RECONSTRUCTING GROUP ACTIONS

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Abstract. We give a general structure theory for reconstructing non-trivial group actions on sets without any further assumptions on the group, the action, or the set on which the group acts. Using certain 'local data' $\mathcal{D}$ from the action we build a group $\mathcal{G}(\mathcal{D})$ of the data and a space $\mathcal{X}(\mathcal{D})$ with an action of $\mathcal{G}(\mathcal{D})$ on $\mathcal{X}(\mathcal{D})$ that arise naturally from the data $\mathcal{D}$. We use these to obtain an approximation to the original group G, the original space X and the original action $G\times X \longrightarrow X$. The data $\mathcal{D}$ is distinguished by the property that it may be chosen from the action locally.

For a large enough set of local data $\mathcal{D}$, our definition of $\mathcal{G}(\mathcal{D})$ in terms of generators and relations allows us to obtain a presentation for the group G. We demonstrate this on several examples. When the local data $\mathcal{D}$ is chosen from a graph of groups, the group $\mathcal{G}(\mathcal{D})$ is the fundamental group of the graph of groups and the space $\mathcal{X}(\mathcal{D})$ is the universal covering tree of groups.

For general non properly discontinuous group actions our local data allows us to imitate a fundamental domain, quotient space and universal covering for the quotient. We exhibit this on a non properly discontinuous free action on $\mathbb{R}$. For a certain class of non properly discontinuous group actions on the upper half plane, we use our local data to build a space on which the group acts discretely and cocompactly.

Our combinatorial approach to reconstructing abstract group actions on sets is a generalization of the Bass-Serre theory for reconstructing group actions on trees. Our results also provide a generalization of the notion of developable complexes of groups by Haefliger.

Contents

The first author was supported in part by NSF grant #DMS-1101282. The authors were supported in part by a grant from the US-Israel Bi-national Science Foundation.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Section 0.
INTRODUCTION TO RECONSTRUCTING GROUP ACTIONS

Let G be a group and X a set with an action $G \times X \longrightarrow X$. The fundamental question we consider is the following:

(0.1) Question. *Can we extract 'data' $\mathcal{D}$ from this setting in order to reconstruct G, X and the action $G \times X \longrightarrow X$? Can we use this to obtain a structure theorem for G?*

In this work we give a detailed answer to this question. Consider first a non-trivial action of a group Γ on a simplicial tree X without inversions and without fixed points. We may reconstruct Γ, X and the action $\Gamma \times X \longrightarrow X$, choosing the data $\mathcal{D}$ to be

$$\mathcal{D} \quad = \quad \{A = \Gamma \backslash X\},$$

the quotient graph of X modulo Γ. Then the group Γ is isomorphic to the fundamental group $\Pi = \pi_1(A)$ of the quotient graph A, a free group. The tree X is isomorphic to the universal covering $Y = \widetilde{A}$ of A, and the action of Π on Y by covering transformations commutes with the action of Γ on X.

Our task here is to give an analogous structure theory for reconstructing group actions on sets without any assumptions on the group, the action or the structure of the space where the group is acting.

To describe the known results which address Question (0.1) we will use the following terminology.

A subset $Z \subseteq X$ (respectively a point $x \in X$) is called a *set of fixed points* (respectively a *fixed point*) if there exists $1 \neq g \in G$ such that $gz = z$ for each $z \in Z$ (respectively $gx = x$). We say that G acts *freely* on X if there are no fixed points. The *pointwise stabilizer* of $Z \subseteq X$ (respectively $x \in X$) is

$$G_Z \quad := \quad \{g \in G \mid gz = z \text{ for every } z \in Z\},$$

(respectively $G_x := \{g \in G \mid gx = x\}$.)

If G acts freely on X then $G_x = \{1\}$ for every $x \in X$. The *G-orbit* of $x \in X$ is the set

$$G \cdot x \quad := \quad \{gx \mid g \in G\}.$$

The *quotient* of X modulo G is the set of all G-orbits, denoted $G\backslash X$.

Now we take X to be a topological space, and we assume that G acts on X as homeomorphisms. We say that G acts *strongly properly discontinuously* on X if for each $x \in X$ there is a neighborhood U containing x such that

$$gU \quad \cap \quad U \quad = \quad \varnothing$$

for each $1 \neq g \in G$. Since the non-trivial translates of U are all disjoint, a strongly properly discontinuous action is fixed point free. The following gives an answer to Question (0.1) for strongly properly discontinuous group actions on topological spaces.

(2) Reconstruction theorem I ([Mas]). *Let X be a topological space. Suppose that X is path connected and locally path connected. Let G be a group acting on X as homeomorphisms. Then the action of G on X is strongly properly discontinuous if and only if the quotient map $\pi : X \longrightarrow G\backslash X$ is a covering map. In this case*

$$G \quad = \quad Deck(X, \pi) \quad := \quad \{g \in Homeom(X) \mid \pi \circ g = \pi\}$$

which is the group of covering transformations of $\pi : X \longrightarrow G\backslash X$.

Example 1. *A strongly properly discontinuous group action.*

Let $X = \mathbb{R}$, $G = \mathbb{Z}$. Then G acts on X by translation $\sigma_n : x \longrightarrow x + n$, for $n \in \mathbb{Z}$, $x \in \mathbb{R}$, with quotient $\mathbb{Z}\backslash\mathbb{R} \cong S^1$. Then G is a strongly properly discontinuous group of homeomorphisms of $\mathbb{R}$ since for any $x \in \mathbb{R}$, and $U_x = (x - \frac{1}{3}, x + \frac{1}{3}) \subset \mathbb{R}$, the translates $\sigma_n(U_x)$ are pairwise disjoint. By Theorem (0.2), $p : \mathbb{R} \longrightarrow \mathbb{Z}\backslash\mathbb{R}$ is a covering space, and

$$\mathbb{Z} \quad = \quad Deck(\mathbb{R}, p) \quad \cong \quad \pi_1(\mathbb{Z}\backslash\mathbb{R}, x_0) \quad \cong \quad \pi_1(S^1),$$

for $x_0 \in X$, and $\mathbb{Z}$ is the group of deck transformations of the covering.

For free and strongly properly discontinuous actions the notion of reconstructing a group action and obtaining a group presentation from a fundamental domain is due to Poincaré in the early 1900's. For example, the 'Poincaré Fundamental Polyhedron Theorem' for groups acting discontinously on hyperbolic spaces provides a method to obtain a presentation of a Kleinian group from a collection of combinatorial conditions on a polyhedron which is a fundamental domain for the action ([Mask]).

This idea was further exploited by Borel and Harish-Chandra who obtained fundamental domains for arithmetic subgroups of algebraic groups (in some cases in the presence of fixed points) and showed ([BH]) that these arithmetic subgroups are finitely generated. A description for the defining relations for arithmetic subgroups of algebraic groups was given in [PR]. Their proof is sufficiently general that it holds for an arbitrary group of transformations of a topological space that is connected, locally connected and simply connected provided there is a fundamental domain for the action.

In [Br], Brown gives a graph of group type presentation for groups of homeomorphisms of CW-complexes where there is a fundamental domain and the action permutes the cells. This improves on an earlier work of Soulé ([So]). Other examples of group presentations obtained from a fundamental domain have been given by Abels ([Ab]), Behr ([Be]), Gerstenhaber ([Ge]) and Macbeath ([Mac]).

Many group actions are not fixed point free or strongly properly discontinuous. We say that a group G acts *properly discontinuously* on a topological space X if for each $x, y \in X$ there are neighborhoods U containing x, V containing y such that

$$\{g \in G \mid gU \cap V \neq \varnothing\}$$

is a finite set.

All finite group actions are automatically properly discontinuous. If we take $x = y$ in the above, we see that $\{g \in G \mid gU \cap U \neq \varnothing\}$ is a finite set. Hence properly discontinuous group actions may have fixed points. It is easy to see that for properly discontinuous group actions on topological spaces X, the point stabilizers G_x for $x \in X$ are finite groups. However, there are fixed point free actions that are not properly discontinuous (the action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}$ by translations as in Section (5.3)).

The following proposition is easy to see.

(3) Proposition. *Let X be a Hausdorff topological space. Let G be a group acting on X as homeomorphisms. If G acts freely on X, then the following conditions are equivalent.*

(1) The action of G on X is strongly properly discontinuous.

(2) The action of G on X is properly discontinuous.

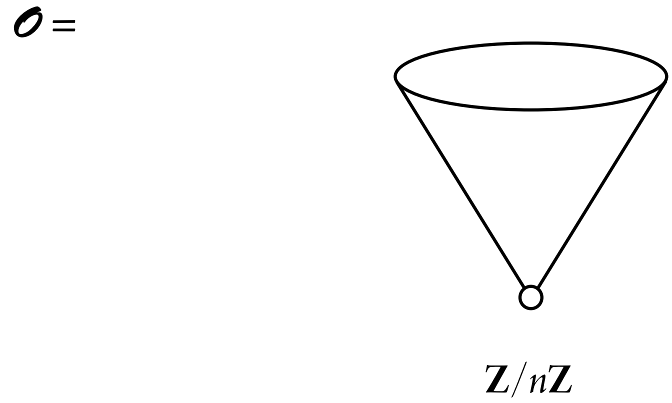
Example 2. *An action that is properly discontinuous but not strongly properly discontinuous.*

Let $X = \mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ be the open unit disk in the plane. Then $G = \mathbb{Z}/n\mathbb{Z}$ acts on X by rotation of $2\pi/n$ fixing the origin. We have

$$Stab_G(x) \quad = \begin{cases} \mathbb{Z}/n\mathbb{Z}, \ x = (0,0) \\ \{1\}, \ x \in \mathbb{D}^2\{(0,0)\}, \end{cases}$$

so the action of G is properly discontinuous but not strongly properly discontinuous.

