

UNIQUENESS OF REPRESENTATION–THEORETIC HYPERBOLIC KAC–MOODY GROUPS OVER \mathbb{Z}

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ABSTRACT. For a simply laced and hyperbolic Kac–Moody group $G = G(R)$ over a commutative ring R with 1, we consider a map from a finite presentation of $G(R)$ obtained by Allcock and Carbone to a representation–theoretic construction $G^\lambda(R)$ corresponding to an integrable representation V^λ with dominant integral weight λ . When $R = \mathbb{Z}$, we prove that this map extends to a group homomorphism $\rho_{\lambda, \mathbb{Z}} : G(\mathbb{Z}) \rightarrow G^\lambda(\mathbb{Z})$. We prove that the kernel K^λ of $\rho_{\lambda, \mathbb{Z}}$ lies in $H(\mathbb{C})$ and if the natural group homomorphism $\varphi : G(\mathbb{Z}) \rightarrow G(\mathbb{C})$ is injective, then $K^\lambda \leq H(\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\text{rank}(G)}$.

1. INTRODUCTION

Here we consider and compare two distinct constructions of a simply laced and hyperbolic Kac–Moody group $G = G(R)$ over a commutative ring R with 1. One of these groups is Tits’s construction of $G(R)$, though we work only with a finite presentation of $G(R)$ obtained by Allcock and Carbone ([AC]). The other is a Kac–Moody group $G^\lambda(R)$ constructed in [Ca] (following [CG]) using an integrable representation V^λ of the underlying Kac–Moody algebra with dominant integral weight λ .

We consider a natural map between generators of $G(R)$ and $G^\lambda(R)$. When $R = \mathbb{Z}$, we prove that this map extends to a group homomorphism $\rho_{\lambda, \mathbb{Z}} : G(\mathbb{Z}) \rightarrow G^\lambda(\mathbb{Z})$. We prove that the kernel K^λ of the map $\rho_{\lambda, \mathbb{Z}} : G(\mathbb{Z}) \rightarrow G^\lambda(\mathbb{Z})$ lies in $H(\mathbb{C})$ and if the natural group homomorphism $\varphi : G(\mathbb{Z}) \rightarrow G(\mathbb{C})$ is injective, then $K^\lambda \leq H(\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\text{rank}(G)}$. Injectivity of the natural map $\varphi : G(\mathbb{Z}) \rightarrow G(\mathbb{C})$ is not currently known and depends on functorial properties of Tits’ group ([Ti]).

Our notion of a ‘Kac–Moody group’ is the infinite dimensional analog of an *elementary Chevalley group* over the ring R , namely, a group generated by real root group generators. The notion of *Chevalley group* over R , that is, the infinite dimensional analog of the Chevalley–Demazure group scheme, has not yet been fully formulated, though the groundwork for establishing it was given in [Ti] and discussed further in [A] and [C].

The methods for constructing Kac–Moody groups over fields are numerous, though we will only refer to constructions of [CG] (following [G]) and [RR]). When R is a field, Garland constructed affine Kac–Moody groups as central extensions of loop groups and characterized the dependence of a completion of $G^\lambda(R)$ on λ in terms of the Steinberg cocycle ([G]).

Over fields, the group constructed in [RR] coincides with the Kac–Moody group $G(R)$ of Tits ([Ti]). Let $K^\lambda(R) = \text{Ker}(\rho_\lambda : G(R) \rightarrow G^\lambda(R))$. There are a number of known results which determine the natural extension of K^λ to completions of $G(R)$ and $G^\lambda(R)$ when R is a field.

For symmetrizable Kac–Moody groups over fields, [BR] showed that K^λ is contained in the center of a completion of $G(R)$. Recent work of Rousseau ([Rou]) shows that over fields of characteristic zero, completions $G(R)$ (as in [RR]) and $G^\lambda(R)$ (as in [CG]) are isomorphic as

topological groups (see also [Mar]). The complete Kac–Moody group of [CG] is constructed with respect to a choice of dominant integral weight λ . Thus Rousseau’s work also shows that over fields of characteristic zero, the complete Kac–Moody groups of [CG] are independent of the choice of λ up to isomorphism, as topological groups.

Here our interest lies in *minimal* or incomplete Kac–Moody groups over rings, where the methods for determining K^λ are less transparent. Thus we work only over $R = \mathbb{Z}$ and study the kernel of $\rho_{\lambda, \mathbb{Z}}$ when G is simply laced and hyperbolic. We hope to eventually have the techniques to extend these results to a wider class of Kac–Moody groups over more general commutative rings.

The authors wish to thank Daniel Allcock for his comments which helped to clarify the results in the last section.

2. TITS’ KAC–MOODY GROUP

Let \mathfrak{g} be a Kac–Moody algebra with Cartan subalgebra \mathfrak{h} and root space decomposition:

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{h} \oplus \mathfrak{g}^-$$

The roots $\Delta \subset \mathfrak{h}^*$ are the eigenvalues of the simultaneous adjoint action of \mathfrak{h} on \mathfrak{g} , and $\mathfrak{g}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$ where the root spaces

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, h \in \mathfrak{h}\}$$

are the corresponding eigenspaces. When \mathfrak{g} is infinite dimensional, $|\Delta| = \infty$. In this case, \mathfrak{g} has 2 types of roots: *real roots* with positive norm and *imaginary roots* with negative or zero norm. The imaginary roots will not play a role in what follows, as we will work with Tits’ presentation for Kac–Moody groups, which uses only the real root groups as generators.

Tits showed how Kac–Moody groups can be presented by generators and relations, generalizing the Steinberg presentation for finite dimensional Lie groups ([Ti]).

In the finite dimensional case, there is a Chevalley type commutation relation of the form

$$[\chi_\alpha(u), \chi_\beta(v)] = \prod_{m,n} \chi_{m\alpha+n\beta}(C_{mn\alpha\beta} u^m v^n)$$

between every pair of root groups U_α, U_β . Here $u, v \in R$, $C_{mn\alpha\beta}$ are integers (structure constants) and the χ_α are viewed as formal symbols in

$$U_\alpha = \{\chi_\alpha(u) \mid \alpha \in \Delta, u \in R\} \cong (R, +).$$

However, in the infinite dimensional case, Tits’ presentation of Kac–Moody groups has infinitely many Chevalley commutation relations. To describe these, we give the following definition.

Let (α, β) be a pair of real roots and let W be the Weyl group. Then (α, β) is called a *prenilpotent pair*, if there exist $w, w' \in W$ such that

$$w\alpha, w\beta \in \Delta_+^{re} \text{ and } w'\alpha, w'\beta \in \Delta_-^{re}.$$

A pair of roots $\{\alpha, \beta\}$ is prenilpotent if and only if $\alpha \neq -\beta$ and

$$(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Delta_+^{re}$$

is a finite set. For every prenilpotent pair of roots $\{\alpha, \beta\}$, Tits defined the Chevalley commutation relation

$$[\chi_\alpha(u), \chi_\beta(v)] = \prod_{m\alpha+n\beta \in (\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Delta_+^{re}} \chi_{m\alpha+n\beta}(C_{mn\alpha\beta} u^m v^n)$$

where $u, v \in R$ and $C_{mn\alpha\beta}$ are integers (structure constants).

