CHARACTERIZATION OF NON-MINIMAL TREE ACTIONS

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ABSTRACT. Let Γ be a group acting without inversions on a tree X. If there is no proper Γ-invariant subtree, we call the action of Γ on X *minimal*. Here we give a characterization of *non-minimal* tree actions. For a non-minimal action of Γ on X, we give structure theorems for the quotient graph of groups for Γ on X, its associated edge-indexed graph and its group of deck transformations.

1. Introduction.

Let Γ be a group acting without inversions on a locally finite tree X. We say that Γ acts *minimally* on X if there is no proper Γ-invariant subtree. As is the case with well known results concerning group actions on $\mathbb{R}$-trees, when the hyperbolic length function $l(\Gamma)$ of Γ is non-zero, there is a unique minimal Γ-invariant subtree $X_0 \subset X$ (see for example [B]). Now set $G = Aut(X)$. If X is uniform, that is, X is the universal cover of a finite connected graph, then by ([BL], (9.7)), $l(G) \neq 0$ and hence there is a unique minimal G-invariant subtree $X_0 \subset X$.

Here we give a characterization of *non-minimal* actions in terms of the quotient graph of groups for Γ on X, its associated edge-indexed graph (Section 2) and its group of deck transformations, that is, the group of automorphisms of X that commute with the quotient morphism (Section 5). In each case, we give a 'decomposition' of the action into 'minimal and non-minimal components' (Section 4). This gives rise to a structure theorem for the group of deck transformations as a semi-direct product of its minimal component and a product of finite groups (Section 5).

When the edge-indexed quotient graph is finite, we show that it contains a unique minimal invariant subgraph which we describe geometrically (Section 4). Moreover, we give a procedure to find the minimal invariant subgraph.It follows that the minimal Γ-invariant subtree X_0 of X is the universal covering of the minimal component of the quotient (Section 4).

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Our motivation for this work is two-fold. Primarily the results here of Section 2 and Section 4 are an important ingredient for the proof in [C] of Bass-Lubotzky's conjecture for the existence of non-uniform lattices on uniform trees ([BL], Ch 8). Let X be a uniform tree and let $G = Aut(X)$. Let X_0 be the unique minimal G-invariant subtree of X. Let $G_0 = Aut(X) \mid_{X_0}$. Here we show that that $G_0 = Aut(X_0)$ (Section 5). In [C], the first author showed how to extend non-uniform lattice subgroups $\Gamma_0 \leq G_0 = Aut(X_0)$ to non-uniform lattices $\Gamma \leq G = Aut(X)$. An important ingredient in this proof is the geometric characterization of an edge-indexed graph in terms of its unique minimal subgraph obtained here in Section 4.

Secondly, this work fills a gap in the literature concerning non-minimal actions on simplicial trees. Our results in Sections 3 and 4 are essentially a reworking, in terms of edge-indexed graphs, of known results (see for example [B] and [BL]). Our structure theorem (Section 5) for the deck transformation group of a non-minimal action is a straightforward consequence of [BL] Section 6, but does not seem to appear in the literature.

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2. Group actions, edge-indexed quotients and minimality.

Let Γ be a group acting without inversions on a tree X. The fundamental theorem of Bass and Serre states that Γ is encoded (up to isomorphism) in a quotient graph of groups $\mathbb{A} = \Gamma\backslash\backslash X$ ([B], [S]). Conversely a graph of groups $\mathbb{A}$ gives rise to a group $\Gamma = \pi_1(\mathbb{A}, a)$, $a \in V\mathbb{A}$, acting on a tree $X = \widetilde{(\mathbb{A}, a)}$ without inversions.

Now assume that X is locally finite, and that Γ acts on X with quotient graph of groups $\mathbb{A} = \Gamma\backslash\backslash X$. Then $\mathbb{A}$ naturally gives rise to an 'edge-indexed' graph (A, i), defined as follows. The graph A is the underlying graph of $\mathbb{A}$ with vertex set VA, edge set EA, initial and terminal functions $\partial_0, \partial_1 : EA \longmapsto VA$ which pick out the endpoints of an edge, and with fixed point free involution $- : EA \longmapsto EA$ which reverses the orientation. The indexing $i : EA \longmapsto \mathbb{Z}_{>0}$ of (A, i) is defined to be the group theoretic index

$$i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e(\mathcal{A}_e)],$$

where

$$(\mathcal{A}_a)_{a \in VA} \text{ and } (\mathcal{A}_e = \mathcal{A}_{\overline{e}})_{e \in EA}$$