Supose now that X is a manifold and that a group G acts properly discontinuously on X as homeomorphisms. If S is the set of fixed points for the action then the projection $G\backslash S$ to the quotient $G\backslash X$ is the 'singular locus' of the quotient 'orbifold' $\mathcal{O}$. In the orbifold the singular points are labelled with a finite group, the stabilizer of a lifting of the point to X, which is determined up to conjugacy. In Example 2, the quotient orbifold $\mathcal{O}$ for $G = \mathbb{Z}/n\mathbb{Z}$ on $X = \mathbb{D}^2$ is a cone:



In this case, the quotient orbifold plays the role of a fundamental domain for the action. The following theorem gives an answer to Question (0.1) for properly discontinuous group actions on (smooth) manifolds.

(4) Reconstruction theorem II ([Sc]). *Let X be a smooth manifold and suppose that a group G acts on X as diffeomorphisms. If G acts properly discontinuously on X with quotient orbifold $\mathcal{O}$ then the quotient map $\pi : X \longrightarrow \mathcal{O}$ is an orbifold covering map. If X is simply connected then*

$$G \quad = \quad Deck_{Orb}(X, \pi) \quad := \quad \pi_1(\mathcal{O}).$$

In Example 2 the quotient projection $\pi : \mathbb{D}^2 \longrightarrow \mathcal{O}$ *locally* is the quotient map modulo the action of the point stabilizers. Hence $\pi_1(\mathcal{O}) = \mathbb{Z}/n\mathbb{Z}$.

Our work primarily concerns the problem of finding generators and relations and approximating a fundamental domain for a group whose action is not properly discontinuous, particularly when there is no fundamental domain for the action. Some examples are:

(1) Group actions on simplicial trees without inversions and with infinite vertex stabilizers.
(2) Group actions on negatively curved simply connected simplicial complexes with infinite vertex stabilizers.
(3) Group actions on $\mathbb{R}$-trees with 'non discrete component' (see Example 2 in Section (5.3)).
(4) Non discrete subgroups of $PSL_2(\mathbb{R})$ acting on the upper half plane (see Section (7)).

The Bass-Serre theory for reconstructing group actions on simplicial trees ([B], [S]) and Haefliger's theory of developable complexes of groups ([H]) give complete answers to Question (0.1) in cases (1) and (2) respectively. Our combinatorial approach to reconstructing abstract group actions on sets may be viewed as a generalization of the Bass-Serre theory. Our work also provides a generalization of Haefliger's complexes of groups. A knowledge of the basics of the Bass-Serre theory is assumed here and can be found in [B] and [S].

In order to give a general structure theory for reconstructing group actions on sets we choose certain 'local data' $\mathcal{D}$ arising from a group action $G \times X \longrightarrow X$ that encodes information about a natural pseudogroup induced by the action, pointwise stabilizers from the action with natural inclusions and isomorphisms between them, and fixed points for the induced pseudogroup.

By a *pseudogroup* we mean a collection of bijective mappings on subsets of X, containing the identity map, where compositions may be taken wherever they are defined. The notion of a pseudogroup originally grew out of the early theory of Sophus Lie and was developed by Élie Cartan in the early 1900's ([C]). Lie's original concept of what we now call a Lie group consisted of local pseudogroups of functions defined in a neighbourhood of the origin. More recently diffeomorphism pseudogroups have arisen naturally in differential geometry as transition functions of an atlas for a manifold. Our notion of pseudogroup is a combinatorial analog of the notion of a geometric pseudogroup.

Using our local data as a model, we build abstract machinery for reconstructing group actions that are not necessarily free or properly discontinuous. In particular we build a group $\mathcal{G}(\mathcal{D})$ of the data and a space $\mathcal{X}(\mathcal{D})$ with an action of $\mathcal{G}(\mathcal{D})$ on $\mathcal{X}(\mathcal{D})$ that arise naturally from the data $\mathcal{D}$. We use these to obtain an approximation to the original group, the original space and the original action.

Our data mimics the notion of a presheaf of sets, though we do not assume a basis for the topology of the underlying space. Rather, we work only with *local* pseudogroups of sets and maps between them that satisfy axioms analogous to presheaf axioms.

Our main theorem is the following.

(5) Reconstruction theorem III (Section (5.1)). *Let G be a group acting non-trivially on a set X. Let $\mathcal{D}$ be any choice of local data for the action of G on X (in the sense of Section (1.5)). There is a canonical homomorphism $\mu : \mathcal{G}(\mathcal{D}) \longrightarrow G$ and canonical set map $\nu : \mathcal{X}(\mathcal{D}) \longrightarrow X$ such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{G}(\mathcal{D}) & \times & \mathcal{X}(\mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{D}) \\ \downarrow_{\mu} & & \downarrow_{\nu} & & \downarrow_{\nu} \\ G & \times & X & \longrightarrow & X \end{array}$$

That is, $\nu(\sigma \cdot y) = \mu(\sigma) \cdot \nu(y)$, for $\sigma \in \mathcal{G}(\mathcal{D})$, $y \in \mathcal{X}(\mathcal{D})$. If further $\mathcal{D}$ is 'complete' in the sense of Section 3, then the map μ is a group isomorphism, and ν is a set bijection. Moreover a complete set of local data $\mathcal{D}$ always exists for the action of G on X.

The data $\mathcal{D}$ is distinguished by the property that it may be chosen from the action locally. For a complete set of data $\mathcal{D}$, our definition of $\mathcal{G}(\mathcal{D})$ in terms of generators and relations allows us to obtain a presentation for the group G. In Section 5.3, we demonstrate this on several examples. In Section (5.3) we show how to reconstruct Fuchsian groups acting either discretely or non discretely on the Poincaré disk by choosing data from a fundamental polygon in the Poincaré disk.

Our Reconstruction theorem III is particularly useful for obtaining a presentation for a finitely presented group G that acts on a set when the action of the generators is known explicitly on a large enough finite subset. A local pseudogroup describing the restriction of the whole action to the action of the generators on a finite subset gives rise to the 'monodromy groupoid' $\mathcal{M}(\mathcal{D})$ we describe in Section 2.3. If the finite subset is sufficiently large, $\pi_1(\mathcal{M}(\mathcal{D}))$ is isomorphic to G (Section 5.1).

When the local data $\mathcal{D}$ is chosen from a graph of groups, the group $\mathcal{G}(\mathcal{D})$ is isomorphic to the fundamental group of the graph of groups and the space $\mathcal{X}(\mathcal{D})$ coincides with the universal covering tree of groups (Section 6). Thus when our local data is chosen from a graph of groups, we recover the Bass-Serre correspondence between actions (without inversions) on trees and quotient graphs of groups.

For non properly discontinuous group actions our local data allows us to imitate a fundamental domain, quotient space and universal covering for the quotient. For a certain class of non properly discontinuous group actions on the upper half plane, we may use our local data to build a space on which the group acts discretely and cocompactly (Section 7). We exhibit this construction explicitly for a free action on $\mathbb{R}$ with 'non discrete component' (Example 2 of Section 5.3) and on the non discrete subgroup

$$G = \langle x, y \mid yxy^{-1} = x^2 \rangle$$

of $PSL_2(\mathbb{R})$ which acts on the upper half plane $\mathbb{H}^2$ by translation $\sigma_x : z \mapsto z+1$ and homothety $\sigma_y : z \mapsto 2z$ (Section 7).

Another application of our general structure theory is the following. We may choose local data $\mathcal{D}$ from a free action of a finitely presented group on an $\mathbb{R}$-tree in order to deduce the following theorem, conjectured by Morgan and Shalen ([MS1]-[MS4]).

Rips' Theorem. *If G is finitely presented and acts freely on an $\mathbb{R}$-tree by isometries, then G is the free product of free abelian groups and surface groups.*

Let G be a finitely presented group acting freely on an $\mathbb{R}$-tree T by isometries. We choose a sufficiently large finite subtree K of T. This subtree K should be subject to the condition that it is large enough to 'capture' the defining relations of G with respect to the action of the generators of G.

The local data $\mathcal{D}$ is obtained by taking the partial maps $K \longrightarrow K$ corresponding to the restrictions of the action of the generators of G on K. The multiplicity structure is trivial since G acts freely, hence all monodromy elements are also trivial.

Given this local data $\mathcal{D}$, we build the group $\mathcal{G}(\mathcal{D})$. Since K is large enough, we obtain an isomorphism $G \cong \mathcal{G}(\mathcal{D})$. We then build the space $\mathcal{X}(\mathcal{D})$ of $\mathcal{D}$ on which $\mathcal{G}(\mathcal{D})$ acts.