3. TITS' PRESENTATION

Tits defined the *Steinberg group*

$$\mathfrak{St}(R) = *_{\alpha \in \Delta^{re}} U_{\alpha} / \text{Chev. comm. relns. on prenilpotent pairs}$$

Tits' Kac–Moody group $G(R)$ is a quotient of the Steinberg group $\mathfrak{St}(R)$ by some additional relations which are easy to describe.

Tits' Kac–Moody group has an infinite set of generators and an infinite set of defining relations. The sets of generators and relations are redundant and can be reduced significantly, as shown in [A] and [AC].

In [A], Allcock defined a new functor, the *pre–Steinberg group* $\mathfrak{PSt}(R)$ and showed that in many cases $\mathfrak{PSt}(R) \cong \mathfrak{St}(R)$. The presentation of $\mathfrak{PSt}(R)$ is defined in terms of the Dynkin diagram rather than the full infinite root system.

4. SIMPLY LACED HYPERBOLIC TYPE

Carbone and Allcock obtained a further simplification of the isomorphism $\mathfrak{PSt}(R) \cong \mathfrak{St}(R)$ for root systems that are simply laced and hyperbolic ([AC]).

A Dynkin diagram is *simply laced* if it consists only of single bonds between nodes.

It is *hyperbolic* if it is neither of affine nor of finite dimensional type, but every proper connected subdiagram is either of affine or finite dimensional type.

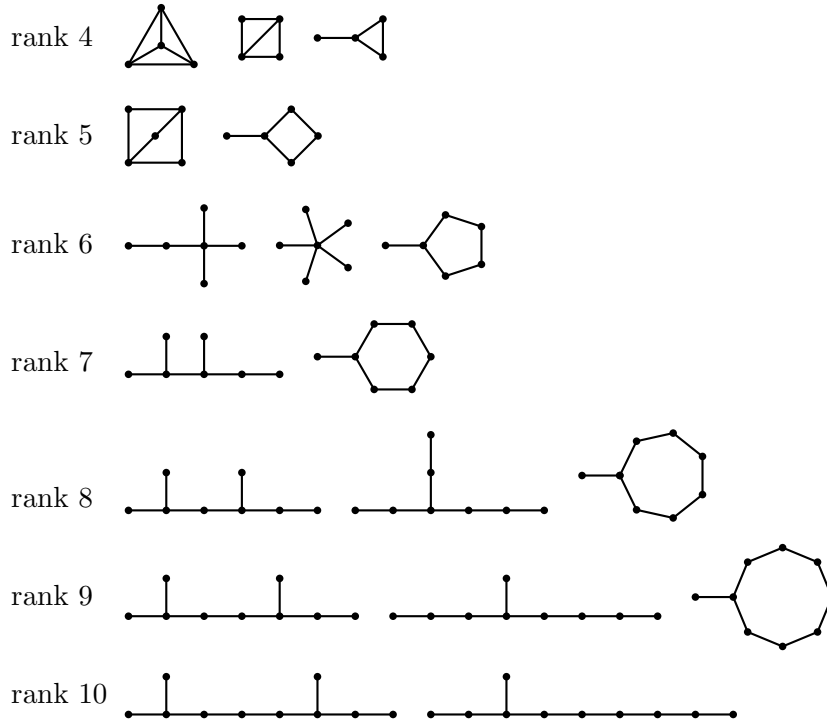


TABLE 1. The simply-laced hyperbolic Dynkin diagrams. The *rank* equals the number of nodes.

5. FINITELY MANY DEFINING RELATIONS PARAMETRIZED OVER R

The Kac–Moody group $G(R)$ is generated by elements $X_i(t)$, $t \in R$, S_i , $i \in \{1, \dots, \ell\}$. The elements $\tilde{h}_i(a)$ in the first column of Table 1 are

$$\begin{aligned}\tilde{h}_i(a) &:= \tilde{s}_i(a)\tilde{s}_i(-1), \\ \tilde{s}_i(a) &:= X_i(a)S_iX_i(a^{-1})S_i^{-1}X_i(a).\end{aligned}$$

Any i , all $a, b \in R^\times$	Each $i, t, u \in R$	$i \neq j$ not adjacent $t, u \in R$	$i \neq j$ adjacent $t, u \in R$
$\tilde{h}_i(a)\tilde{h}_i(b) = \tilde{h}_i(ab)$	$X_i(t)X_i(u) = X_i(t+u)$ $[S_i^2, X_i(t)] = 1$ $S_i = X_i(1)S_iX_i(1)S_i^{-1}X_i(1)$	$S_iS_j = S_jS_i$ $[S_i, X_j(t)] = 1$ $[X_i(t), X_j(u)] = 1$	$S_iS_jS_i = S_jS_iS_j$ $S_i^2S_jS_i^{-2} = S_j^{-1}$ $X_i(t)S_jS_i = S_jS_iX_j(t)$ $S_i^2X_j(t)S_i^{-2} = X_j(t)^{-1}$ $[X_i(t), X_j(u)] = S_iX_j(tu)S_i^{-1}$ $[X_i(t), S_iX_j(u)S_i^{-1}] = 1$

TABLE 2. The defining relations for $G(R)$, G simply laced and hyperbolic, R a commutative ring with 1

Over $R = \mathbb{Z}$, the generators $X_i(u)$ are obtained from $X_i = X_i(1)$ via $X_i(u) = X_i^u$.

Thus we can rewrite the presentation without the scalars from the underlying ring and obtain a *finite* presentation for $G(\mathbb{Z})$ as in Table 2.

Any i	Each $i = \{1, \dots, \ell\}$	$i \neq j$ not adjacent	$i \neq j$ adjacent
(1) $S_i^4 = 1$	(2) $[S_i^2, X_i] = 1$ (3) $S_i = X_iS_iX_iS_i^{-1}X_i$	(4) $S_iS_j = S_jS_i$ (5) $[S_i, X_j] = 1$ (6) $[X_i, X_j] = 1$	(7) $S_iS_jS_i = S_jS_iS_j$ (8) $S_i^2S_jS_i^{-2} = S_j^{-1}$ (9) $X_iS_jS_i = S_jS_iX_j$ (10) $S_i^2X_jS_i^{-2} = X_j^{-1}$ (11) $[X_i, X_j] = S_iX_jS_i^{-1}$ (12) $[X_i, S_iX_jS_i^{-1}] = 1$

TABLE 3. The defining relations for $G(\mathbb{Z})$, G simply laced and hyperbolic

6. REPRESENTATION–THEORETIC KAC–MOODY GROUPS OVER RINGS

Here we describe the construction of a representation theoretic Kac–Moody group $G^\lambda(R)$, over any commutative ring R with 1, constructed using an integrable highest weight module V^λ for any symmetrizable Kac–Moody algebra and a \mathbb{Z} –form $\mathcal{U}_\mathbb{Z}$ of the universal enveloping algebra \mathcal{U}_R . This construction was developed in [Ca] following the methods of [CG] and is a natural generalization of the theory of elementary Chevalley groups over commutative rings (see for example [VP]).