are the vertex and edge groups of $\mathbb{A}$, and $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}$ are the boundary monomorphisms of $\mathbb{A}$. We write $(A, i) = I(\mathbb{A})$ when $i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e(\mathcal{A}_e)]$ for data

$$\{\mathcal{A}_a,\ \mathcal{A}_e = \mathcal{A}_{\overline{e}},\ \alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}\}$$

from $\mathbb{A}$. Conversely, an *edge-indexed* graph (A, i) is defined to be a graph A and an assignment $i(e)$ ($\neq i(\overline{e})$ in general) of a positive integer to each oriented edge. Then (A, i) determines a universal covering tree $X = \widetilde{(A,i)}$ up to isomorphism, and every edge-indexed graph arises from a tree action [BL]. Here we assume $i(e)$ is finite for each $e \in EA$. Under this assumption the universal covering tree $X = \widetilde{(A,i)}$ is locally finite [BL].

Let (A,i) be an edge-indexed graph. We say that (A,i) is *minimal* if it is the edge-indexed quotient of a minimal tree action. A vertex $a \in VA$ is a *terminal vertex* of (A,i) if $deg_{(A,i)}(a) = 1$, where

$$deg_{(A,i)}(a) = \sum_{e \in E_0(a)} i(e),$$

and $E_0(a) = \{e \in EA \mid \partial_0 e = a\}$. A terminal vertex in (A,i) is then a geometrically terminal vertex in the graph A. That is, there is a unique edge e with $\partial_0 e = a$. If (A,i) consists of a single vertex v, we set $deg_{(A,i)}(v) = 0$ and so v is not a terminal vertex.

Now let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups. If $A' \subset A$ is a connected subgraph of A, we write $\mathbb{A} \mid_{A'}$ for the subgraph of groups obtained by restriction to A'. Following ([B], (7.10)) we call $\mathbb{A}$ a *minimal graph of groups* if

$$\pi_1(\mathbb{A} \mid_{A'}, a_0) = \pi_1(\mathbb{A}, a_0)$$

only when $A' = A$, where $a_0 \in A'$. We call a vertex $a \in VA$ a *terminal vertex* of $\mathbb{A}$ if there is a unique edge e with $\partial_0 e = a$ and $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_a$ is an isomorphism.

(2.1) Lemma. *Let $\mathbb{A}$ be a graph of groups and let $(A,i) = I(\mathbb{A})$. Then $\mathbb{A}$ has no terminal vertices if and only if (A,i) has no terminal vertices.*

Proof: Let a be a terminal vertex of $\mathbb{A}$. Then there is a unique edge e with $\partial_0 e = a$ and $i(e) = [\mathcal{A}_a : \alpha_e(\mathcal{A}_e)] = 1$. That is, a is a terminal vertex of (A,i).

Conversely suppose (A,i) has a terminal vertex $a \in VA$, that is, $deg_{(A,i)}(a) = 1$ and a is a geometrically terminal vertex. This forces $[\mathcal{A}_a : \alpha_e(\mathcal{A}_e)] = 1$, which occurs if and only if $\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_a$ is an isomorphism. Thus a is a terminal vertex of $\mathbb{A}$. $\square$

3. Characterizing minimal actions.

Our applications in [C] and section 5 require a detailed description of a non-minimal action on a simplicial tree. We use the formulation of Bass ([B]) of a minimal graph of groups to describe minimality of a tree action $\Gamma \times X \longrightarrow X$ in terms of the edge-indexed quotient graph $(A,i) = I(\mathbb{A})$ of the graph of groups $\mathbb{A} = \Gamma \backslash\backslash X$.

Let $\mathbb{A}$ be a graph of groups. The notion of a minimal graph of groups of Bass ([B], (7.10)) is given in terms of intrinsic properties of $\mathbb{A}$. Bass then showed that minimal graphs of groups encode minimal tree actions. His result is the following:

(3.1) Theorem ([B], (7.12)). *Let* $\mathbb{A}$ *be a graph of groups, let* $a_0 \in VA$, $\Gamma = \pi_1(\mathbb{A}, a_0)$, *and* $X = \widetilde{(\mathbb{A}, a_0)}$. *The following conditions are equivalent.*

(a) The action of Γ *on* X *is minimal.*

(b) $\mathbb{A}$ *is minimal.*

These conditions imply:

(c) $\mathbb{A}$ *has no terminal vertices.*

If A *is finite then (c) is equivalent to the other conditions.*

The following proposition gives a restatement of Bass' theorem in terms of the edge-indexed graph arising from $\mathbb{A}$.

