_1}(s)\chi_{-\alpha_1}(-s^{-1})\chi_{\alpha_1}(s)\chi_{\alpha_1}(1)\chi_{-\alpha_1}(1^{-1})\chi_{\alpha_1}(1). \end{aligned}$$

Hence,

$$\psi_{P_1^-}[h_{\alpha_1}(s)] = \chi_{-\alpha_1}(-s^{-1})\chi_{-\alpha_1}(1^{-1}). \ \square$$

Lemma 4.8. *For $k = \mathbb{F}_q$ and $q = 2^s$*

$$K_{P_1^-} \cap H_{\alpha_1} = \{1\}$$

Lemma 4.8 is a consequence of Lemma 4.7.

Lemma 4.9. *For $s \in k = \mathbb{F}_q$, $q = 2^s$, we have*

$$\psi_{P_1^-}[\chi_{\alpha_1}(s)w_1 H_{\alpha_2} w_1^{-1}\chi_{\alpha_1}(-s)] = \{1\}.$$

Proof. For $s \in \mathbb{F}_q$ and $t \in \mathbb{F}_q^\times$, we have

$$\begin{aligned} & w_1 h_{\alpha_2}(t) w_1^{-1} \\ &= w_1(\widetilde{w}_2(t)\widetilde{w}_2(1)^{-1})w_1^{-1} \\ &= w_1[\chi_{\alpha_2}(t)\chi_{-\alpha_2}(-t^{-1})\chi_{\alpha_2}(t)\chi_{\alpha_2}(1)\chi_{-\alpha_2}(1^{-1})\chi_{\alpha_2}(1)]w_1^{-1} \\ &= (w_1\chi_{\alpha_2}(t)w_1^{-1})(w_1\chi_{-\alpha_2}(-t^{-1})w_1^{-1})(w_1\chi_{\alpha_2}(t)w_1^{-1}) \\ &\quad \cdot (w_1\chi_{\alpha_2}(1)w_1^{-1})(w_1\chi_{-\alpha_2}(1^{-1})w_1^{-1})(w_1\chi_{\alpha_2}(1)w_1^{-1}) \\ &= \chi_{w_1\alpha_2}(t)\chi_{-w_1\alpha_2}(-t^{-1})\chi_{w_1\alpha_2}(t)\chi_{w_1\alpha_2}(1)\chi_{-w_1\alpha_2}(1^{-1})\chi_{w_1\alpha_2}(1). \end{aligned}$$

So

$$\begin{aligned} & \chi_{\alpha_1}(s)w_1 h_{\alpha_2}(t) w_1^{-1}\chi_{\alpha_1}(-s) \\ &= \chi_{\alpha_1}(s)\chi_{w_1\alpha_2}(t)\chi_{-w_1\alpha_2}(-t^{-1})\chi_{w_1\alpha_2}(t)\chi_{w_1\alpha_2}(1)\chi_{-w_1\alpha_2}(1^{-1})\chi_{w_1\alpha_2}(1)\chi_{\alpha_1}(-s). \end{aligned}$$

Hence

$$\psi_{P_1^-}[\chi_{\alpha_1}(s)w_1 H_{\alpha_2} w_1^{-1}\chi_{\alpha_1}(-s)] = \{1\},$$

by the definition of $\psi_{P_1^-}$. $\square$

Lemma 4.10. *For $n \geq 1$, $s \in \mathbb{F}_q$, $q = 2^s$,*

$$\psi_{P_1^-}[\chi_{\alpha_1}(s)w_1 U_{n,1}^- w_1^{-1}\chi_{\alpha_1}(-s)] = \{1\}.$$

Proof. For $n \geq 1$, $u_\alpha, s \in \mathbb{F}_q$, $q = 2^s$,

$$\begin{aligned} \chi_{\alpha_1}(s) w_1 U_{n,1}^{-} w_1^{-1} \chi_{\alpha_1}(-s) &= \chi_{\alpha_1}(s) w_1 [\{ \prod_{\substack{\alpha \in \Delta_{1,-}^{re} \\ 1 \leq \ell(\alpha) \leq n}} \chi_\alpha(u_\alpha)\}] w_1^{-1} \chi_{\alpha_1}(-s) \\ &= \chi_{\alpha_1}(s) \{ \prod_{\substack{\alpha \in \Delta_{1,-}^{re} \\ 1 \leq \ell(\alpha) \leq n}} [w_1 \chi_\alpha(u_\alpha) w_1^{-1}] \} \chi_{\alpha_1}(-s) \\ &= \chi_{\alpha_1}(s) \{ \prod_{\substack{\alpha \in \Delta_{1,-}^{re} \\ 1 \leq \ell(\alpha) \leq n}} \chi_{w_1 \alpha}(u_\alpha) \} \chi_{\alpha_1}(-s). \end{aligned}$$