One can prove that $\mathcal{X}(\mathcal{D})$ is also an $\mathbb{R}$-tree and that it is Hausdorff. However, $\mathcal{X}(\mathcal{D})$ may not be G-equivariantly isometric to T. Our reconstruction theorem gives a G-equivariant map from $\mathcal{X}(\mathcal{D})$ to T but this map is not necessarily an isometry. The tree $\mathcal{X}(\mathcal{D})$ is essentially equivalent to the 'resolution tree' of [BF] and the notion of 'geometric action' of [LP], since $\mathcal{X}(\mathcal{D})$ converges to T as K gets larger and larger.

Any pseudogroup of isometries of $\mathbb{R}$ can be put into normal form, decomposing it into a union of components on which the orbits are either finite or dense. This corresponds to the decomposition of $\mathcal{G}(\mathcal{D})$ as a free product. The action of $\mathcal{G}(\mathcal{D})$ on $\mathcal{X}(\mathcal{D})$, viewed as an approximation of the original action, is then analyzed via the 'Rips machine'. This is a sequence of processes, called Process I and Process II that were inspired by the Makanin-Ravborov elimination process for solutions of equations in free groups ([Ma], [Ra1], [Ra2]). The Rips machine is used for further analysis of the factors with dense orbits and reveals pseudogroups of various types: axial, interval exchange types and Levitt types. We then deduce that $\mathcal{G}(\mathcal{D})$ is a free product of free abelian groups, surface groups and free groups, with the axial type giving rise to free abelian groups, the interval exchange type giving rise to surface groups and the Levitt type corresponding to free groups.

Our general structure theory has allowed us to reduce the problem of obtaining a presentation of a finitely presented group acting freely on an $\mathbb{R}$-tree to consideration of only local data and not the whole action.

The 1991 lectures of the second author at the Isle of Thorns inspired Bestvina and Feighn to generalize Rips' Theorem to stable actions of finitely generated groups on $\mathbb{R}$-trees ([BF]) using geometric methods in place of combinatorial arguments. Gaboriau, Levitt and Paulin also proved Rips' Theorem for finitely generated groups acting freely on $\mathbb{R}$-trees ([GLP2]) using the methods of dynamical systems and measured foliations. In [LP] the authors refined the notion of a geometric action on an $\mathbb{R}$-tree to give a definition in terms of measured foliations and they showed, using a geometric interpretation of the second author's method for approximating actions on $\mathbb{R}$-trees, that every finitely supported action of a finitely generated group G on an $\mathbb{R}$-tree T is a strong limit of geometric actions.

We shall not say more about the Rips machine or the classification of free actions on $\mathbb{R}$-trees in this work, but we refer the interested reader to [BF] and [GLP2] (see also [GLP1] , [LP] and [L]).

The lectures of the second author inspired several other applications. Actions of finitely generated groups on $\mathbb{R}$-trees are also a main ingredient in Sela's approach to acylindrical accessibility ([Se]) and the JSJ-decomposition of finitely presented groups ([RS]). Sela's approach to classifying free actions of finitely generated groups on $\mathbb{R}$-trees was generalized by Guirardel under some stability hypotheses ([G1]). Guirardel's work has also found applications by Drutu and Sapir ([DS]). The problem of classifying finitely presented and finitely generated groups acting freely on Λ-trees is an active area of current research ([AB], [B], [G2], [KMS1], [KMS2], [KMRS]). In particular, Kharlampovich, Miasnikov and Serbin have developed a non-standard version of the Rips machine which gives an elimination process for arbitrary non-Archimedean actions such as free actions on Λ-trees ([KMS1]).

This work was completed in ongoing discussions with Ilya Kapovich. We are indebted to Ilya for his substantial input and for the time he spent considering the ideas and constructions in this work. Many of his ideas are contained within. We take great pleasure in thanking him. We are extremely grateful to the editor and the referee for their careful and persistent attention to the details of this work and for several corrections. We thank Hyman Bass, Mladen Bestvina, Mark Feighn, Gilbert Levitt, Olga Kharlampovich, Alexei Miasnikov, Frédéric Paulin and Mark Sapir for helpful discussions, encouragement and suggestions. We are grateful to Diego Penta for assistance with preparing the diagrams.

Section I.
GENERAL STRUCTURE THEORY FOR RECONSTRUCTING GROUP ACTIONS

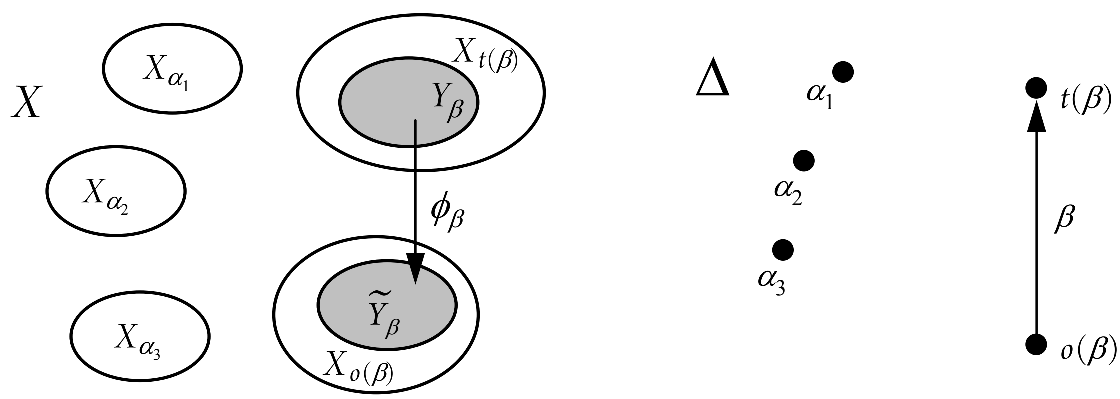
1. LOCAL DATA.

Let $(X_\alpha)_{\alpha \in V}$ be a family of sets, and let X denote the disjoint union $\bigsqcup_{\alpha \in V} X_\alpha$ of the sets X_α. We describe natural data associated with X, satisfying certain axioms. Let $\mathcal{P}(X)$ denote the power set of X, and let $\Phi \subseteq \mathcal{P}(X)$.

1.1 Local pseudogroup data.

(1) Let $\Delta = (V, E, o, t, -)$ be an oriented graph with vertex set V, edge set E, initial and terminal functions, o and t respectively, which pick out the origin and terminus of an edge, and an involution, $-$, on the edge set which is fixed point free and is a reversal of orientation. We do not assume that Δ is finite or locally finite.
(2) A pair $((X_\alpha)_{\alpha \in V}, (\phi_\beta)_{\beta \in E})$ is a Φ-*pseudogroup presentation* on X if $(\phi_\beta)_{\beta \in E}$ is a family of mappings between subsets Y_β and $\widetilde{Y_\beta}$ of X such that:

(1) for any $\beta \in E$, $\phi_\beta : Y_\beta \longrightarrow \widetilde{Y_\beta}$ is a bijection, where $Y_\beta \subseteq X_{t(\beta)}$ and $\widetilde{Y_\beta} \subseteq X_{o(\beta)}$,
(2) for any $\beta \in E$, $\widetilde{Y_\beta} = Y_{\overline{\beta}}$ and $\phi_\beta^{-1} = \phi_{\overline{\beta}}$,
(3) the partial mappings $(\phi_\beta)_{\beta \in E}$ preserve Φ-subsets of X; that is if $U \subseteq Y_\beta$ then $U \in \Phi$ if and only if $\phi_\beta(U) \in \Phi$.