(3.2) Proposition. *Let* Γ *be a group acting without inversions on a tree* X *with quotient graph of groups* $\mathbb{A} = \Gamma \backslash\backslash X$ *and edge-indexed quotient* $(A, i) = I(\mathbb{A})$.

(1) If (A, i) *is minimal then* (A, i) *has no terminal vertices.*

(2) If (A, i) *is finite and has no terminal vertices then* (A, i) *is minimal.*

Proof: (1) Suppose that (A, i) is minimal. Then (A, i) is the edge-indexed quotient graph of a minimal action Γ' x $X' \longmapsto X'$ which corresponds to the action of Γ on X up to isomorphism. Without loss of generality we take $\Gamma' = \Gamma, X' = X$. Then by Theorem (3.1), $\mathbb{A} = \Gamma \backslash\backslash X$ has no terminal vertices, which by Lemma (2.1) is equivalent to (A, i) having no terminal vertices.

(2) Since (A, i) has no terminal vertices, $\mathbb{A}$ has no terminal vertices by Lemma (2.1). Assume that (A, i) is finite. Then by Theorem (3.1), Γ acts minimally on X. That is, $(A, i) = I(\Gamma \backslash\backslash X)$ is minimal. □

We remark that (2) in Proposition (3.2) is false without the assumption of finiteness of (A, i). This can be seen from the following example. Let (A, i) be the graph $A = \mathbb{Z}$ of the integer line with $i(e) = 1$ for all edges except for a finite number of $e_1, e_2, \ldots, e_n \in EA$ with $\partial_1 e_j = \partial_0 e_{j+1}$, $j = 1, \ldots, n-1$, so that $i(e_j) > 1$, $j = 1, \ldots, n$. If $\mathbb{A}$ is any graph of groups with $(A, i) = I(\mathbb{A})$, then $\mathbb{A}$ cannot be minimal, since $\pi_1(\mathbb{A} \mid_\gamma, a_0) = \pi_1(\mathbb{A}, a_0)$, where γ is the path $\gamma = (e_1, e_2, \ldots, e_n)$ and $a_0 \in \gamma$.

In summary, we see that the property of minimality can be detected equivalently from a graph of groups $\mathbb{A}$ or its edge-indexed graph (A, i) with $(A, i) = I(\mathbb{A})$. Combining the results of Lemma (2.1), Theorem (3.1) and Proposition (3.2), we have:

(3.3) Corollary. *Let* $\mathbb{A}$ *be a graph of groups and let* $(A, i) = I(\mathbb{A})$. *Assume that* A *is finite. Then the following conditions are equivalent:*

(1) $\mathbb{A}$ *is minimal,*

(2) $\mathbb{A}$ *has no terminal vertices,*

(3) (A, i) *is minimal,*

(4) (A, i) *has no terminal vertices.*

4. Structure of non-minimal actions.

In this section we show that a finite edge-indexed quotient graph (A, i) of a non-minimal action has a unique minimal subgraph $(A_0, i_0) \subset (A, i)$. We describe (A, i) and $X = \widetilde{(A, i)}$ geometrically in terms of their 'minimal components'. We first recall the following results of [B] and [BL] which give criteria for the existence of minimal subtrees.

(4.1) Proposition ([B], (7.5)). *Let Γ be a group acting on a tree X and let $l(\Gamma)$ be the hyperbolic length function of the action. If $l(\Gamma) \neq 0$ then there is a unique minimal Γ-invariant subtree X_Γ of X. We have*

$$X_\Gamma = \bigcup_{\gamma \in \Gamma,\ l(\gamma) > 0} X_\gamma,$$

where X_γ is the axis of translation of the hyperbolic element $\gamma \in \Gamma$.

If we consider subgroups of $Aut(X)$ rather than general group actions, we obtain the following criterion of Bass and Lubotzky for the existence of a unique minimal invariant subtree of X.

(4.2) Proposition ([BL], (9.7)). *Let X be a tree and let $G = Aut(X)$. If X is uniform then $l(G) \neq 0$ and there is a unique minimal G-invariant subtree $X_0 \subset X$.*

Our aim is to obtain analogous results to Propositions (4.1) and (4.2) in terms of edge-indexed quotient graphs.