So, for $n \geq 1$, $s \in \mathbb{F}_q$,

$$\psi_{P_1^-}[\chi_{\alpha_1}(s) w_1 U_{n,1}^{-} w_1^{-1} \chi_{\alpha_1}(-s)] = \{1\},$$

by the definition of $\psi_{P_1^-}$.

In order to give the graph of groups presentation for $K_{P_1^-}$, we take the following coset representatives for P_1/B,

$$\mathcal{S} = \{g_j = \chi_{\alpha_1}(s_j) w_1 \mid s_j \in \mathbb{F}_q, j = 0, 1, 2, \ldots, q-1, with\; s_0 = 0\}.$$

For each $j = 0, 1, 2, \ldots, q-1$, we take translates $g_j \mathcal{A}_0^+$ of the positive half $\mathcal{A}_0^+$ of the fundamental apartment.

Then for $n \geq 1$, $j = 0, 1, 2, \ldots, q-1$ the half-ray $g_j \mathcal{A}_0^+$ has the following sequence of vertices

$$\{a_{0,j}^{1,-} = P_2, a_{n,j}^{1,-} = \chi_{\alpha_1}(s_j) w_1 u_{n-1} P_i \mid s_j \in \mathbb{F}_q, j = 0, 1, 2, \ldots, q-1, s_0 = 0\},$$

where u_{n-1} is unique element of Ω_1, $\ell(u_{n-1}) = n-1$ with $u_0 = 1$ and $i = 1$ if n is even and $i = 2$ if n is odd.

For $n \geq 1$, we define subgroups of P_1^-

$$K_{n,1}^{-} = \langle U_{n,1}^{-}, H_{\alpha_2} \rangle$$

which satisfy the following ascending chain of inclusions:

$$K_{1,1}^{-} \subseteq K_{2,1}^{-} \subseteq K_{3,1}^{-} \subseteq \ldots$$

Next lemma shows that translates of these subgroups by the cosets representatives of P_1/B are contained in the principal congruence subgroup $K_{P_1^-}$.

Lemma 4.11. *For $s \in k = \mathbb{F}_q$, $q = 2^s$,*

$$\chi_{\alpha_1}(s) w_1 K_{n,1}^{-} w_1^{-1} \chi_{\alpha_1}(-s) = \chi_{\alpha_1}(s) w_1 \langle U_{n,1}^{-}, H_{\alpha_2} \rangle w_1^{-1} \chi_{\alpha_1}(-s) \subseteq K_{P_1^-}.$$

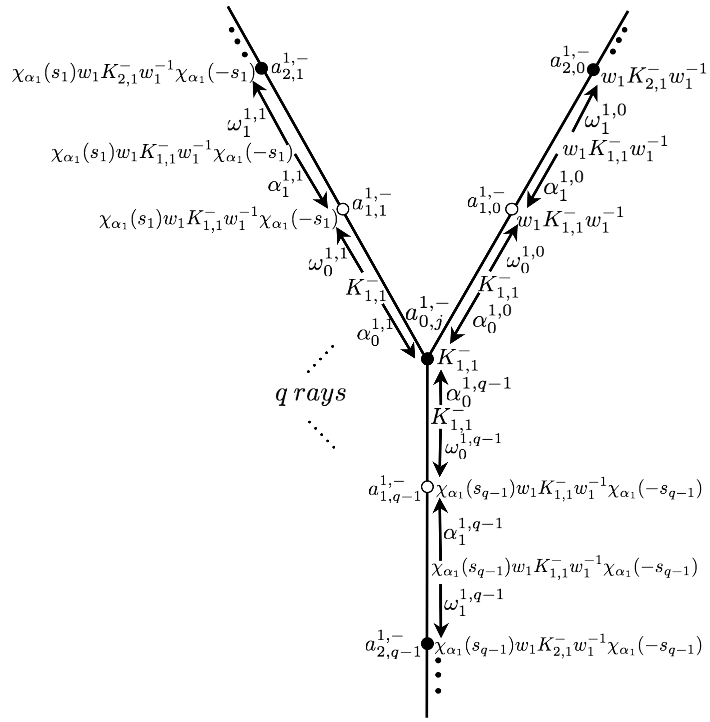
This follows from Lemma 4.9 and Lemma 4.10.