Let $\mathfrak{g} = \mathfrak{g}(A)$ be Kac–Moody algebra corresponding to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$. Let V^λ be the unique irreducible highest weight module for \mathfrak{g} corresponding to dominant integral weight λ .

We let $\Lambda \subseteq \mathfrak{h}^*$ be the linear span of the simple roots α_i , for $i \in I$, and let $\Lambda^\vee \subseteq \mathfrak{h}$ be the linear span of the simple coroots α_i^\vee , for $i \in I$.

Let e_i and f_i be the Chevalley generators of \mathfrak{g} . Let $\mathcal{U}_{\mathbb{C}}$ be the universal enveloping algebra of \mathfrak{g} . Let

$\mathcal{U}_{\mathbb{Z}} \subseteq \mathcal{U}_{\mathbb{C}}$ be the \mathbb{Z} -subalgebra generated by $\frac{e_i^m}{m!}$, $\frac{f_i^m}{m!}$ for $i \in I$ and $\binom{h}{m}$, for $h \in \Lambda^\vee$ and $m \geq 0$,

$\mathcal{U}_{\mathbb{Z}}^+$ be the \mathbb{Z} -subalgebra generated by $\frac{e_i^m}{m!}$ for $i \in I$ and $m \geq 0$,

$\mathcal{U}_{\mathbb{Z}}^-$ be the \mathbb{Z} -subalgebra generated by $\frac{f_i^m}{m!}$ for $i \in I$ and $m \geq 0$,

$\mathcal{U}_{\mathbb{Z}}^0 \subseteq \mathcal{U}_{\mathbb{C}}(\mathfrak{h})$ be the \mathbb{Z} -subalgebra generated by $\binom{h}{m}$, for $h \in \Lambda^\vee$ and $m \geq 0$. We set

$$\begin{aligned}\mathfrak{g}_{\mathbb{Z}} &= \mathfrak{g}_{\mathbb{C}} \cap \mathcal{U}_{\mathbb{Z}}, \\ \mathfrak{g}_{\mathbb{Z}}^\pm &= \mathfrak{g}_{\mathbb{C}}^\pm \cap \mathcal{U}_{\mathbb{Z}}, \\ \mathfrak{h}_{\mathbb{Z}} &= \mathfrak{h}_{\mathbb{C}} \cap \mathcal{U}_{\mathbb{Z}}.\end{aligned}$$

Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathfrak{h}_{\mathbb{Z}}$ is generated by $\binom{h}{m}$, for $h \in \Lambda^\vee$ and $m \geq 0$.

For R a commutative ring with 1, set $\mathfrak{g}_R = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. Then \mathfrak{g}_R is the infinite dimensional analog of the *Chevalley algebra* over R . Set $\mathfrak{h}_R = \mathfrak{h}_{\mathbb{Z}} \otimes R$ and for $\alpha \in \Delta$, let $\mathfrak{g}_R^\alpha = \mathfrak{g}_{\mathbb{Z}}^\alpha \otimes R$. Then

$$\mathfrak{g}_R = \mathfrak{n}_R^- \oplus \mathfrak{h}_R \oplus \mathfrak{n}_R^+$$

is the root space decomposition of \mathfrak{g}_R relative to \mathfrak{h}_R , where

$$\mathfrak{n}_R^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{n}_R^\alpha, \quad \mathfrak{n}_R^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{n}_R^\alpha.$$

We now consider the orbit of our highest weight vector $v_\lambda \in V$ under $\mathcal{U}_{\mathbb{Z}}$. We have

$$\mathcal{U}_{\mathbb{Z}}^+ v_\lambda = \mathbb{Z} v_\lambda$$

since all elements of $\mathcal{U}_{\mathbb{Z}}^+$ except for 1 annihilate v_λ . Also

$$\mathcal{U}_{\mathbb{Z}}^0 v_\lambda = \mathbb{Z} v_\lambda$$

since $\mathcal{U}_{\mathbb{Z}}^0$ acts as scalar multiplication on v_λ by a \mathbb{Z} -valued scalar.

Thus

$$\mathcal{U}_{\mathbb{Z}} \cdot v_\lambda = \mathcal{U}_{\mathbb{Z}}^- \cdot (\mathbb{Z} v_\lambda) = \mathcal{U}_{\mathbb{Z}}^- \cdot (v_\lambda).$$

Define $\mathcal{U}(\mathfrak{g})_R = \mathcal{U}(\mathfrak{g})_{\mathbb{Z}} \otimes R$. This is the infinite dimensional analog of a *hyperalgebra* as in [C] (see also [Hu]). We set

$$V_{\mathbb{Z}}^\lambda = \mathcal{U}_{\mathbb{Z}} \cdot v_\lambda = \mathcal{U}_{\mathbb{Z}}^- \cdot (v_\lambda)$$

Then $V_{\mathbb{Z}}^\lambda$ is a lattice in $V_R^\lambda = R \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^\lambda$ and a $\mathcal{U}_{\mathbb{Z}}$ -module. For each weight μ of V , let V_μ^λ be the corresponding weight space. Set

$$V_R^\lambda = R \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^\lambda.$$

and

$$V_{\mu,R}^\lambda = R \otimes_{\mathbb{Z}} V_{\mu,\mathbb{Z}}^\lambda$$

so that

$$V_R^\lambda = \bigoplus_{\mu} V_{\mu,R}^\lambda.$$

Here, $V_R = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$ free R -module with basis $v_{\lambda} \otimes 1$ and a module over \mathfrak{g}_R called the *Weyl module*. For $s, t \in R$ and $i \in I$, set

$$\begin{aligned}\chi_{\alpha_i}(t) &= \exp(\rho(se_i)), \\ \chi_{-\alpha_i}(t) &= \exp(\rho(tf_i)),\end{aligned}$$

where ρ is the defining representation for V , and we set

$$\begin{aligned}\tilde{w}_{\alpha_i} &= \chi_{\alpha_i}(t)\chi_{-\alpha_i}(-t^{-1})\chi_{\alpha_i}(t), \\ h_{\alpha_i}(t) &= \tilde{w}_{\alpha_i}(t)\tilde{w}_{\alpha_i}(1)^{-1}.\end{aligned}$$

Then these are elements of $\text{Aut}(V^{\lambda}R)$, thanks to the local nilpotence of e_i, f_i .

The group \widetilde{W} is known as the *extended Weyl group*. We will use a presentation of \widetilde{W} given in [KP]. Let $H^{\lambda} \leq G_{\lambda}$ be the subgroup generated by the elements $h_{\alpha_i}(t), t \in R^{\times}, i \in I$.

We let $G^{\lambda}(R) \leq \text{Aut}(V_R^{\lambda})$ be the group:

$$G^{\lambda}(R) = \langle \chi_{\alpha_i}(s), \chi_{-\alpha_i}(t) \mid i \in I, s, t \in R \rangle.$$

We may refer to $G^{\lambda}(R)$ as a ‘representation-theoretic Kac–Moody group’. We summarize the construction in the following.