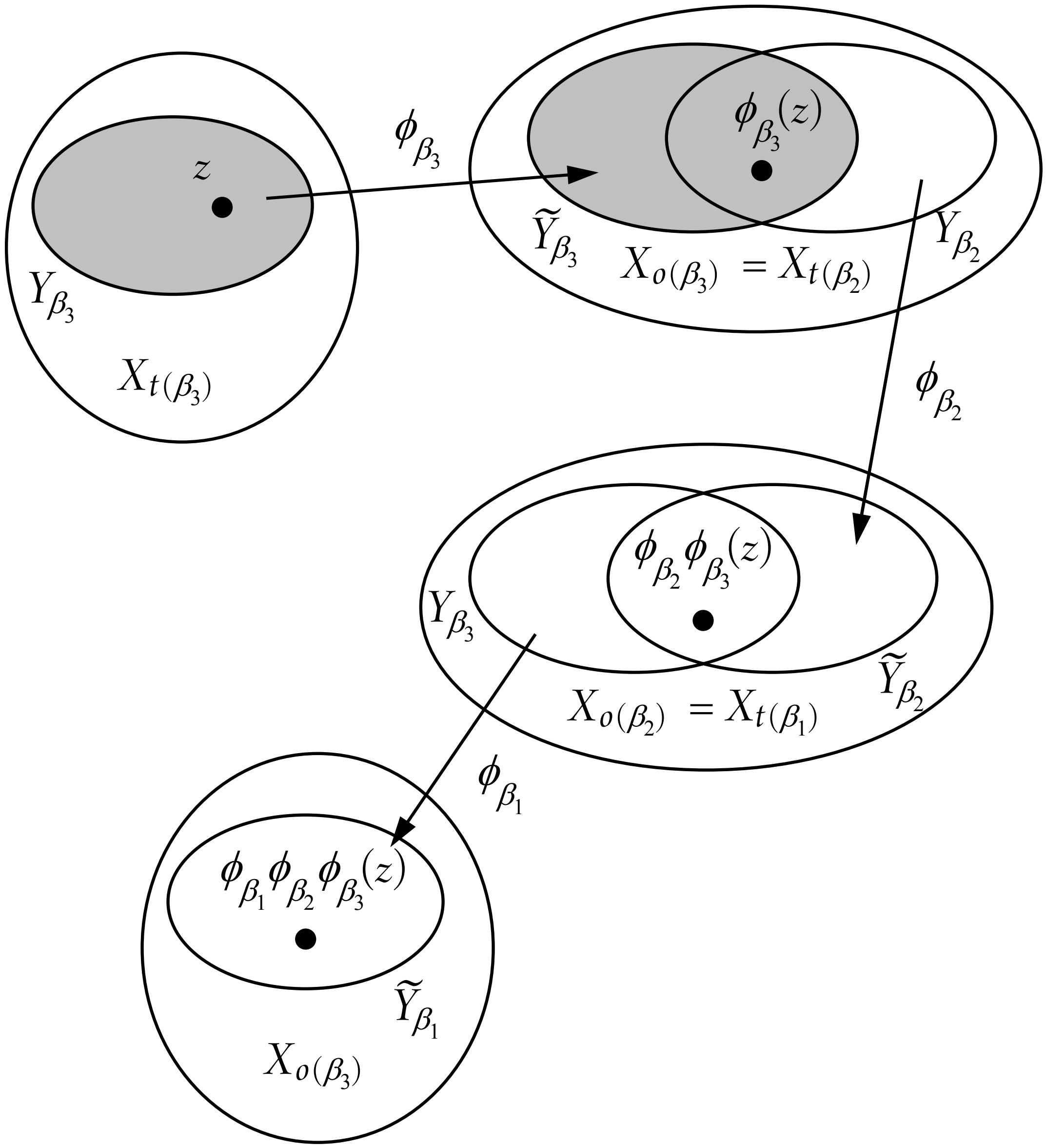
(3) The *local* Φ-*pseudogroup*, Γ, defined by a Φ- pseudogroup presentation, $((X_\alpha)_{\alpha \in V}, (\phi_\beta)_{\beta \in E})$, is the collection of all maps

$$\phi_{\gamma,Z} : Z \longrightarrow \widetilde{Z},$$

where $Z \in \Phi$, $Z \subseteq Y_{\beta_k}$, $\widetilde{Z} \subseteq Y_{\beta_1}$, $\gamma = \beta_1\beta_2 \ldots \beta_k$ is a path in Δ such that the composition $(\phi_{\beta_1} \circ \cdots \circ \phi_{\beta_k})(z)$ is defined for each $z \in Z$,

$$\phi_{\gamma,Z} : z \longmapsto (\phi_{\beta_1} \cdot \ldots \cdot \phi_{\beta_k})(z)$$

for any $z \in Z$, and $\widetilde{Z} = \phi_{\gamma,Z}(Z)$.



(4) We say that the maps ϕ_β are *distinguished generators* of the local Φ-pseudogroup, Γ. Notice that Γ has the following properties:

(a) If $Z \in \Phi$ and $Z \subseteq Y_\beta$ for some $\beta \in E$, then $\phi_{\overline{\beta}} \cdot \phi_\beta \mid_Z = Id_Z$ and so $\phi_{\beta\overline{\beta},Z} = Id_Z$ belongs to Γ.

(b) If $\phi_{\gamma,Z} : Z \longrightarrow \widetilde{Z}$ belongs to Γ and $Z' \in \Phi$ with $Z' \subseteq Z$ then the restriction of $\phi_{\gamma,Z}$ to Z'

$$\phi_{\gamma,Z'} = \phi_{\gamma,Z} \mid_{Z'}$$

belongs to Γ.

(c) If $\gamma = \beta_1\beta_2\ldots\beta_k$ is a path in Δ and $\phi_{\gamma,Z} : Z \longrightarrow \widetilde{Z}$ belongs to Γ, then $\phi_{\gamma,Z}$ is a bijection, $\widetilde{Z} \in \Phi$ and the inverse of $\phi_{\gamma,Z}$ belongs to Γ and equals $\phi_{\gamma^{-1},\widetilde{Z}}$ where $\gamma^{-1} = \overline{\beta}_k\overline{\beta}_{k-1}\ldots\overline{\beta}_1$.

1.2 Multiplicity structure.

(1) A *multiplicity structure on X* consists of

(1) a family of groups $(G_U)_{U\in\Phi}$ called *multiplicity groups*
(2) a family of homomorphisms $\rho_U^V : G_V \longrightarrow G_U$ for all $U, V \in \Phi$, with $U \subseteq V$ called *restriction mappings*
(3) a family of group isomorphisms $\lambda_{\beta,U} : G_U \longrightarrow G_{\phi_\beta(U)}$ where $\beta \in E$, and $U \in \Phi$ with $U \subseteq Y_\beta$

satisfying the following axioms:

(2) Multiplicity axioms.

Axiom 1 (Identity).

For any $U \in \Phi$,

$$\boxed{\rho_U^U = Id_{G_U}}$$

Axiom 2 (Transitivity).

If $U, V, W \in \Phi$, $U \subseteq V \subseteq W$, then

$$\boxed{\rho_U^V \cdot \rho_V^W = \rho_U^W}$$

Axiom 3 (Compatibility with Restrictions).

If $U_1 \subseteq U_2 \subseteq Y_\beta$, for $\beta \in E$ and $U_1, U_2 \in \Phi$ then the following diagram commutes:

$$\begin{array}{ccc} G_{U_2} & \xrightarrow{\rho_{U_1}^{U_2}} & G_{U_1} \\ \lambda_{\beta,U_2}\downarrow & & \downarrow\lambda_{\beta,U_1} \\ G_{\phi_\beta(U_2)} & \xrightarrow{\rho_{\phi_\beta(U_1)}^{\phi_\beta(U_2)}} & G_{\phi_\beta(U_1)} \end{array}$$

1.3 Monodromy data.

Let $((X_\alpha)_{\alpha\in V}, (\phi_\beta)_{\beta\in E})$ be a local Φ-pseudogroup presentation on X with associated pseudogroup, Γ, and let

$$((G_U)_{U\in\Phi}, (\rho_U^V \mid U, V \in \Phi, U \subseteq V), (\lambda_{\beta,U} \mid U \in \Phi, U \subseteq Y_\beta, \beta \in E))$$

be a multiplicity structure on X.

Monodromy data for X is a choice of $g_{\gamma,Z} \in G_Z$, the *monodromy element*, for each pair (γ, Z) such that $\phi_{\gamma,Z} \in \Gamma$, $\phi_{\gamma,Z} \mid_Z = Id_Z$ and the following axioms are satisfied:

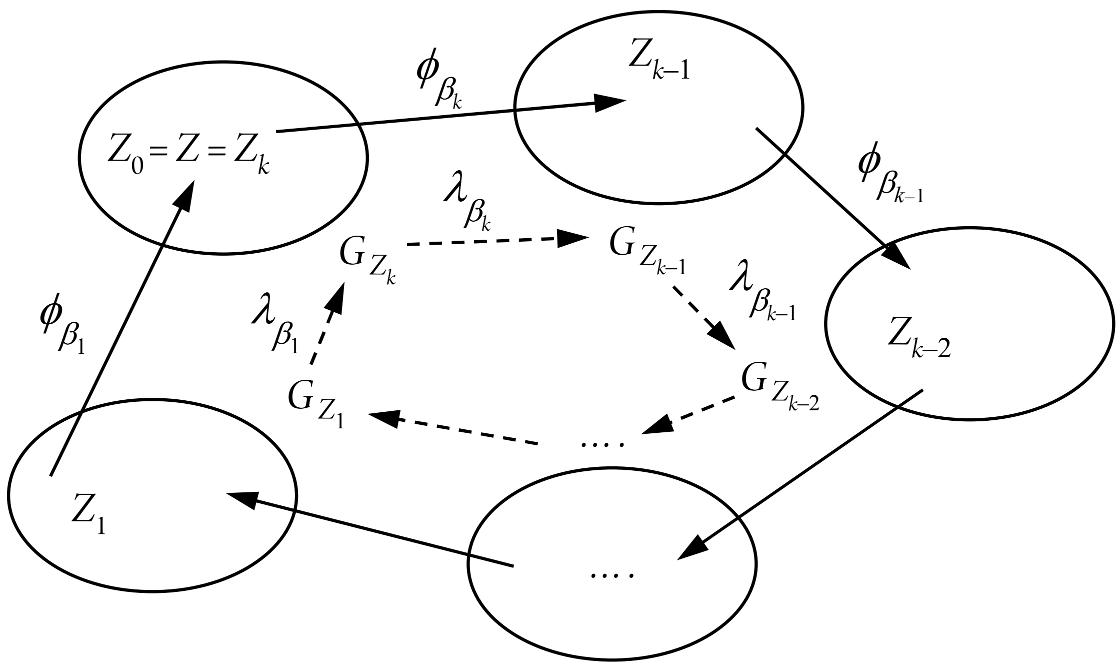
(1) Monodromy axioms.

Axiom 1 (Inner Automorphism).