Let (T, i) be an edge-indexed graph. As in [BL] and [R] we say that (T, i) is a *dominant edge-indexed tree* if T is a tree, $|VT| > 1$ and there is a vertex $a \in VT$ such that for all $e \in ET$

$$d(\partial_0 e, a) > d(\partial_1 e, a) \text{ implies } i(e) = 1.$$

We call such a vertex $a \in VT$ a *dominant root* of (T, i) and we write (T, i, a) when (T, i) is a dominant edge-indexed tree, rooted at $a \in VT$.

If (T, i, a) is a dominant rooted edge-indexed tree then it is clear from the definition that (T, i, a) contains a terminal vertex.

Let (A, i) be any finite edge-indexed graph. We may view (A, i) as the edge-indexed quotient of a tree action $\Gamma \text{ x } X \longrightarrow X$, with $X = \widetilde{(A, i)}$ $I(\Gamma \backslash\backslash X) = (A, i)$, that is in general non-minimal. The following theorem establishes the existence of a minimal subgraph $(A_0, i_0) \subset (A, i)$. The edge-indexed graph (A_0, i_0) will then coincide (up to isomorphism) with the edge-indexed quotient graph of any action $\Gamma_0 \text{ x } X_0 \longrightarrow X_0$ on the minimal subtree $X_0 \subset X$. Our assumption that (A, i) is finite guarantees that $X = \widetilde{(A, i)}$ is uniform and hence by Proposition (4.2) there is a unique minimal subtree $X_0 \subset X$.

Moreover, we will show that the minimal subgraph (A_0, i_0) is essesntially unique. That is, (A_0, i_0) is unique unless (A, i) is a tree, in which case (A_0, i_0) is unique up to isomorphism (Theorem (4.5) below).

(4.3) Theorem. *Let (A,i) be a finite edge-indexed graph. Then*

(1) There is a connected subgraph (A_0,i_0) of (A,i), maximal with respect to the property of having no terminal vertices, such that (A_0,i_0) contains the union of all closed paths in (A,i) and $(A,i)\setminus(A_0,i_0)$ is a disjoint union of edge-indexed trees.

(2) Let (A_0,i_0) be the subgraph of (A,i) as in (1). Then (A,i) is a union of (A_0,i_0) and dominant rooted edge-indexed trees with roots in (A_0,i_0). That is,

$$(A,i) = (A_0,i_0) \amalg \coprod_{a_j\in\Delta} (T_j,i_j,a_j),$$

where $\Delta = \{a_1,\ldots,a_n\}$ is a subset of VA_0, (T_j,i_j,a_j) are finite dominant-rooted edge-indexed trees with root vertices $a_j \in VA_0$, $j = 1\ldots n$.

Proof: (1) If (A,i) has no terminal vertices, then (A,i) is minimal and we take $(A_0,i_0) = (A,i)$. Now suppose (A,i) has terminal vertices. In order to obtain (A_0,i_0) we will use a *minimalization procedure* which we now describe.

Suppose (A,i) has terminal vertices $v_{11},\ldots,v_{1n_1}$, where n_1 is some positive integer. Then for each v_{1j} there is a unique edge e_{1j} with $\partial_0 e_{1j} = v_{1j}$, $j = 1\ldots n_1$. Since v_{1j} is terminal, e_{1j} cannot belong to a closed path in (A,i). Let (A_2,i_2) be the subgraph of $(A_1,i_1) = (A,i)$ obtained by removing v_{1j} and e_{1j} but not $\partial_1 e_{1j}$, $j = 1,\ldots n_1$. If (A_2,i_2) has no terminal vertices then set $(A_0,i_0) = (A_2,i_2)$ and we are done. Note that removing v_{1j} and e_{1j} may give rise to terminal vertices in (A_2,i_2). Suppose that (A_2,i_2) has terminal vertices. Let (A_3,i_3) be the subgraph of (A_2,i_2) obtained by removing the terminal vertices of (A_2,i_2) and their unique incident edges.

If at some stage of the procedure, the subgraph (A_n,i_n) consists of exactly two vertices connected by an edge, we can remove either of the two vertices and obtain a subgraph consisting of only one vertex.

Since (A,i) is finite, the successive 'pruning' of terminal vertices eventually produces a non-empty, connected subgraph (A_0,i_0) of (A,i) that is maximal with respect to the property of having no terminal vertices. Moreover, (A_0,i_0) contains the union of all closed paths in (A,i).