We now give the graph of groups for $K_{P_1^-}$.

Theorem 4.12. *Let G be an affine or hyperbolic complete simply–connected Kac–Moody group over the finite field $k = \mathbb{F}_q$ and assume that $q = 2^s$. Let P_1^- be the negative standard parabolic subgroup of G. Then the principal congruence subgroup*

$$K_{P_1^-} = Ker\{\psi_{P_1^-} : P_1^- \to P_0 \mid U_\alpha \mapsto \{1\} \text{ if } \ell(\alpha) \geq 1, \alpha \in \Delta_-^{re} \setminus \{-\alpha_1\}\}.$$

of P_1^- has the following graph of groups:



where where the total number of rays is $q+1$, indexed over the set $\mathcal{S}$,

$$\alpha_0^{1,j} : K^-_{1,1} \to K^-_{1,1}$$

and

$$\alpha_n^{1,j} : \chi_{\alpha_1}(s_j)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j) \to \chi_{\alpha_1}(s_j)w_iK^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j)$$

are identity maps, for $n \geq 1$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$ setting $s_0 = 0$,

$$\omega_0^{1,j} : K^-_{1,1} \to \chi_{\alpha_i}(s_j)w_iK^-_{1,1}w_i^{-1}\chi_{\alpha_i}(-s_j)$$

$$\omega_n^{1,j} : \chi_{\alpha_1}(s_j)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j) \hookrightarrow \chi_{\alpha_i}(s_j)w_iK^-_{n+1,1}w_i^{-1}\chi_{\alpha_i}(-s_j)$$

are inclusion maps, for $n \geq 1$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots, q-1$ setting $s_0 = 0$. □

Theorem 4.12 is a consequence of the following.

Proposition 4.13. *Let $K_{P_1^-}$ be the principal congruence subgroup of the standard parabolic subgroup P_1^-. Let $k = \mathbb{F}_q$ and assume that $q = 2^s$.*

(1) We have $K_{P_1^-} \cap P_2 = K^-_{1,1}$.

(2) For $n \geq 1$, $s_j \in k = \mathbb{F}_q$, $q = 2^s$, $j = 0, 1, 2, \dots, q-1$, with $s_0 = 0$ we have

$$Stab_{K_{P_1^-}}(a^{1,-}_{n,j}) = \chi_{\alpha_1}(s_j)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j).$$

(3) The vertices $a^{1,-}_{n,j}$, $n \geq 1$, $j = 0, 1, 2, \dots, q-1$ are not equivalent mod $K_{P_1^-}$.

(4) For $n \geq 1$, $s_j \in k = \mathbb{F}_q$, $q = 2^s$, $j = 0, 1, 2, \dots, q-1$ with $s_0 = 0$, the coset $\chi_{\alpha_1}(s_j)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j)$ fixes the edge $(a^{1,-}_{n,j}, a^{1,-}_{n+1,j})$ and acts transitively on the set of edges with origin $a^{1,-}_{n,j}$ distinct from

$(a^{1,-}_{n,j}, a^{1,-}_{n+1,j})$ *taking all edges to* $\chi_{\alpha_1}(s_j)w_1u_{n-1}B$*, where* $u_{n-1} \in \Omega_1$ *with* $\ell(u_{n-1}) = n-1$*, setting* $u_0 = 1$.

Proof. We have

$$K_{P_1^-} \cap P_2 \subseteq P_1^- \cap P_2 = B^-_{1,1} = \langle H, U_{-\alpha_2}\rangle.$$

But $K_{P_1^-} \cap H = H_{\alpha_2}$ and $U_{-\alpha_2} \subset K_{P_1^-}$, so $K_{P_1^-} \cap P_2 = K^-_{1,1}$. This proves (1).