Let $Z \in \Phi$ and let $\gamma = \beta_1\beta_2 \dots \beta_k$ be a closed path in Δ such that $\phi_{\gamma,Z} \in \Gamma$, with $\phi_{\gamma,Z} |_Z = \phi_{\beta_1} \cdot \dots \cdot \phi_{\beta_k} |_Z = Id_Z$.

Set

$$\begin{aligned}
&Z_k = Z \\
&Z_{k-1} = \phi_{\beta_k}(Z_k) \\
&Z_{k-2} = \phi_{\beta_{k-1}}(Z_{k-1}) = \phi_{\beta_{k-1}}\phi_{\beta_k}(Z_k) \\
&\dots \\
&Z_1 = \phi_{\beta_2}(Z_2) = \phi_{\beta_2}\phi_{\beta_3} \dots \phi_{\beta_k}(Z_k) \\
&Z_0 = \phi_{\beta_1}(Z_1) = \phi_{\beta_1} \dots \phi_{\beta_k}(Z_k) = \phi_{\gamma,Z_k}(Z_k) = Z_k = Z
\end{aligned}$$



Consider the following sequence of maps:

$$G_Z = G_{Z_k} \xrightarrow{\lambda_{\beta_k,Z_k}} G_{Z_{k-1}} \xrightarrow{\lambda_{\beta_{k-1},Z_{k-1}}} \dots \longrightarrow G_{Z_1} \xrightarrow{\lambda_{\beta_1,Z_1}} G_{Z_0} = G_{Z_k} = G_Z.$$

Put $\lambda_{\gamma,Z} = \lambda_{\beta_1,Z} \dots \lambda_{\beta_{k-1},Z}\lambda_{\beta_k,Z} : G_Z \longrightarrow G_Z$. Then

$$\boxed{\lambda_{\gamma,Z} = ad(g_{\gamma,Z})}$$

where $ad(g_{\gamma,Z})(g) = g_{\gamma,Z} \cdot g \cdot g_{\gamma,Z}^{-1}$ for each $g \in G_Z$.

Axiom 2 (Compatibility with Restrictions).

If $Z' \subseteq Z$, then restriction mappings take the monodromy element of (γ, Z) to the monodromy element of (γ, Z'):

$$\boxed{\rho^Z_{Z'} : G_Z \longrightarrow G_{Z'}, \text{ with } g_{\gamma,Z} \longmapsto g_{\gamma,Z'}}$$

Axiom 3 (Multiplication).

Suppose that $\phi_{\gamma,Z}, \phi_{\delta,Z} \in \Gamma$, with $\phi_{\gamma,Z} \mid_Z = Id_Z$, $\phi_{\delta,Z} \mid_Z = Id_Z$, and so

$$\phi_{\gamma\delta,Z} = \phi_{\gamma,Z} \cdot \phi_{\delta,Z} \in \Gamma$$

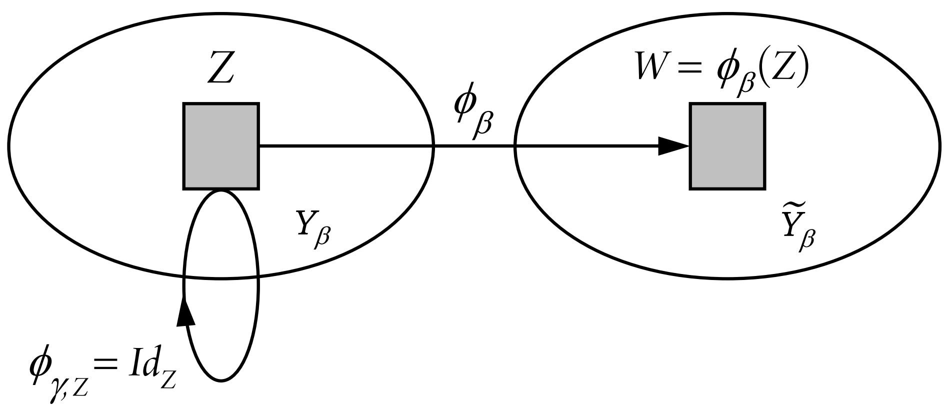
with $\phi_{\gamma\delta,Z} \mid_Z = Id_Z$. Then the monodromy element $g_{\gamma\delta,Z}$ of $(\gamma\delta, Z)$ is the product in G_Z

$$\boxed{g_{\gamma\delta,Z} = g_{\gamma,Z} \cdot g_{\delta,Z}}$$

Axiom 4 (Covariance).

Let $Z \subseteq Y_\beta$ for some $\beta \in E$ and let $\phi_{\gamma,Z} \in \Gamma$, with $\phi_{\gamma,Z} \mid_Z = Id_Z$. Set $W = \phi_\beta(Z)$ and observe that

$$\phi_{\beta\gamma\beta^{-1},W} = \phi_\beta \cdot \phi_{\gamma,Z} \cdot \phi_\beta^{-1} \mid_W = Id_W.$$



Then we have

$$\boxed{\lambda_{\beta,Z}(g_{\gamma,Z}) = g_{\beta\gamma\beta^{-1},W}.}$$

1.4 Local data on X.

We say that X is a set with *local data* $\mathcal{D}$, where

$$\mathcal{D} = ((X_\alpha)_{\alpha\in V}, (\phi_\beta)_{\beta\in E}, (G_U)_{U\in\Phi}, (\rho^V_U), (\lambda_{\beta,U}), (g_{\gamma,Z}))$$

if

(1) $((X_\alpha)_{\alpha\in V}, (\phi_\beta)_{\beta\in E})$ is a Φ-pseudogroup presentation on X defining a local Φ-pseudogroup, Γ,

(2) $((G_U)_{U\in\Phi}, (\rho_U^V \mid U, V \in \Phi, U \subseteq V), (\lambda_{\beta,U} \mid U \in \Phi, U \subseteq Y_\beta, \beta \in E))$ is a multiplicity structure on X,

(3) $(g_{\gamma,Z})$ is monodromy data for X, where the monodromy element $g_{\gamma,Z} \in G_Z$ is given for each pair (γ, Z) such that $\phi_{\gamma,Z} \in \Gamma$ and $\phi_{\gamma,Z} |_Z = Id_Z$.

A set X with local data $\mathcal{D}$ will sometimes be denoted $(X, \mathcal{D})$.

1.5 Data from a group action on a set.

(1) Suppose that a group G acts on a set X. If G fixes any element $x \in X$, we say that the action of G on X is trivial. We will assume that the action of G on X is non-trivial.

Let $(X_\alpha)_{\alpha\in V}$ be a family of subsets of X. Let $(g_k)_{k\in K}$ be a family of elements of G.

We construct a local pseudogroup that imitates the action of G on X as follows:

(2) Set

$$E^+ = \{\beta = (\alpha', k, \alpha) \mid \alpha, \alpha' \in V, k \in K \text{ and } X_{\alpha'} \cap g_k^{-1} X_\alpha \neq \varnothing\}$$

For each $\beta = (\alpha', k, \alpha) \in E^+$ put

$$Y_\beta = X_{\alpha'} \cap g_k^{-1} X_\alpha \subseteq X_{\alpha'},$$

$$\widetilde{Y_\beta} = g_k X_{\alpha'} \cap X_\alpha \subseteq X_\alpha,$$

and $\phi_\beta = g_k |_{Y_\beta} \colon Y_\beta \longrightarrow \widetilde{Y_\beta}$. Then ϕ_β is a bijection. We set $E = E^+ \bigsqcup \overline{E^+}$ where $\overline{E^+} = \{\overline{\beta} : \beta \in E^+\}$. For each $\overline{\beta} \in \overline{E^+}$ we define $\phi_{\overline{\beta}} = \phi_\beta^{-1} : \widetilde{Y_\beta} \longrightarrow Y_\beta$ and $Y_{\overline{\beta}} = \widetilde{Y_\beta}$, $\widetilde{Y_{\overline{\beta}}} = Y_\beta$, $\overline{\overline{\beta}} = \beta$. For every $\beta = (\alpha', k, \alpha) \in E^+$, we set $o(\beta) = \alpha = t(\overline{\beta})$, $t(\beta) = \alpha' = o(\overline{\beta})$. Then $\Delta = (V, E, o, t, -)$ is a graph with corresponding pseudogroup presentation

$$((X_\alpha)_{\alpha\in V}, (\phi_\beta)_{\beta\in E}).$$

(3) Let Φ be a family of subsets of X, invariant under the ϕ_β, $\beta \in E$. For each Φ-subset $Z \subseteq X_\alpha$ we define the multiplicity group G_Z to be the pointwise stabilizer of Z

$$G_Z = \{g \in G \mid g(z) = z \text{ for every } z \in Z\}.$$

Then if $Z \subseteq Z'$, we have $G_{Z'} \subseteq G_Z$, and we define the restriction mappings

$$\rho_Z^{Z'} : G_{Z'} \longrightarrow G_Z$$

to be the inclusions $G_{Z'} \hookrightarrow G_Z$.