We remark that this minimalization procedure is not unique. For example, one can 'prune a branch at a time'. That is, we can successively remove all vertices on a path from a given terminal vertex that are not terminal in (A,i) but become terminal at some stage of the procedure.

(2) Let (A_0,i_0) be the minimal subgraph of (A,i) as in (1), obtained by a minimalization procedure. We claim that (A,i) is a union of (A_0,i_0) and a finite number of dominant rooted trees (T_j,i_j), $j = 1\ldots n$, rooted at $a_j \in VA_0$, $j = 1\ldots n$. From (1) we have that $(A,i)\setminus(A_0,i_0)$ is a disjoint union of edge-indexed trees, say (T_j,i_j), $j = 1\ldots n$. It remains to show that each (T_j,i_j) is a dominant rooted edge-indexed tree with root vertices in VA_0.

Let $e \in EA\setminus EA_0$ be such that $d(\partial_0 e, A_0) = 1$ and e points 'towards' (A_0,i_0) where for $a \in VA$

$$d(a,A_0) = \min_{x\in VA} d(a,x),$$

and $d : VA \longrightarrow \mathbb{Z}_{>0}$ is the edge-path distance. Then we must have $i(e) = 1$, otherwise $(A_0, i_0) \cup \{e\}$ is minimal, contradicting the fact that (A_0, i_0) is maximal with respect to having no terminal vertices. Let

$$S(A_0, r) = \{e \in EA \setminus EA_0 \mid d(\partial_0 e, A_0) = r \text{ and } e \text{ points towards } A_0\}$$

be the sphere of radius r in (A, i) about (A_0, i_0). Let m be the maximum radius of any sphere in (A, i) about (A_0, i_0). For each $k = 1 \dots m$, let $(A_k, i_k) = (A_0, i_0) \cup S(A_0, k)$. Then for each $e \in (A_k, i_k) \setminus (A_0, i_0)$ we must have $i(e) = 1$. Otherwise

$$(A_0, i_0) \cup \gamma \cup \{e\}$$

is minimal, where γ is the unique path from $\partial_1 e$ to (A_0, i_0), contradicting the maximality of (A_0, i_0). Now let (T_j, i_j) denote the connected components of $(A, i) \setminus (A_0, i_0)$, $j = 1 \dots n$. For each $j = 1 \dots n$, let $a_j \in VA_0$ be such that $a_j = \partial_1 e$ for some $e \in S(A_0, 1)$. Then for each $e \in ET_j$ we have shown that $d(\partial_0 e, a_j) > d(\partial_1 e, a_j)$ implies $i(e) = 1$. That is, (T_j, i_j) is a dominant rooted tree, rooted at $a_j \in VA_0$. In general

$$(A, i) = (A_0, i_0) \amalg \coprod_{a_j \in \Delta} (T_j, i_j, a_j),$$

where $\Delta = \{a_1, \dots, a_n\}$ is a subset of VA_0. $\square$

The proof of Theorem (4.3) shows that any dominant rooted edge-indexed tree (T, i, a) in (A, i) is removed during minimalizatiation, while the root a remains in (A_0, i_0). We obtain the following corollary.

(4.4) Corollary. *Let (A, i) be a finite edge-indexed graph. Let (A_0, i_0) be as in Theorem (4.3). If $v \in VA$ is the root of a dominant rooted edge-indexed tree in (A, i), v is a vertex of (A_0, i_0).* $\square$

(4.5) Theorem. *Let (A, i) be a finite edge-indexed graph. Let (A_0, i_0) be a connected subgraph, maximal with respect to having no terminal vertices, as in (1) of Theorem (4.4). If (A, i) is a tree then (A_0, i_0) may not be unique, but is unique up to isomorphism. Otherwise (A_0, i_0) is unique.*

Proof: Suppose that $|V_{A_0}| = 1$. Then either (A, i) is a tree in which case A_0 consists of a single vertex and is unique up to isomorphism, or (A, i) is not a tree in which case (A_0, i_0) is a bouquet of loops attached at a single vertex, and is unique. This completes the proof when $|V_{A_0}| = 1$.