Theorem 6.1. *Let \mathfrak{g} be a symmetrizable Kac–Moody algebra over a commutative ring R with 1. Let $\alpha_i, i \in I$, be the simple roots and e_i, f_i the Chevalley generators of \mathfrak{g} . Let V_R^{λ} be an R -form of an integrable highest weight module V^{λ} for \mathfrak{g} , corresponding to dominant integral weight λ and defining representation $\rho : \mathfrak{g} \rightarrow \text{End}(V_R^{\lambda})$. Then*

$$G^{\lambda}(R) = \langle \chi_{\alpha_i}(s) = \exp(\rho(se_i)), \chi_{-\alpha_i}(t) = \exp(\rho(tf_i)) \mid s, t \in R \rangle \leq \text{Aut}(V_R^{\lambda})$$

is a representation-theoretic Kac–Moody group associated to \mathfrak{g} .

A similar construction for G^{λ} over arbitrary fields was used in [CG] to construct Kac–Moody groups over finite fields.

When $R = \mathbb{C}$ (or $R = \mathbb{Q}$), we define the integral subgroup $G^{\lambda}(\mathbb{Z})$ to be the group

$$G^{\lambda}(\mathbb{Z}) = \langle \chi_{\alpha_i}(s), \chi_{-\alpha_i}(t) \mid s, t \in \mathbb{Z}, i \in I \rangle.$$

It is non-trivial to prove that this group coincides with the ‘Chevalley group’ over \mathbb{Z} , namely the subgroup of $G^{\lambda}(\mathbb{C})$ preserving $V^{\lambda}(\mathbb{Z})$. This is proven in [Ca]. The following was also proven in [Ca].

Theorem 6.2. *([Ca]) As a subgroup of $G^{\lambda}(\mathbb{C})$ (or $G^{\lambda}(\mathbb{Q})$) the group $G^{\lambda}(\mathbb{Z})$ has the following generating sets:*

(1) $\chi_{\alpha_i}(1)$ and $\chi_{-\alpha_i}(1)$,

and

(2) $\chi_{\alpha_i}(1)$ and $\tilde{w}_{\alpha_i}(1) = \chi_{\alpha_i}(1)\chi_{-\alpha_i}(-1)\chi_{\alpha_i}(1)$.

We now take λ to be a regular weight, and we define the *weight topology* on $G^{\lambda}(R)$ by taking stabilizers of elements of V_R^{λ} as a sub-base of neighborhoods of the identity. The completion of $G^{\lambda}(R)$ in this topology will be referred to as the *Carbone–Garland completion* and denoted by $\widehat{G}^{\lambda}(A)$. Since $G^{\lambda}(R)$ is a homomorphic image of $G(R)$, we can think of $\widehat{G}^{\lambda}(R)$ as a completion of $G(R)$ rather than $G^{\lambda}(R)$.

7. UNIQUENESS OF REPRESENTATION–THEORETIC KAC–MOODY GROUPS OVER \mathbb{Z}

For G simply laced and hyperbolic, we consider a map from the finite presentation of $G(R)$ of [AC] to the representation–theoretic group $G^\lambda(R)$, over a commutative ring R with 1.

We define the map ρ_λ :

$$G(R) \xrightarrow{\rho_\lambda} G^\lambda(R)$$

from generators of $G(R)$ to generators of $G^\lambda(R)$:

$$\begin{aligned} X_i(t) &\mapsto \chi_{\alpha_i}(t) \\ S_i &\mapsto \tilde{w}_{\alpha_i} \\ \tilde{h}_i(u) &\mapsto h_{\alpha_i}(u) \end{aligned}$$

for $t \in R$, $u \in R^\times$, $i \in \{1, \dots, \ell\}$.

Theorem 7.1. *When $R = \mathbb{Z}$, this map extends to a group homomorphism*

$$\rho_{\lambda, \mathbb{Z}} : G(\mathbb{Z}) \rightarrow G^\lambda(\mathbb{Z}).$$

Remark. The existence of an abstract group homomorphism from Tits' group $G(\mathbb{Q}) \rightarrow G^\lambda(\mathbb{Q})$ and its restriction $G(\mathbb{Z}) \rightarrow G^\lambda(\mathbb{Z})$ can be deduced from [Ti] and [CER]. However, we are interested only in this homomorphism from the finite presentation of $G(\mathbb{Z})$ in [AC] and in determining the generators of its kernel. Thus we prove that the relations in the finite presentation of $G(\mathbb{Z})$ are satisfied in $G^\lambda(\mathbb{Z})$.

Proof. The presentation in [AC] for $G(\mathbb{Z})$ does not make signs of structure constants explicit, but rather holds formally for any choice of sign of the structure constants. The known relations in $G^\lambda(\mathbb{Z})$ also hold up to a sign in the structure constants. Hence we verify that $\rho_{\lambda, \mathbb{Z}}$ is a homomorphism up to a sign in the structure constants. From now on, we will write ρ_λ to denote $\rho_{\lambda, \mathbb{Z}}$ if the context allows.

We show that the relations in $G(\mathbb{Z})$ are satisfied by their images in $G^\lambda(\mathbb{Z})$ under ρ_λ . We refer to the labelling of the relations in Table 2.

(1) Consider the image of $S_i^4 = 1$ under ρ_λ . For any i we have:

$$\rho_\lambda(S_i^4) = \tilde{w}_{\alpha_i}^4.$$

Since the \tilde{w}_{α_i} generate the extended Weyl group \tilde{W} and $\tilde{w}_{\alpha_i}^4 = 1$ in \tilde{W} , it follows that

$$\rho_\lambda(S_i^4) = \tilde{w}_{\alpha_i}^4 = 1.$$

Thus, the image of the relation $S_i^4 = 1$ is satisfied in $G^\lambda(\mathbb{Z})$.

(2) In $G^\lambda(\mathbb{Z})$, we have

$$\tilde{w}_{\alpha_i} \chi_{\alpha_j} \tilde{w}_{\alpha_i}^{-1} = \chi_{w_{\alpha_i}(\alpha_j)}$$

for any pair i, j . For each i we have $w_{\alpha_i}(\alpha_i) = -\alpha_i$. Hence

$$\tilde{w}_{\alpha_i} \chi_{\alpha_i} \tilde{w}_{\alpha_i}^{-1} = \chi_{-\alpha_i}.$$

Consider now the image of $[S_i^2, X_i]$ in $G^\lambda(\mathbb{Z})$:

$$\rho_\lambda([S_i^2, X_i]) = \rho_\lambda(S_i S_i X_i S_i^{-1} S_i^{-1} X_i^{-1}) = \tilde{w}_{\alpha_i} \tilde{w}_{\alpha_i} \chi_{\alpha_i} \tilde{w}_{\alpha_i}^{-1} \tilde{w}_{\alpha_i}^{-1} \chi_{\alpha_i}^{-1}.$$

This reduces to

$$\rho_\lambda([S_i^2, X_i]) = \tilde{w}_{\alpha_i}(\tilde{w}_{\alpha_i}\chi_{\alpha_i}\tilde{w}_{\alpha_i}^{-1})\tilde{w}_{\alpha_i}^{-1}\chi_{\alpha_i}^{-1} = \tilde{w}_{\alpha_i}\chi_{-\alpha_i}\tilde{w}_{\alpha_i}^{-1}\chi_{\alpha_i}^{-1} = \chi_{\alpha_i}\chi_{\alpha_i}^{-1} = 1.$$

Thus the image of the relation $[S_i^2, X_i] = 1$ is satisfied in $G^\lambda(\mathbb{Z})$.