(4) Moreover, if $\beta = (\alpha', k, \alpha) \in E^+$, $g_k \in G$, $Z \subseteq Y_\beta$ and $\phi_\beta(Z) = g_k \cdot Z$, then $h \in G_Z$ if and only if $g_k \cdot h \cdot g_k^{-1} \in G_{\phi_\beta(Z)}$. Therefore, we set $\lambda_{\beta,Z} : G_Z \longrightarrow G_{\phi_\beta(Z)}$ to be the conjugation $\lambda_{\beta,Z} = ad(g_k) : h \mapsto g_k \cdot h \cdot g_k^{-1}$. It follows that the isomorphisms $\lambda_{\beta,Z} : G_Z \longrightarrow G_{\phi_\beta(Z)}$ commute with the restriction mappings $\rho_Z^{Z'} : G_{Z'} \longrightarrow G_Z$.

(5) Suppose that for some $Z \in \Phi$, there is a composition of partial mappings

$$\phi_\gamma = \phi_{\beta_1} \circ \cdots \circ \phi_{\beta_t}$$

that acts identically on Z: that is; $\phi_{\beta_1} \circ \cdots \circ \phi_{\beta_t} |_Z = Id_Z$, where

$$\begin{aligned} \beta_t &= (\alpha_t, k_t, \alpha_{t-1}) \\ \beta_{t-1} &= (\alpha_{t-1}, k_{t-1}, \alpha_{t-2}) \\ &\vdots \\ \beta_1 &= (\alpha_1, k_1, \alpha_0 = \alpha_t) \end{aligned}$$

and

$$\begin{aligned} Z_t &= Z \\ Z_{t-1} &= \phi_{\beta_t}(Z_t) = g_{k_t} Z_t \\ &\vdots \\ Z_1 &= \phi_{\beta_2}(Z_2) = \phi_{\beta_2}\phi_{\beta_3}\ldots\phi_{\beta_t}(Z_t) = g_{k_2}g_{k_3}\ldots g_{k_t}Z_t \\ Z = Z_0 &= \phi_{\beta_1}(Z_1) = \phi_{\beta_1}\ldots\phi_{\beta_t}(Z_t) = g_{k_1}\ldots g_{k_t}Z_t = g_{k_1}\ldots g_{k_t}Z \end{aligned}$$

(6) Observe that

$$\lambda_{\beta_1, Z_1} \circ \lambda_{\beta_2, Z_2} \circ \cdots \circ \lambda_{\beta_t, Z_t} = ad(g_1 \ldots g_t) : G_Z \longrightarrow G_Z$$

and that the product $g_1 \ldots g_t$ stabilizes Z pointwise.
(7) We set the monodromy element $g_{\gamma,Z}$ to be the product $g_1 \ldots g_t \in G_Z$. It is easy to see that the monodromy axioms are satisfied for this choice of $g_{\gamma,Z}$. We conclude that when a group G acts on a set X,

$$((X_\alpha)_{\alpha \in V}, (\phi_\beta)_{\beta \in E}, (G_Z)_{Z \in \Phi}, (\rho_Z^{Z'}), (\lambda_{\beta,Z}), (g_{\gamma,Z}))$$

is local data on $\bigsqcup_{\alpha \in V} X_\alpha$ in the sense of 1.4.

2. COVERING THEORY FOR A SET WITH LOCAL DATA.

2.1 The graph of groups for a set with local data.

(1) Let $(X, \mathcal{D})$ be a set with local data, where

$$\mathcal{D} = ((X_\alpha)_{\alpha \in V}, (\phi_\beta)_{\beta \in E}, (G_U)_{U \in \Phi}, (\rho_U^V), (\lambda_{\beta,U}), (g_{\gamma,Z}))$$

as in 1.4, and let $\Delta = (V, E, o, t, -)$ be the corresponding graph. In this section, we construct a graph of groups $\mathcal{G}(V, E)$ naturally associated with $\mathcal{D}$.
(2) We define $\mathcal{G}(V, E)$ as follows: the underlying graph is the oriented graph $\Delta = (V, E, o, t, -)$. For each $\alpha \in V$, we define the vertex group $\mathcal{G}(\alpha)$ to be the direct limit of the multiplicity groups and restriction mappings:

$$\mathcal{G}(\alpha) = \varinjlim_{Z \subseteq X_\alpha} (G_Z, \rho_Z^{Z'}).$$

We note that the further condition that $Z \subset \Phi$ is implied but will not be explicitly stated throughout the paper.
(3) By a direct limit $\mathcal{G}(\alpha)$ of the family of groups G_Z and the set of homomorphisms $\rho_Z^{Z'} : G_{Z'} \longrightarrow G_Z$ for $Z \subseteq Z'$ we mean the following. For each $Z \subseteq X_\alpha$, $Z \in \Phi$, there are canonical homomorphisms $\rho^Z : G_Z \longrightarrow \mathcal{G}(\alpha)$ such that $\rho^Z \cdot \rho_Z^{Z'} = \rho^{Z'}$ whenever $Z \subseteq Z'$; that is; the following diagram commutes:

$$\begin{array}{ccc} & G_{Z'} & \\ \rho_Z^{Z'} \swarrow & & \searrow \rho^{Z'} \\ G_Z & \xrightarrow{\rho^Z} & \mathcal{G}(\alpha) \end{array}$$

(4) Moreover, $\mathcal{G}(\alpha)$ satisfies the following universal property: if H is any group and there is a system of homomorphisms

$$\Psi^Z : G_Z \longrightarrow H$$

for each $Z \subseteq X_\alpha$ such that $\Psi^Z \cdot \rho_Z^{Z'} = \Psi^{Z'}$ whenever $Z \subseteq Z' \subseteq X_\alpha$, then there is a unique homomorphism $\Psi : \mathcal{G}(\alpha) \longrightarrow H$ such that for each $Z \subseteq X_\alpha$ the following diagram commutes

$$\begin{array}{ccc} & G_Z & \\ \rho^Z \swarrow & & \searrow \Psi^Z \\ \mathcal{G}(\alpha) & \xrightarrow{\Psi} & H \end{array}$$

This is the usual notion of direct limit as in ([S], Sec 1.1).
(5) For $g \in G_Z$, let $\overline{g}$ denote the image $\rho_Z(g)$ of g in $\mathcal{G}(\alpha) = \varinjlim_{Z \subseteq X_\alpha} (G_Z, \rho_Z^{Z'})$.
(6) We choose an orientation E^+ of E such that $E = E^+ \bigsqcup \overline{E^+}$. For each $\beta \in E^+$ with $o(\beta) = \alpha$, $t(\beta) = \alpha'$:

$$t(\beta) = \alpha' \xleftarrow[\beta]{\hspace{3cm}} o(\beta) = \alpha$$

$$Y_\beta \subseteq X_{\alpha'} \xrightarrow[\phi(\beta)]{\hspace{8em}} \widetilde{Y_\beta} \subseteq X_\alpha$$

(7) We define the edge groups $\mathcal{G}(\beta)$ of $\mathcal{G}(V,E)$ as follows:

$$\mathcal{G}(\beta) = \varinjlim_{Z \subseteq Y_\beta} (G_Z, \rho_Z^{Z'})$$

where $Y_\beta \subseteq X_{\alpha'} = X_{t(\beta)}$.
(8) For $\beta \in E^+$ we set $\mathcal{G}(\overline{\beta}) = \mathcal{G}(\beta)$. Observe that for an edge $\beta \in E^+$ with $o(\beta) = \alpha$ and $t(\beta) = \alpha'$ there is a canonical homomorphism

$$\mathcal{G}(\beta) = \varinjlim_{Z \subseteq Y_\beta \subseteq X_{\alpha'}} (G_Z, \rho_Z^{Z'}) \xrightarrow{\mu} \varinjlim_{Z \subseteq X_{\alpha'}} (G_Z, \rho_Z^{Z'}) = \mathcal{G}(\alpha').$$

We therefore define the edge homomorphism

$$\omega_\beta : \mathcal{G}(\beta) \longrightarrow \mathcal{G}(\alpha') = \mathcal{G}(t(\beta))$$

to be the homomorphism μ.
(9) We recall that for each $Z \subseteq Y_\beta \subseteq X_{\alpha'}$ and $\phi_\beta(Z) \subseteq \widetilde{Y_\beta} \subseteq X_\alpha$ there are isomorphisms of multiplicity groups $\lambda_{\beta,Z} : G_Z \longrightarrow G_{\phi_\beta(Z)}$ that commute with restriction mappings.
(10) Therefore, there is a canonical isomorphism of direct limits:

$$\lambda_\beta : \varinjlim_{Z \subseteq Y_\beta \subseteq X_{\alpha'}} (G_Z, \rho_Z^{Z'}) \longrightarrow \varinjlim_{Z \subseteq \widetilde{Y_\beta} \subseteq X_\alpha} (G_Z, \rho_Z^{Z'}).$$

(11) We also have a canonical homomorphism

$$\varinjlim_{Z \subseteq \widetilde{Y_\beta} \subseteq X_\alpha} (G_Z, \rho_Z^{Z'}) \xrightarrow{\widetilde{\mu}} \varinjlim_{Z \subseteq X_\alpha} (G_Z, \rho_Z^{Z'})$$

induced by the inclusion $\widetilde{Y_\beta} \subseteq X_\alpha$.
(12) We define the edge homomorphism:

$$\omega_{\overline{\beta}} : \mathcal{G}(\beta) \longrightarrow \mathcal{G}(\alpha) = \mathcal{G}(o(\beta))$$

to be the composition $\widetilde{\mu} \cdot \lambda_\beta$:

$$\mathcal{G}(\beta) \xrightarrow{\lambda_\beta} \varinjlim_{Z \subseteq \widetilde{Y_\beta} \subseteq X_\alpha} (G_Z, \rho_Z^{Z'}) \xrightarrow{\widetilde{\mu}} \mathcal{G}(\alpha) = \mathcal{G}(o(\beta)).$$

(13) Remark.