For (2), for $n \geq 1$ and $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \ldots, q-1$, setting $s_0 = 0$ we have

$$\begin{aligned} \chi_{\alpha_1}(s_j)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j) &= \chi_{\alpha_1}(s_j)w_1\langle H_{\alpha_2}, U^-_{n,1}\rangle w_1^{-1}\chi_{\alpha_1}(-s_j) \\ &\subset \chi_{\alpha_1}(s_j)w_1B^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j). \end{aligned}$$

Also from Lemma 4.11

$$\chi_{\alpha_1}(s)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s) = \chi_{\alpha_1}(s)w_1\langle U^-_{n,1}, H_{\alpha_2}\rangle w_1^{-1}\chi_{\alpha_1}(-s) \subseteq K_{P_1^-}$$

So,

$$\begin{aligned} \chi_{\alpha_1}(s_j)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j) &\subseteq \chi_{\alpha_1}(s_j)w_1B^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j) \cap K_{P_1^-} \\ &= Stab_{P_1^-}(a^{1,-}_{n,j}) \cap K_{P_1^-} \\ &= Stab_{K_{P_1^-}}(a^{1,-}_{n,j}). \end{aligned}$$

Conversely, for $n \geq 1$, $s_j \in k = \mathbb{F}_q$, $q = 2^s$, $j = 0, 1, 2, \ldots, q-1$, with $s_0 = 0$ and $t = 1$ if n is odd and $t = 2$ if n is even,

$$\begin{aligned} Stab_{K_{P_1^-}}(a^{1,-}_{n,j}) &= \chi_{\alpha_1}(s_j)w_1u_{n-1}P_tw_1^{-1}\chi_{\alpha_1}(-s_j) \cap K_{P_1^-} \\ &\subseteq \chi_{\alpha_1}(s_j)w_1u_{n-1}P_tw_1^{-1}\chi_{\alpha_1}(-s_j) \cap P_1^- \\ &= \chi_{\alpha_1}(s_j)w_1(u_{n-1}P_t \cap P_1^-)w_1^{-1}\chi_{\alpha_1}(-s_j) \\ &= \chi_{\alpha_1}(s_j)w_1\langle H, U^-_{n,1}\rangle w_1^{-1}\chi_{\alpha_1}(-s_j) \\ &= \chi_{\alpha_1}(s_j)w_1B^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j). \end{aligned}$$

So $Stab_{K_{P_1^-}}(a^{1,-}_{n,j}) \subseteq \chi_{\alpha_1}(s_j)w_1B^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j) \cap K_{P_1^-}$. Hence $Stab_{K_{P_1^-}}(a^{1,-}_{n,j}) = \chi_{\alpha_1}(s_j)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j)$.

Part (3) is obvious.

For (4), the chain of inclusions $K^-_{1,1} \subseteq K^-_{2,1} \subseteq K^-_{3,1} \subseteq \ldots$ yields

$$\chi_{\alpha_1}(s_j)w_1K^-_{1,1}w_1^{-1}\chi_{\alpha_1}(-s_j) \subseteq \chi_{\alpha_1}(s_j)w_1K^-_{2,1}w_1^{-1}\chi_{\alpha_1}(-s_j) \subseteq \chi_{\alpha_1}(s_j)w_1K^-_{3,1}w_1^{-1}\chi_{\alpha_1}(-s_j)\ldots$$

which shows that for $n \geq 1$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, 3, \ldots, q-1$, with $s_0 = 0$, the coset $\chi_{\alpha_1}(s_j)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j)$ fixes the edge $(a^{1,-}_{n,j}, a^{1,-}_{n+1,j})$.

For $n \geq 1$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \ldots, q-1$ with $s_0 = 0$, the edge $(a^{1,-}_{n,j}, a^{1,-}_{n+1,j})$ corresponds to the coset $\chi_{\alpha_1}(s_j)w_1u_{n-1}w_tB$ where u_{n-1} is the unique element of Ω_1 with $\ell(u_{n-1}) = n-1$ having set $u_0 = 1$ and $t = 1$ if n is even and $t = 2$ if n is odd. The coset $\chi_{\alpha_1}(s_j)w_1u_{n-1}w_tB$ for $j = 0, 1, 2, 3, \ldots, q-1$ is an elements of the set

$$E_0^X(a^{1,-}_{n,j}) = \{\chi_{\alpha_1}(s_j)w_1u_{n-1}B\} \cup \{\chi_{\alpha_1}(s_j)w_1u_{n-1}\chi_{\alpha_t}(s)w_tB \mid s_j \in \mathbb{F}_q, s \in \mathbb{F}_q\}$$

for each $n \geq 1$, $s_j \in \mathbb{F}_q$, $j = 0, 1, 2, \dots q-1$ with $s_0 = 0$. We are left to consider