(3) Similarly, consider the image of $X_i S_i X_i S_i^{-1} X_i$. Applying ρ_λ , we have

$$\rho_\lambda(X_i S_i X_i S_i^{-1} X_i) = \chi_{\alpha_i} \tilde{w}_{\alpha_i} \chi_{\alpha_i} \tilde{w}_{\alpha_i}^{-1} \chi_{\alpha_i} = \chi_{\alpha_i} \chi_{-\alpha_i} \chi_{\alpha_i}.$$

This last term equals \tilde{w}_{α_i} , so that

$$\rho_\lambda(X_i S_i X_i S_i^{-1} X_i) = \tilde{w}_{\alpha_i} = \rho_\lambda(S_i).$$

Thus, the image of the relation $S_i = X_i S_i X_i S_i^{-1} X_i$ is satisfied in $G^\lambda(\mathbb{Z})$.

(4) Suppose now that $i \neq j$ are not adjacent on the Dynkin diagram. Using the presentation of \tilde{W} in [KP], we have

$$\tilde{w}_{\alpha_i} \tilde{w}_{\alpha_j} \tilde{w}_{\alpha_i}^{-1} \tilde{w}_{\alpha_j}^{-1} = 1.$$

Hence, $\tilde{w}_{\alpha_i} \tilde{w}_{\alpha_j} = \tilde{w}_{\alpha_j} \tilde{w}_{\alpha_i}$, so that $\rho_\lambda(S_i S_j) = \rho_\lambda(S_j S_i)$. Thus the image of the relation $S_i S_j = S_j S_i$ for $i \neq j$, i, j not adjacent on the Dynkin diagram, is satisfied in $G^\lambda(\mathbb{Z})$.

(5) Consider the image of $[S_i, X_j]$ under ρ_λ :

$$\rho_\lambda([S_i, X_j]) = \rho_\lambda(S_i X_j S_i^{-1} X_j^{-1}) = \tilde{w}_{\alpha_i} \chi_{\alpha_j} \tilde{w}_{\alpha_i}^{-1} \chi_{\alpha_j}^{-1} = \chi_{w_{\alpha_i}(\alpha_j)} \chi_{\alpha_j}^{-1}.$$

In this last term, we evaluate $w_{\alpha_i}(\alpha_j)$:

$$w_{\alpha_i}(\alpha_j) = \alpha_j - \alpha_j(\alpha_i^\vee)\alpha_i = \alpha_j - (a_{ij})\alpha_i.$$

However, for $i \neq j$, i, j not adjacent on the Dynkin diagram, we have $a_{ij} = 0$, so that

$$w_{\alpha_i}(\alpha_j) = \alpha_j - (a_{ij})\alpha_i = \alpha_j.$$

If $i \neq j$ are not adjacent, then

$$\rho_\lambda([S_i, X_j]) = \chi_{\alpha_j} \chi_{\alpha_j}^{-1} = 1.$$

Thus, the image of $[S_i, X_j] = 1$ for $i \neq j$, i, j not adjacent on the Dynkin diagram is satisfied in $G^\lambda(\mathbb{Z})$.

(6) For $i \neq j$, and i, j not adjacent, we may compute in the rank 2 root subsystem of type $A_1 \times A_1$. We have $[\chi_{\alpha_i}, \chi_{\alpha_j}] = 1$, so that $\rho_\lambda([X_i, X_j]) = 1$. Thus the image of the relation $[X_i, X_j] = 1$ for $i \neq j$ not adjacent, is satisfied in $G^\lambda(\mathbb{Z})$.

(7) We now suppose that $i \neq j$ are adjacent on the Dynkin diagram. Using the presentation of \tilde{W} in [KP], we have

$$\tilde{w}_{\alpha_i} \tilde{w}_{\alpha_j} \tilde{w}_{\alpha_i} \tilde{w}_{\alpha_j}^{-1} \tilde{w}_{\alpha_i}^{-1} \tilde{w}_{\alpha_j}^{-1} = 1.$$

Hence, $\tilde{w}_{\alpha_i} \tilde{w}_{\alpha_j} \tilde{w}_{\alpha_i} = \tilde{w}_{\alpha_j} \tilde{w}_{\alpha_i} \tilde{w}_{\alpha_j}$, so that

$$\rho_\lambda(S_i S_j S_i) = \tilde{w}_{\alpha_i} \tilde{w}_{\alpha_j} \tilde{w}_{\alpha_i} = \tilde{w}_{\alpha_j} \tilde{w}_{\alpha_i} \tilde{w}_{\alpha_j} = \rho_\lambda(S_j S_i S_j).$$

Thus the image of the relation $S_i S_j S_i = S_j S_i S_j$ for $i \neq j$, i, j adjacent is satisfied in $G^\lambda(\mathbb{Z})$.

(8) We have the relation $\tilde{w}_{\alpha_j} \tilde{w}_{\alpha_i}^2 \tilde{w}_{\alpha_j}^{-1} = \tilde{w}_{\alpha_i}^2 (\tilde{w}_{\alpha_j})^{-2a_{ij}}$ in $G^\lambda(\mathbb{Z})$ (see [CER]). If $i \neq j$ are adjacent, then $a_{ij} = -1$. Hence

$$\tilde{w}_{\alpha_j} \tilde{w}_{\alpha_i}^2 \tilde{w}_{\alpha_j}^{-1} = \tilde{w}_{\alpha_i}^2 \tilde{w}_{\alpha_j}^2.$$

Multiplying by \tilde{w}_{α_j} on both sides yields $\tilde{w}_{\alpha_j}\tilde{w}_{\alpha_i}^2 = \tilde{w}_{\alpha_i}^2\tilde{w}_{\alpha_j}^3$. Since $\tilde{w}_{\alpha_i}^4 = 1$ for all i in $G^\lambda(\mathbb{Z})$, we can rewrite the right hand side as $\tilde{w}_{\alpha_i}^2\tilde{w}_{\alpha_j}^{-1}$. We have $\tilde{w}_{\alpha_i}^2 = \tilde{w}_{\alpha_i}^{-2}$ for all i , so $\tilde{w}_{\alpha_j}\tilde{w}_{\alpha_i}^{-2} = \tilde{w}_{\alpha_i}^{-2}\tilde{w}_{\alpha_j}^{-1}$. Finally, multiplying by $\tilde{w}_{\alpha_i}^2$ on both sides gives

$$\rho_\lambda(S_i^2 S_j S_i^{-2}) = \tilde{w}_{\alpha_i}^2 \tilde{w}_{\alpha_j} \tilde{w}_{\alpha_i}^{-2} = \tilde{w}_{\alpha_j}^{-1} = \rho_\lambda(S_j^{-1}).$$

Thus, the image of the relation $S_i^2 S_j S_i^{-2} = S_j^{-1}$ for i, j adjacent is satisfied in $G^\lambda(\mathbb{Z})$.