Now suppose that $|V_{A_0}| \geq 2$, and that (B_0, j_0) is another minimal subgraph of (A, i). Suppose that $(B_0, j_0) \cap (A_0, i_0) = \varnothing$. This implies that (A, i) is a tree, since by Theorem (4.3) both (B_0, j_0) and (A_0, i_0) contain the union of all closed paths in (A, i). If (A, i) is a tree, and $(B_0, j_0) \cap (A_0, i_0) = \varnothing$, then (B_0, j_0) and (A_0, i_0) are joined by a separating path

γ that is removed during minimalization, since $|V_{A_0}| \geq 2$ and $|V_{B_0}| \geq 2$. This is impossible as no vertex of γ is a terminal vertex at any stage of minimalization. Hence we must have $(B_0, j_0) \cap (A_0, i_0) \neq \varnothing$. We no longer assume that (A, i) is a tree.

Let $x_0 \in VB_0 \setminus VA_0$. Then x_0 belongs to a dominant rooted edge-indexed tree (T, i, a) in (A, i) removed during a minimalization procedure that produces (A_0, i_0). In particular x_0 is *not* the root vertex a, since by Corollary (4.4), (A_0, i_0) contains all dominant roots of (A, i). But x_0 *is* the root vertex of the subtree of (T, i, a) consisting of all paths from x_0 to terminal vertices of (A, i). Hence x_0 is removed during a minimalization procedure that produces (B_0, j_0), which is a contradiction. Hence $(B_0, j_0) = (A_0, i_0)$ and the minimal subgraph is unique. $\square$

We call (A_0, i_0) in Theorems (4.3) and (4.5) the *unique minimal subgraph* of (A, i). Let (A, i) be finite and $X = \widetilde{(A, i)}$. Suppose we apply the above minimalization procedure to X and (A, i) simultaneously. That is, when we remove an edge e from A, we remove all edges in X in the fiber above e. Once we obtain the unique minimal subgraph (A_0, i_0) we have also obtained its covering tree $\widetilde{(A_0, i_0)}$. It can be easily shown that $\widetilde{(A_0, i_0)}$ coincides with the unique minimal subtree $X_0 \subset X$. That is,

$$(A, i) = (A_0, i_0) \amalg \coprod_{a_j \,\in\, \Delta \,\subseteq\, VA_0} (T_j, i_j, a_j),$$

and X has the form

$$X \quad = \quad X_0 \quad \amalg \coprod_{x_{j,k} \in p^{-1}(a_j),\ a_j \in VA_0} (Y_{j,k}, x_{j,k}), \tag{4.6}$$

where $X_0 = \widetilde{(A_0, i_0)}$, $(Y_{j,k}, x_{j,k}) = \widetilde{(T_j, i_j, a_j)}$, p is the covering map, $p(x_{j,k}) = a_j$, $j = 1 \ldots n$, $k \geq 1$. Since T_j is a dominant rooted edge-indexed tree and X is uniform, for $j = 1 \ldots n$, $k \geq 1$, $Y_{j,k}$ is a finite rooted tree, rooted at $x_{j,k} \in p^{-1}(a_j)$, $a_j \in VA_0$. Combining the results in this section we obtain the following.

(4.7) Corollary. *Let X be a uniform tree, let $G = Aut(X)$, and let $(A, i) = I(G\backslash\backslash X)$. Then there is a unique minimal G-invariant subtree $X_0 \subset X$ and a unique minimal subgraph $(A_0, i_0) \subset (A, i)$. Moreover $X_0 = \widetilde{(A_0, i_0)}$, and $(A_0, i_0) = I(G\backslash\backslash X_0)$. Conversely, for any finite edge-indexed graph (A, i) and universal covering tree $X = \widetilde{(A, i)}$, there is a unique minimal subgraph (A_0, i_0), of (A, i), unique minimal subtree X_0 of X, and $X_0 = \widetilde{(A_0, i_0)}$.* $\square$

5. Structure of deck transformation groups for non-minimal actions.

Let (A, i) be a finite edge-indexed graph, and let (A_0, i_0) be the unique minimal subgraph of (A, i). Let $X_0 = \widetilde{(A_0, i_0)}$ and $X = \widetilde{(A, i)}$ with corresponding covering maps $p_0 : X_0 \longrightarrow$