$$\{\chi_{\alpha_1}(s_j)w_1u_{n-1}B\} \cup \{\chi_{\alpha_1}(s_j)w_1u_{n-1}\chi_{\alpha_t}(s)w_tB \mid s_j \in \mathbb{F}_q, s \in \mathbb{F}_q^\times\}$$

which are the edges with origin $a^{1,-}_{n,j}$ distinct from $(a^{1,-}_{n,j}, a^{1,-}_{n+1,j})$, $j = 0, 1, 2, \dots, q-1$. We claim that for some $u_i \in K_{P_1^-}$

$$u_i \cdot [\chi_{\alpha_1}(s_j)w_1u_{n-1}\chi_{\alpha_t}(s)w_tB] = \chi_{\alpha_1}(s_j)w_1u_{n-1}B = (a^{1,-}_{n-1,j}, a^{1,-}_{n,j})$$

then by Theorem 4.4

$$u_i \in \chi_{\alpha_1}(s_j)w_1U^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j).$$

Also,

$$u_{n-1}^{-1}w_1^{-1}\chi_{\alpha_1}(-s_j)u_i\chi_{\alpha_1}(s_j)w_1u_{n-1}\chi_{\alpha_t}(s)w_t \in B,$$

which implies

$$u_{n-1}^{-1}w_1^{-1}\chi_{\alpha_1}(-s_j)u_i\chi_{\alpha_1}(s_j)w_1u_{n-1}\chi_{\alpha_t}(s)w_t \in B \cap K_{P_1^-} = H_{\alpha_2}.$$

So,

$$u_i \in \chi_{\alpha_1}(s_j)w_1H_{\alpha_2}w_1^{-1}\chi_{\alpha_1}(-s_j).$$

Hence

$$u_i \in \chi_{\alpha_1}(s_j)w_1\langle H_{\alpha_2}, U^-_{n,1}\rangle w_1^{-1}\chi_{\alpha_1}(-s_j) = \chi_{\alpha_1}(s_j)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j).$$

This completes the proof. □

The following corollary is immediate.

Corollary 4.14. *For $s_j \in k = \mathbb{F}_q$, $q = 2^s$ we have*

$$K_{P_1^-} \quad = \quad *_{K^-_{1,1}}[\bigcup_{n\geq 1, s_j \in \mathbb{F}} \chi_{\alpha_1}(s_j)w_1K^-_{n,1}w_1^{-1}\chi_{\alpha_1}(-s_j)].$$

Where $j = 0, 1, 2, \dots, q-1$ with $s_0 = 0$. □

We note that $K_{P_1^-}$ is a nonuniform lattice subgroup of G whose graph of groups has volume a scalar multiple of $1 + \sum_{n\geq 1} \dfrac{1}{q^n}$.

5. Other directions

5.1. **Higher level congruence subgroups.** The congruence subgroups K_{i,U^-} and $K_{P_1^-}$ defined in the previous sections may be viewed as principal or 'level 1' congruence subgroups in an infinite sequence of congruence subgroups. In this subsection we define higher level congruence subgroups of P_1^-. We recall that

$$K_{P_1^-} = Ker\{\psi_{P_1^-} : P_1^- \to P_0 \mid U_\alpha \mapsto \{1\} \text{ if } \ell(\alpha) \geq 1, \alpha \in \Delta^{re}_- \setminus \{-\alpha_1\}\},$$

where $P_0 = \phi_1(SL_2(\mathbb{F}_q))$ and ϕ_1 is the map defined in Lemma 3.6.

For $n \geq 2$, we define

$$K_{P_1^-}(n) = Ker\{\rho : P_1^- \to Q_n \mid U_\alpha \mapsto 1 \text{ if } \ell(\alpha) \geq n\},$$

where $Q_n = \phi_n(SL_2(\mathbb{F}_q[t]/(t^n)))$, where ϕ_n is the map that extends naturally from $SL_2(\mathbb{F}_q)$ to $SL_2(\mathbb{F}_q[t]/(t^n))$.

Higher level congruence subgroups of U^- can be defined similarly. By analogy with the classical case, we expect the graphs of groups of $K_{P_1^-}(n)$ to be significantly more complicated than that of $K_{P_1^-}$ (*cf.* [CCM]). Instead of a core graph consisting of a single vertex, as is the case for $K_{P_1^-}$, we expect to find a core graph consisting of a complete bipartite graph with the number of cusps growing exponentially with the size of the core.

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