(9) Now consider the words $\tilde{w}_{\alpha_i}\chi_{\alpha_j}\tilde{w}_{\alpha_i}^{-1}$ and $\tilde{w}_{\alpha_j}^{-1}\chi_{\alpha_i}\tilde{w}_{\alpha_j}$ in $G^\lambda(\mathbb{Z})$. As before, we have

$$\tilde{w}_{\alpha_i}\chi_{\alpha_j}\tilde{w}_{\alpha_i}^{-1} = \chi_{w_{\alpha_i}(\alpha_j)}$$

for any i, j . Using the relation $\tilde{w}_{\alpha_i}^4 = 1$ for all i , we may rewrite the second word above as

$$\tilde{w}_{\alpha_j}^{-1}\chi_{\alpha_i}\tilde{w}_{\alpha_j} = \tilde{w}_{\alpha_j}^3\chi_{\alpha_i}\tilde{w}_{\alpha_j}^{-3}.$$

For i, j adjacent on the Dynkin diagram, $a_{ij} = -1$, so

$$w_{\alpha_j}(\alpha_i) = \alpha_i - \alpha_i(\alpha_j^\vee)\alpha_j = \alpha_i - (a_{ij})\alpha_j = \alpha_i + \alpha_j.$$

Thus

$$\tilde{w}_{\alpha_j}\chi_{\alpha_i}\tilde{w}_{\alpha_j}^{-1} = \chi_{w_{\alpha_j}(\alpha_i)} = \chi_{\alpha_i + \alpha_j}.$$

Similarly,

$$w_{\alpha_i}(\alpha_j) = \alpha_j - \alpha_j(\alpha_i^\vee)\alpha_i = \alpha_j - (a_{ji})\alpha_i = \alpha_j + \alpha_i = \alpha_i + \alpha_j.$$

We may simplify the first word:

$$\tilde{w}_{\alpha_i}\chi_{\alpha_j}\tilde{w}_{\alpha_i}^{-1} = \chi_{\alpha_i + \alpha_j}$$

and the second word:

$$\tilde{w}_{\alpha_j}^{-1}\chi_{\alpha_i}\tilde{w}_{\alpha_j} = \tilde{w}_{\alpha_j}^3\chi_{\alpha_i}\tilde{w}_{\alpha_j}^{-3} = \tilde{w}_{\alpha_j}^2(\tilde{w}_{\alpha_j}\chi_{\alpha_i}\tilde{w}_{\alpha_j})\tilde{w}_{\alpha_j}^{-2} = \tilde{w}_{\alpha_j}^2\chi_{\alpha_i + \alpha_j}\tilde{w}_{\alpha_j}^{-2}.$$

This yields

$$\tilde{w}_{\alpha_j}^{-1}\chi_{\alpha_i}\tilde{w}_{\alpha_j} = \tilde{w}_{\alpha_j}^2\chi_{\alpha_i + \alpha_j}\tilde{w}_{\alpha_j}^{-2} = \tilde{w}_{\alpha_j}(\tilde{w}_{\alpha_j}\chi_{\alpha_i + \alpha_j}\tilde{w}_{\alpha_j}^{-1})\tilde{w}_{\alpha_j}^{-1} = \tilde{w}_{\alpha_j}\chi_{w_{\alpha_j}(\alpha_i + \alpha_j)}\tilde{w}_{\alpha_j}^{-1}.$$

We have

$$w_{\alpha_j}(\alpha_i + \alpha_j) = w_{\alpha_j}(\alpha_i) + w_{\alpha_j}(\alpha_j).$$

Since $w_{\alpha_j}(\alpha_i) = \alpha_i + \alpha_j$ and $w_{\alpha_j}(\alpha_j) = -\alpha_j$, we have

$$w_{\alpha_j}(\alpha_i + \alpha_j) = \alpha_i.$$

Hence

$$\tilde{w}_{\alpha_j}^{-1}\chi_{\alpha_i}\tilde{w}_{\alpha_j} = \tilde{w}_{\alpha_j}\chi_{w_{\alpha_j}(\alpha_i + \alpha_j)}\tilde{w}_{\alpha_j}^{-1} = \tilde{w}_{\alpha_j}\chi_{\alpha_i}\tilde{w}_{\alpha_j}^{-1}$$

which yields

$$\tilde{w}_{\alpha_j}^{-1}\chi_{\alpha_i}\tilde{w}_{\alpha_j} = \tilde{w}_{\alpha_j}\chi_{\alpha_i}\tilde{w}_{\alpha_j}^{-1} = \chi_{w_{\alpha_j}(\alpha_i)} = \chi_{\alpha_i + \alpha_j}.$$

Hence

$$\tilde{w}_{\alpha_j}\chi_{\alpha_i}\tilde{w}_{\alpha_j}^{-1} = \chi_{\alpha_i + \alpha_j} = \tilde{w}_{\alpha_j}^{-1}\chi_{\alpha_i}\tilde{w}_{\alpha_j}$$

Multiplying by \tilde{w}_{α_i} and then \tilde{w}_{α_j} on both sides gives the equality $\tilde{w}_{\alpha_j}\tilde{w}_{\alpha_i}\chi_{\alpha_j} = \chi_{\alpha_i}\tilde{w}_{\alpha_j}\tilde{w}_{\alpha_i}$, which is the image of $X_i S_j S_i = S_j S_i X_j$ under ρ_λ . Thus, the image of the relation $X_i S_j S_i = S_j S_i X_j$ is satisfied in $G^\lambda(\mathbb{Z})$.

(10) To determine the image of $S_i^2 X_j S_i^{-2}$ under ρ_λ , note that $S_i^2 = \tilde{h}_i(-1)$, so that

$$\rho_\lambda(S_i^2 X_j S_i^{-2}) = \rho_\lambda(\tilde{h}_i(-1)X_j\tilde{h}_i(-1)^{-1}) = h_i(-1)\chi_{\alpha_j}(1)h_i(-1)^{-1}.$$

We have the relation $h_i(u)\chi_\alpha(v)h_i(u)^{-1} = \chi_\alpha(vu^{aj})$ in $G^\lambda(\mathbb{Z})$ for any i, j ([CER]). Hence, for i, j adjacent, we see that

$$\rho_\lambda(S_i^2 X_j S_i^{-2}) = h_i(-1)\chi_{\alpha_j}(1)h_i(-1)^{-1} = \chi_{\alpha_j}(-1) = \chi_{\alpha_j}^{-1} = \rho_\lambda(X_j)^{-1}.$$

Thus, the image of the relation $S_i^2 X_j S_i^{-2} = X_j^{-1}$ for i, j adjacent is satisfied in $G^\lambda(\mathbb{Z})$.