(A_0, i_0) and $p : X \longrightarrow (A, i)$. By (4.3) we have

$$(A,i) = (A_0, i_0) \amalg \coprod_{a_j \,\in\, \Delta \,\subseteq\, VA_0} (T_j, i_j, a_j),$$

and by (4.6) the universal covering tree $X = \widetilde{(A,i)}$ has the form

$$X \quad = \quad X_0 \quad \amalg \coprod_{x_{j,k} \in p^{-1}(a_j),\ a_j \in VA_0} (Y_{j,k}, x_{j,k}),$$

where $X_0 = \widetilde{(A_0, i_0)}$, $(Y_{j,k}, x_{j,k}) = \widetilde{(T_j, i_j, a_j)}$, p is the covering map, $p(x_{j,k}) = a_j$, $j = 1 \ldots n$, $k \geq 1$. The subgroups of $Aut(X)$

(5.1) $$G_{(A_0,i_0)} = \{g \in Aut(X_0) \mid g \circ p_0 = p_0\}$$

(5.2) $$G_{(A,i)} = \{g \in Aut(X) \mid g \circ p = p\}$$

are the groups of deck transformations of (A_0, i_0), and (A, i) respectively. The following lemma shows that deck transformations of the minimal subtree X_0 extend naturally to deck transformations of X.

(5.3) Lemma. *Let $g_0 \in G_{(A_0,i_0)}$. Then g_0 extends to $g \in G_{(A,i)}$.*

Proof: Let $a_0 \in VA_0$ and let (T_{a_0}, i, a_0) be a dominant rooted edge-indexed tree attached at a_0. Let $x_0 \in p_0^{-1}(a_0) \in VX_0$, and let $g_0 \in G_{(A_0,i_0)}$. Let $y_0 = g_0(x_0)$. Then

$$p_0(y_0) \quad = \quad p_0(g_0(x_0)) \quad = \quad p_0(x_0) \quad = \quad a_0.$$

Thus $y_0 \in p_0^{-1}(a_0)$. Let $(Y_{x_0}, x_0) = \widetilde{(T_{a_0}, i, a_0)}$. Then Y_{x_0} is a finite rooted tree, rooted at x_0. Moreover since $y_0 \in p_0^{-1}(a_0)$, there is a finite rooted tree, Y_{y_0}, rooted at y_0, such that

$$(Y_{y_0}, y_0) \quad = \quad \widetilde{(T_{a_0}, i, a_0)} \quad \cong \quad (Y_{x_0}, x_0).$$

That is, we can extend $g_0 \in G_{(A_0,i_0)}$ to $g \in G_{(A,i)}$ in such a way that $g\mid_{X_0} = g_0$, and g carries Y_{x_0} isomorphically to $Y_{g(x_0)} = Y_{y_0}$. $\square$

(5.4) Corollary. *We have*

$$G_{(A,i)} \mid_{X_0} = G_{(A_0,i_0)}.$$

Proof: Let $g \in G_{(A,i)} \mid_{X_0}$. Then $g(X_0) = X_0$ and $p(g(X_0)) = A_0$. So $g \mid_{X_0} \in G_{(A_0,i_0)}$. Conversely, let $g_0 \in G_{(A_0,i_0)}$. By Lemma (5.3) g_0 extends to $g \in G_{(A,i)}$ in such a way that $g \mid_{X_0} = g_0$. $\square$

Let $G_{X_0} = Stab_G(X_0) = \{g \in G \mid gx = x,\ x \in VX_0\}$, and set

$$Stab_{G_{(A,i)}}(X_0) \quad = \quad G_{(A,i)} \cap G_{X_0}.$$

(5.5) Theorem. *There is a split short exact sequence*

(5.6) $$1 \longrightarrow Stab_{G_{(A,i)}}(X_0) \longrightarrow G_{(A,i)} \longrightarrow G_{(A,i)} \mid_{X_0} \longrightarrow 1 \ .$$

Proof: Since X_0 is $G_{(A,i)}$-invariant, by ([BL], (6.6) (6)) there is an exact sequence

$$1 \longrightarrow Stab_{G_{(A,i)}}(X_0) \longrightarrow G_{(A,i)} \longrightarrow G_{(A,i)} \mid_{X_0} \longrightarrow 1.$$

By Lemma (5.3), given $g_0 \in G_{(A,i)} \mid_{X_0} = G_{(A_0,i_0)}$, g_0 extends to $g \in G_{(A,i)}$ in such a way that $g \mid_{X_0} = g_0$. Hence there is a map $\mu : G_{(A,i)} \mid_{X_0} \longrightarrow G_{(A,i)}$ such that $\mu \circ \phi = 1$, where ϕ is the restriction map $\phi : G_{(A,i)} \longrightarrow G_{(A,i)} \mid_{X_0}$ Thus (5.6) is a split exact sequence. □
An immediate consequence of Theorem (5.5) is the following.