In this setting, the edge homomorphisms ω_β and $\omega_{\overline{\beta}}$ are not necessarily monomorphisms. In Section 4 we shall see that under certain conditions on $\mathcal{D}$, these maps are indeed one-to-one.

(14) Example.

Let A, B and C be groups with $A \cap B = C$. Let data $\mathcal{D}$ be as follows:

$$X_\alpha \quad = \quad [0,1] \quad \subseteq \quad \mathbb{R}.$$

For an interval $Z = [a,b] \subseteq [0,1]$ we set:

$$G_Z \quad = \quad G_{[a,b]} = \quad \begin{cases} A, & \text{if } [a,b] = \{0\} \\ B, & \text{if } [a,b] = \{1\} \\ C, & \text{otherwise,} \end{cases}$$

and for $Z \subseteq Z'$, the restriction mappings $\rho_Z^{Z'}$ are the natural inclusion mappings.

The graph of groups $\mathcal{G}(V,E)$ associated to this data $\mathcal{D}$ is described as follows. The underlying graph Δ consists of a single vertex α. The vertex group $G(\alpha) = \varinjlim_{Z \subseteq X_\alpha}(G_{Z'}, \rho_Z^{Z'})$ is the amalgamated product $A *_C B$, which coincides with the fundamental group of an 'edge of groups' with vertex groups A and B and edge group C (see [B] or [S]).

2.2 The groupoid of a graph of groups.

In this section we describe a natural *groupoid* that can be constructed from a graph of groups.

(1) We recall that a *groupoid* is a small category where every morphism is invertible. In this way, we can view a group as a groupoid with a single object, and elements of the group as morphisms from this object to itself.

(2) Groups can be given by generators and defining relations. In a similar way, a groupoid can be described by specifying the set of objects, the set of generating morphisms, and the set of defining relations between these morphisms.

(3) Let $A = (VA, EA, o, t, -)$ be an oriented graph, and let $\mathbb{A}$ be a graph of groups with underlying graph A, vertex groups $\mathcal{A}_v$, $v \in VA$, edge groups $\mathcal{A}_e$, $e \in EA$ and boundary homomorphisms $\omega_e : \mathcal{A}_e \longrightarrow \mathcal{A}_{t(e)}$, $\omega_{\overline{e}} : \mathcal{A}_e \longrightarrow \mathcal{A}_{o(e)}$, $e \in EA$.

(4) We define the *Bass-Serre groupoid* of $\mathbb{A}$ to be the groupoid with presentation

$$\mathcal{B}(\mathbb{A}) = \langle (\mathcal{A}_v)_{v \in VA}, EA \mid e\omega_e(g)e^{-1} = \omega_{\overline{e}}(g), g \in \mathcal{A}_e, e\overline{e} = 1_{o(e)}, \overline{e}e = 1_{t(e)} \rangle$$

where the set of objects of $\mathcal{B}(\mathbb{A})$ is VA, each $g \in \mathcal{A}_v$ for $v \in VA$ is viewed as a morphism from v to v, and every edge $e \in EA$ is viewed as a morphism from $o(e)$ to $t(e)$. We let $Mor_{\mathcal{B}(\mathbb{A})}(v_1, v_2)$ denote the collection of all morphisms from $v_1, v_2 \in VA$.

(5) Every morphism $\sigma \in Mor_{\mathcal{B}(\mathbb{A})}(v_1, v_2)$, $v_1, v_2 \in VA$ can be expressed as a product $\sigma = g_0 e_1 g_1 e_2 \dots g_{k-1} e_k g_k$ of generating morphisms, where $\gamma = e_1 e_2 \dots e_k$ is a path in A from v_1 to v_2, $g_0 \in \mathcal{A}_{o(e_1)}$ and $g_i \in \mathcal{A}_{t(e_i)}$, $i = 1, \dots k$.

(6) It follows that for each $v \in VA$

$$Mor_{\mathcal{B}(\mathbb{A})}(v, v) \cong \pi_1(\mathbb{A}, v)$$

where $\pi_1(\mathbb{A}, v)$ is the fundamental group of the graph of groups $\mathbb{A}$ relative to the basepoint v (see [B]).

2.3 The Bass-Serre and monodromy groupoids.

In the previous section we introduced a groupoid associated to a graph of groups. We now apply this construction to the graph of groups associated to a set with local data.

(1) Let $(X, \mathcal{D})$ be a set with local data

$$((X_\alpha)_{\alpha \in V}, (\phi_\beta)_{\beta \in E}, (G_Z)_{Z \in \Phi}, (\rho_Z^{Z'}), (\lambda_{\beta,Z}), (g_{\gamma,Z}))$$

and let $\Delta = (V, E, o, t, -)$ be the corresponding graph.

(2) Recall from 2.1 that $\mathcal{G}(V, E)$ is the graph of groups built on Δ with vertex groups

$$\mathcal{G}(\alpha) = \varinjlim_{Z \subseteq X_\alpha} (G_Z, \rho_Z^{Z'}),$$

edge groups

$$\mathcal{G}(\beta) = \varinjlim_{Z \subseteq Y_\beta} (G_Z, \rho_Z^{Z'})$$

where $Y_\beta \subseteq X_{t(\beta)}$, $\beta \in E^+\Delta$, and boundary homomorphisms

$$\omega_\beta : \mathcal{G}(\beta) \longrightarrow \mathcal{G}(t(\beta))$$

$$\omega_{\overline{\beta}} : \mathcal{G}(\beta) \longrightarrow \mathcal{G}(o(\beta))$$

as defined in 2.1.

(3) We define the Bass-Serre groupoid $\mathcal{B}(\mathcal{D})$ to be the groupoid $\mathcal{B}(\mathcal{G}(V, E))$ of the graph of groups $\mathcal{G}(V, E)$. That is,

$$\mathcal{B}(\mathcal{D}) = \langle (\mathcal{G}(\alpha))_{\alpha \in V\Delta}, (\beta)_{\beta \in E\Delta} \quad | \quad \beta \cdot w_\beta(g) \cdot \beta^{-1} = \omega_{\overline{\beta}}(g), \text{ for every } g \in \mathcal{G}(\beta),\ \beta\overline{\beta} = 1_{o(\beta)}, \overline{\beta}\beta = 1_{t(\beta)} \rangle.$$

(4) We recall that the objects of $\mathcal{B}(\mathcal{D})$ are the vertices $V\Delta$ of Δ. An element $g \in \mathcal{G}(\alpha)$, $\alpha \in V\Delta$ is viewed as a morphism from α to α and every edge β is viewed as a morphism from $o(\beta)$ to $t(\beta)$.

(5) Therefore, any closed path $\gamma = \beta_1 \dots \beta_k$ in Δ with $o(\beta_1) = \alpha = t(\beta_k)$ can be viewed as a morphism from α to α. In particular, if for some $Z \subseteq Y_{\beta_k} \cap \beta_1 \subseteq X_\alpha$, we have

$$\phi_{\gamma,Z} \mid_Z = \phi_{\beta_1} \cdot \dots \cdot \phi_{\beta_k} \mid_Z = Id_Z,$$

then γ and the image $\overline{g_{\gamma,Z}}$ of the monodromy element $g_{\gamma,Z} \in G_Z$ in $\mathcal{G}(\alpha) = \varinjlim_{Z \subseteq X_\alpha} (G_Z, \rho_Z^{Z'})$ can both be viewed as morphisms from α to α.

(6) For $\alpha \in V$ we define

$$\pi_1(\mathcal{B}(\mathcal{D}), \alpha) = Mor_{\mathcal{B}(\mathcal{D})}(\alpha, \alpha).$$

(7) We now introduce the *monodromy groupoid* $\mathcal{M}(\mathcal{D})$ as a quotient of the groupoid $\mathcal{B}(\mathcal{D})$ obtained by identifying the morphisms γ and $\overline{g_{\gamma,Z}}$. That is, $\mathcal{M}(\mathcal{D})$ is given by the presentation

$$\mathcal{M}(\mathcal{D}) = \langle (\mathcal{G}(\alpha))_{\alpha\in V\Delta}, (\beta)_{\beta\in E\Delta} \mid \beta\cdot w_\beta(g)\cdot\beta^{-1} = \omega_{\overline{\beta}}(g) \text{ for every } g\in\mathcal{G}(\beta),$$
$$\beta\overline{\beta} = 1_{o(\beta)}, \overline{\beta}\beta = 1_{t(\beta)}, \gamma = \overline{g_{\gamma,Z}} \text{ for every } \gamma \text{ with } \phi_{\gamma,Z} = Id_Z\rangle.$$

We set $\pi_1(\mathcal{M}(\mathcal{D}),\alpha) = Mor_{\mathcal{M}(\mathcal{D})}(\alpha,\alpha)$, for $\alpha\in V$ and we will refer to $\pi_1(\mathcal{M}(\mathcal{D}),\alpha)$ as $\mathcal{G}(\mathcal{D})$.