(11) For any prenilpotent pair of real roots α and β , by [Ti] and [CG] there is a Chevalley relation in G^λ of the form

$$[\chi_\alpha(u), \chi_\beta(v)] = \prod_{\substack{m\alpha + n\beta \in \Delta_+^{re} \\ m, n > 0}} \chi_{m\alpha + n\beta}(C_{mn\alpha\beta} u^m v^n)$$

between root groups U_α and U_β . Here $u, v \in R$, $C_{mn\alpha\beta}$ are integers (structure constants). We claim that adjacent simple roots α_i and α_j form a prenilpotent pair. By Lemma 6 of [AC], in the simply laced hyperbolic case, a pair of real roots is prenilpotent if $(\alpha, \beta) \geq -1$. But for adjacent simple roots α_i and α_j , $(\alpha_i, \alpha_j) = -1$. Thus the pair α_i, α_j is prenilpotent and we have $C_{mn\alpha\beta} \in \{\pm 1\}$. We have

$$(m\alpha_i + n\alpha_j \mid m\alpha_i + n\alpha_j) = m^2(\alpha_i \mid \alpha_i) + 2mn(\alpha_i \mid \alpha_j) + n^2(\alpha_j \mid \alpha_j) = 2m^2 - 2mn + 2n^2.$$

But $(\cdot \mid \cdot) = 2$ for all elements of Δ^{re} . To determine when $m\alpha_i + n\alpha_j \in \Delta^{re}$ for $m, n > 0$, we seek non-negative solutions of

$$2m^2 - 2mn + 2n^2 = 2(m^2 - mn + n^2) = 2$$

or

$$m^2 - mn + n^2 = 1.$$

It is easy to see that the only solution is

$$m = n = 1.$$

Thus the right hand side of the Chevalley commutation relation is

$$\chi_{\alpha_i + \alpha_j} = \chi_{w_{\alpha_i}(\alpha_j)} = \tilde{w}_{\alpha_i} \chi_{\alpha_j} \tilde{w}_{\alpha_i}^{-1}$$

and the image of the relation $[X_i, X_j] = S_i X_j S_i^{-1}$ for i, j adjacent is satisfied in $G^\lambda(\mathbb{Z})$.

(12) Finally, consider $\rho_\lambda([X_i, S_i X_j S_i^{-1}])$ for i, j adjacent. We have

$$\rho_\lambda([X_i, S_i X_j S_i^{-1}]) = [\rho_\lambda(X_i), \rho_\lambda(S_i X_j S_i^{-1})] = [\chi_{\alpha_i}, \tilde{w}_{\alpha_i} \chi_{\alpha_j} \tilde{w}_{\alpha_i}^{-1}].$$

We have $\tilde{w}_{\alpha_i} \chi_{\alpha_j} \tilde{w}_{\alpha_i}^{-1} = \chi_{w_{\alpha_i}(\alpha_j)}$. As i and j are adjacent, $a_{ij} = -1$, thus it follows that $w_{\alpha_i}(\alpha_j) = \alpha_i + \alpha_j$, so that

$$\rho_\lambda([X_i, S_i X_j S_i^{-1}]) = [\chi_{\alpha_i}, \tilde{w}_{\alpha_i} \chi_{\alpha_j} \tilde{w}_{\alpha_i}^{-1}] = [\chi_{\alpha_i}, \chi_{\alpha_i + \alpha_j}].$$

We note that $[\chi_{\alpha_i}, \chi_{\alpha_i + \alpha_j}] = 1$ when $2\alpha_i + \alpha_j$ is not a root. We have

$$(2\alpha_i + \alpha_j \mid 2\alpha_i + \alpha_j) = 4(\alpha_i \mid \alpha_i) + 2(\alpha_i \mid \alpha_j) + 2(\alpha_j \mid \alpha_i) + (\alpha_j \mid \alpha_j) = 8 - 2 - 2 + 2 = 6.$$

As $(\cdot \mid \cdot) \leq 0$ for all elements of Δ^{im} and $(\cdot \mid \cdot) = 2$ for all elements of Δ^{re} , it follows that $2\alpha_i + \alpha_j \notin \Delta^{im}$ and $2\alpha_i + \alpha_j \notin \Delta^{re}$, thus $2\alpha_i + \alpha_j$ is not a root. Thus

$$\rho_\lambda([X_i, S_i X_j S_i^{-1}]) = [\chi_{\alpha_i}, \chi_{\alpha_i + \alpha_j}] = 1$$

and the image of the relation $[X_i, S_i X_j S_i^{-1}] = 1$ for i, j not adjacent, is satisfied in $G^\lambda(\mathbb{Z})$. □

8. THE KERNEL OF ρ_λ

When $R = \mathbb{K}$ is a field, Rémy and Ronan took a completion $\widehat{G}(\mathbb{K})$ of Tits' Kac–Moody group $G(\mathbb{K})$ in the automorphism group of the building of the BN–pair for G ([RR] and [Mar], Sec 6.1).

The following relationship between $\widehat{G}^\lambda(\mathbb{K})$ and $\widehat{G}(\mathbb{K})$ was established by [BR] and [Mar], (see also [CER]).

Theorem 8.1. *For any regular weight λ , there exists a (canonical) continuous surjective homomorphism $\rho_\lambda : \widehat{G}^\lambda(\mathbb{K}) \rightarrow \widehat{G}(\mathbb{K})$. The kernel of ρ_λ is equal to*

$$\bigcap_{g \in \widehat{G}} g \widehat{B}^\lambda g^{-1}$$

and is contained in $Center(\widehat{G}^\lambda(\mathbb{K}))$.

This result was extended further by [Rou] (see also Prop 6.46 of [Mar]).

Theorem 8.2. *([Rou]) If $Char(\mathbb{K}) = 0$ then $\varepsilon_\lambda : \widehat{G}^\lambda(\mathbb{K}) \rightarrow \widehat{G}(\mathbb{K})$ is an isomorphism of topological groups.*

From now on, we will take $R = \mathbb{C}$.

Lemma 8.3. *The kernel of the map $\rho_\lambda : G(\mathbb{C}) \rightarrow G^\lambda(\mathbb{C})$ lies in $Center(G(\mathbb{C}))$.*

Proof. Consider the sequence of homomorphisms

$$G(\mathbb{C}) \xrightarrow{\rho_\lambda} G^\lambda(\mathbb{C}) \xrightarrow{\iota} \widehat{G}^\lambda(\mathbb{C}) \xrightarrow{\varepsilon_\lambda} \widehat{G}(\mathbb{C})$$

where ι is the inclusion map. The composition of these three maps is the natural map from G to \widehat{G} . By Theorem 8.1, $Ker(\varepsilon_\lambda) \leq Center(\widehat{G}^\lambda(\mathbb{C}))$. By ([RR], 1.B),

$$Ker(\varepsilon_\lambda \circ \iota \circ \rho_\lambda) = Center(G(\mathbb{C})) \leq H(\mathbb{C})$$

where

$$H(\mathbb{C}) = \langle \tilde{h}_i(a) := \tilde{s}_i(a) \tilde{s}_i(-1) \mid a \in \mathbb{C}^\times, \tilde{h}_i(a) \cdot \tilde{h}_i(b) = \tilde{h}_i(ab) \rangle$$

for $\tilde{s}_i(a) := X_i(a) S_i X_i(a^{-1}) S_i^{-1} X_i(a)$. But

$$Ker(\rho_\lambda) \leq Ker(\varepsilon_\lambda \circ \iota \circ \rho_\lambda).$$

The result follows. □

Since $\tilde{h}_{\alpha_i}(1) = 1$, we define

$$H(\mathbb{Z}) = \langle \tilde{h}_{\alpha_i}(-1) = \tilde{s}_i(-1)^2 \mid \tilde{h}_{\alpha_i}(-1)^2 = 1, i \in \{1, \dots, \ell\} \rangle.$$

Then $H(\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{rank(G)}$ and the natural map $H(\mathbb{Z}) \rightarrow H(\mathbb{C})$ is injective.