(5.7) Corollary. *Using the notation in this section, we have*

$$G_{(A,i)} \quad \cong \quad Stab_{G_{(A,i)}}(X_0) \quad \rtimes \quad G_{(A_0,i_0)}. \quad \square$$

Using ([BL], (6.6)) we obtain the following description of $G_{X_0} = Stab_G(X_0)$ as the product of automorphism groups of rooted trees attached at the vertices of X_0.

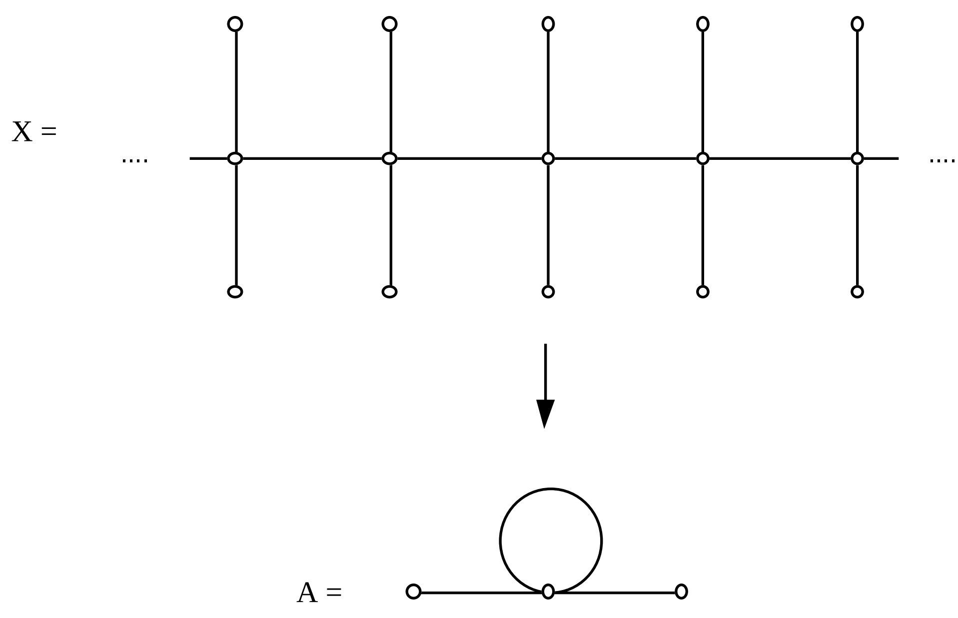
(5.8) Corollary. *Using the notation in this section, by ([BL], (6.6) (3)) we have*

$$G_{X_0} \quad = \quad Stab_G(X_0) \quad \cong \quad \prod_{x_{j,k} \in p^{-1}(a_j), \ a_j \in VA_0} Aut(Y_{j,k}, x_{j,k}),$$

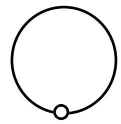
where $j = 1 \ldots n$, $k \geq 1$, *and since each* $Y_{j,k}$ *is a finite tree,* $Aut(Y_{j,k}, x_{j,k})$ *is a finite group. Furthermore, for* $j = 1 \ldots n$, $k \geq 1$ *we have*

$$Stab_{G_{(A,i)}}(X_0) \quad \cong \quad \prod_{x_{j,k} \in p^{-1}(a_j), \ a_j \in VA_0} Aut(Y_{j,k}, x_{j,k}) \ \cap \ G_{(A,i)}.$$

We illustrate the results of this section with the following example.



Let $\Gamma = Aut(X)$. The action of Γ on X leaves the central axis invariant. In the edge-indexed quotient (A, i) we have $i(e) = 1$ for every $e \in EA$. The unique minimal subgraph is:

$A_0 =$ 

We have $G_{(A,i)} = Aut(X)$ and X_0 is (isomorphic to) the integer line $\mathbb{Z}$. Applying Corollary (5.7) and Corollary (5.8) we obtain:

$$Stab_{G_{(A,i)}}(X_0) = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \text{ and } G_{(A_0,i_0)} = D_\infty,$$
$$G_{(A,i)} \quad \cong \quad (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \rtimes D_\infty,$$

where D_∞ denotes the infinite dihedral group, which acts as deck transformations on the central axis, and $Stab_{G_{(A,i)}}(X_0) = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ acts as flips (of order 2) about the central axis.

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