Injectivity of the natural map $G^\lambda(\mathbb{Z}) \rightarrow G^\lambda(\mathbb{C})$ follows from the fact that $G^\lambda(\mathbb{Z})$ is the subgroup of $G^\lambda(\mathbb{C})$ preserving $V_{\mathbb{Z}}$, as proven in [Ca]. Injectivity of the natural map

$$\varphi : G(\mathbb{Z}) \rightarrow G(\mathbb{C})$$

is not currently known and depends on functorial properties of Tits' group ([Ti]).

Lemma 8.4. *Consider the group homomorphism $\varphi : G(\mathbb{Z}) \rightarrow G(\mathbb{C})$. Then*

$$\varphi(\text{Center}(G(\mathbb{Z}))) \subseteq \text{Center}(\varphi(G(\mathbb{Z}))) \leq \text{Center}(G(\mathbb{C})).$$

Proof. Consider $\varphi : G(\mathbb{Z}) \rightarrow G(\mathbb{C})$. Then

$$\varphi(\text{Center}(G(\mathbb{Z}))) \subseteq \text{Center}(\varphi(G(\mathbb{Z}))).$$

and

$$K = \varphi(G(\mathbb{Z})) \cong G(\mathbb{Z})/\text{Ker}(\varphi)$$

is a subgroup of $G(\mathbb{C})$, so φ takes $\text{Center}(G(\mathbb{Z}))$ to the center of a subgroup $K \leq G(\mathbb{C})$. But $\text{Center}(G(\mathbb{C}))$ is abelian so

$$\text{Center}(K) = \text{Center}(\varphi(G(\mathbb{Z}))) \leq \text{Center}(G(\mathbb{C})). \quad \square$$

□

The following diagram commutes:

$$\begin{array}{ccc} G(\mathbb{Z}) & \xrightarrow{\rho_{\lambda, \mathbb{Z}}} & G^\lambda(\mathbb{Z}) \\ \downarrow \varphi & & \downarrow \iota \\ G(\mathbb{C}) & \xrightarrow{\rho_\lambda} & G^\lambda(\mathbb{C}) \end{array}$$

That is,

$$(\iota \circ \rho_{\lambda, \mathbb{Z}})(G(\mathbb{Z})) = (\rho_\lambda \circ \varphi)(G(\mathbb{Z}))$$

so

$$\text{Ker}(\iota \circ \rho_{\lambda, \mathbb{Z}}) = \text{Ker}(\rho_\lambda \circ \varphi).$$

We have

$$\text{Ker}(\rho_{\lambda, \mathbb{Z}}) \leq \text{Ker}(\iota \circ \rho_{\lambda, \mathbb{Z}}) = \text{Ker}(\rho_\lambda \circ \varphi).$$

Thus if φ is injective, then we have

$$\text{Ker}(\iota \circ \rho_{\lambda, \mathbb{Z}}) = \text{Ker}(\rho_\lambda \circ \iota)$$

or

$$\text{Ker}(\rho_{\lambda, \mathbb{Z}}) \leq \text{Ker}(\iota \circ \rho_{\lambda, \mathbb{Z}}) = \text{Ker}(\rho_\lambda \circ \iota) \leq \text{Center}(G(\mathbb{C})) \leq H(\mathbb{C}),$$

using Lemma 8.3. Furthermore, if φ is injective, then by Lemma 8.4 we have

$$\text{Center}(G(\mathbb{Z})) \leq \text{Center}(G(\mathbb{C}))$$

$$\text{Ker}(\rho_{\lambda, \mathbb{Z}}) \leq \text{Center}(G(\mathbb{Z})) \leq \text{Center}(G(\mathbb{C})) \leq H(\mathbb{C})$$

and

$$\text{Center}(G(\mathbb{Z})) = G(\mathbb{Z}) \cap \text{Center}(G(\mathbb{C})) \leq G(\mathbb{Z}) \cap H(\mathbb{C}) = H(\mathbb{Z}).$$

We have proven the following.

Theorem 8.5. *The kernel K^λ of the map $\rho_{\lambda, \mathbb{Z}} : G(\mathbb{Z}) \rightarrow G^\lambda(\mathbb{Z})$ lies in $H(\mathbb{C})$ and if the group homomorphism $\varphi : G(\mathbb{Z}) \rightarrow G(\mathbb{C})$ is injective, then $K^\lambda \leq H(\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\text{rank}(G)}$.*

REFERENCES

- [A] Allcock, D. *Steinberg groups as amalgams*, Preprint, arXiv:1307.2689v5 [math.GR]
- [ACa] Allcock, D. and Carbone, L. *Finite presentation of hyperbolic Kac–Moody groups over rings*, J. Algebra 445, 232–243, (2016), arXiv:1409.5918
- [BR] Baumgartner, U. and Rémy, B. *Private communication*.
- [Ca] Carbone, L. *Infinite dimensional Chevalley groups and Kac–Moody groups over \mathbb{Z}* , Preprint (2016)
- [CER] Carbone, L., Ershov, M. and Ritter, G. *Abstract simplicity of complete Kac–Moody groups over finite fields*, Journal of Pure and Applied Algebra, No. 212, (2008), 2147–2162
- [CG] Carbone, L. and Garland, H., *Existence of lattices in Kac–Moody groups over finite fields*, Communications in Contemporary, Math, Vol 5, No.5, 813–867, (2003)
- [C] Pierre Cartier, *Hyperalgèbres et groupes de Lie formels*, Séminaire ‘Sophus Lie’ 2e: 1955/56, (1957).
- [Hu] Humphreys, J. E. *On the hyperalgebra of a semisimple algebraic group*, Contributions to algebra (collection of papers dedicated to Ellis Kolchin), pp. 203–210. Academic Press, New York, 1977.
- [G] Garland, H., *The arithmetic theory of loop groups*, Publications Mathématiques de l’IHÉS, 52, (1980), 5–136
- [K] Kac, V., *Infinite dimensional Lie algebras*, Cambridge University Press, (1990)
- [KP] Kac, V. and Peterson, D. *Defining relations of certain infinite-dimensional groups*, Astérisque, 1985, Numéro Hors Serie, 165–208
- [Mar] Marquis, T. *Topological Kac–Moody groups and their subgroups*, PhD thesis, Université Catholique de Louvain, (2013)
- [RR] Rémy, B. and Ronan, M. *Topological groups of Kac–Moody type, right-angled twinings and their lattices*. Commentarii Mathematici Helvetici 81 (2006) 191-219.
- [Rou] Rousseau, G. *Groupes de Kac–Moody déployés sur un corps local, II mesures ordonnées*, Preprint (2012), <http://arxiv.org/abs/1009.0138>.
- [Ti] Tits, J. *Uniqueness and presentation of Kac–Moody groups over fields* Journal of Algebra 105, 1987, 542–573
- [VP] Vavilov, N. and Plotkin, E. *Chevalley Groups over Commutative Rings: I. Elementary Calculations*, Acta Applicandae Mathematicae 45: 73–113, (1996).