The group $\mathcal{G}(\mathcal{D})$

In the following special case, we can explicitly give generators and relations for the group

$$\mathcal{G}(\mathcal{D}) = \pi_1(\mathcal{M}(\mathcal{D}),\alpha) = Mor_{\mathcal{M}(\mathcal{D})}(\alpha,\alpha).$$

Let $\mathcal{D}$ be data on a set X with a local pseudogroup of partial isometries

$$(\Psi, X) \quad = \quad \{\phi_\beta : Y_\beta \longrightarrow \widetilde{Y_\beta}\}$$

and assume that (Ψ, X) contains

$$\{\phi_\beta^{-1} : \widetilde{Y_\beta} \longrightarrow Y_\beta\}.$$

Assume that all multiplicity groups of $\mathcal{D}$ are trivial. Assume also that the graph Δ associated to $\mathcal{D}$ has a single vertex α and that $X = X_\alpha$. Let $\mathcal{G}(\mathcal{D})$ be the corresponding group. Then the presentation of $\mathcal{G}(\mathcal{D})$ takes the following form:

$$\mathcal{G}(\mathcal{D}) = \langle (f_\beta)_{\beta\in I} \mid f_\gamma := f_{\beta_1}^{\epsilon_1} f_{\beta_2}^{\epsilon_2}\dots f_{\beta_k}^{\epsilon_k} = 1 \text{ whenever } \phi_\gamma(x) := \phi_{\beta_1}^{\epsilon_1}\phi_{\beta_2}^{\epsilon_2}\dots\phi_{\beta_k}^{\epsilon_k}(x) = x,$$
$$\text{for some } x\in X,\ \epsilon_i = \pm 1\rangle.$$

In the examples that follow in the next sections, we try where possible to choose local data with a single vertex α such that $X = X_\alpha$.

(8) Remark.

It follows from the definitions of $\mathcal{B}(\mathcal{D})$, and $\mathcal{M}(\mathcal{D})$, that if $g\in G_Z$, and $Z\subseteq Y_\beta$, then

$$\beta\cdot\overline{g}\cdot\beta^{-1} \underset{\mathcal{B}(\mathcal{D})}{=} \overline{\lambda_{\beta,Z}(g)}$$

and

$$\beta\cdot\overline{g}\cdot\beta^{-1} \underset{\mathcal{M}(\mathcal{D})}{=} \overline{\lambda_{\beta,Z}(g)}.$$

2.4 Covering spaces.

(1) In this section we build spaces $\widetilde{\mathcal{B}(\mathcal{D},\alpha_0)}$, and $\widetilde{\mathcal{M}(\mathcal{D},\alpha_0)}$ from a set X with local data $\mathcal{D}$ on which the groupoids $\mathcal{B}(\mathcal{D})$ and $\mathcal{M}(\mathcal{D})$ have natural actions.
(2) Recall that $\Delta = (V, E, o, t, -)$ is the graph associated with

$$\mathcal{D} = ((X_\alpha)_{\alpha\in V}, (\phi_\beta)_{\beta\in E}, (G_Z)_{Z\in\Phi}, (\rho_Z^{Z'}), (\lambda_{\beta,Z}), (g_{\gamma,Z})).$$

(3) Fix a basepoint $\alpha_0 \in V\Delta$. We form

$$\widetilde{\mathcal{B}(\mathcal{D},\alpha_0)} = (\bigsqcup_{\alpha\in V\Delta} Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha)\times X_\alpha)/\approx_{\mathcal{B}}$$

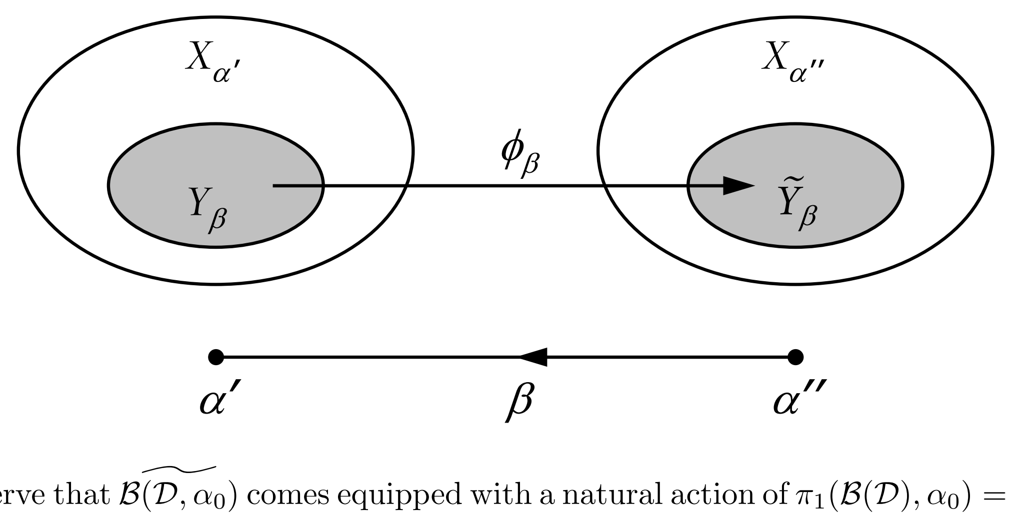
where $Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha)$ denotes the set of all morphisms in $\mathcal{B}(\mathcal{D})$ from α_0 to α and $\approx_{\mathcal{B}}$ is the equivalence relation on $\bigsqcup_{\alpha\in V\Delta} Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha)\times X_\alpha$ generated by the following relations:

$$\boxed{(1)_{\mathcal{B}} \qquad (\sigma g, x) \approx_{\mathcal{B}} (\sigma, x)}$$

where $\sigma \in Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha)$, $x \in X_\alpha$, $g \in G_x$, so that $(\sigma g, x)$ and $(\sigma, x) \in Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha)\times X_\alpha$,

$$\boxed{(2)_{\mathcal{B}} \qquad (\sigma\beta, x) \approx_{\mathcal{B}} (\sigma, \phi_\beta(x))}$$

where $\sigma \in Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha'')$, and $\beta \in E\Delta$ with $o(\beta) = \alpha''$ and $t(\beta) = \alpha'$, so that $\sigma\beta \in Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha')$, and $x \in Y_\beta \subseteq X_{\alpha'}$, $\phi_\beta(x) \in \widetilde{Y_\beta} \subseteq X_{\alpha''}$. (That is, $(\sigma\beta, x) \in Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha')\times X_{\alpha'}$ and $(\sigma, \phi_\beta(x)) \in Mor_{\mathcal{B}(\mathcal{D})}(\alpha,\alpha'')\times X_{\alpha''}$.)



(4) Observe that $\widetilde{\mathcal{B}(\mathcal{D},\alpha_0)}$ comes equipped with a natural action of $\pi_1(\mathcal{B}(\mathcal{D}),\alpha_0) = Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha_0)$. That is; if $\sigma \in Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha_0)$, $\sigma' \in Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha)$ and $x \in X_\alpha$, then

$$\sigma\cdot(\sigma', x)_{\approx_{\mathcal{B}}} = (\sigma\sigma', x)_{\approx_{\mathcal{B}}}.$$

(5) Remark.

The goal in constructing $\widetilde{\mathcal{B}(\mathcal{D},\alpha_0)}$ is to build a space from the data $\mathcal{D}$ on which the groupoid $\mathcal{B}(\mathcal{D})$ has a natural action.

The space $\widetilde{\mathcal{B}(\mathcal{D},\alpha_0)}$ is built from the 'translates' of the X_α by the elements of $\mathcal{B}(\mathcal{D})$ and is therefore the quotient:

$$(\bigsqcup_{\alpha\in V\Delta} Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha)\times X_\alpha)/\approx_{\mathcal{B}} .$$

Here, the equivalence class $(\sigma,x)_{\approx_{\mathcal{B}}}$ represents the σ-translate $\sigma\cdot x$ of x.

(6) We want an element g of the multiplicity group G_x, $x\in X_\alpha$ to stabilize the image of x in $\widetilde{\mathcal{B}(\mathcal{D},\alpha_0)}$, so we impose the relation:

$$(\sigma g,x)\approx_{\mathcal{B}}(\sigma,x)$$

in $\widetilde{\mathcal{B}(\mathcal{D},\alpha_0)}$ for $\sigma\in Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,\alpha)$.

(7) The action of the generating morphism $\beta\in\mathcal{B}(\mathcal{D})$ on $\widetilde{\mathcal{B}(\mathcal{D},\alpha_0)}$ mimics the partial bijection $\phi_\beta : Y_\beta \longrightarrow \widetilde{Y_\beta}$ so we impose the relation:

$$(\sigma\beta,x)\approx_{\mathcal{B}}(\sigma,\phi_\beta(x))$$

in $\mathcal{B}(\mathcal{D})$ for $\sigma\in Mor_{\mathcal{B}(\mathcal{D})}(\alpha_0,o(\beta))$, $x\in Y_\beta\subseteq X_{t(\beta)}$.

(8) We analogously define a space $\widetilde{\mathcal{M}(\mathcal{D},\alpha_0)}$ with a natural action of the group $\pi_1(\mathcal{M}(\mathcal{D}),\alpha_0)$. We set

$$\widetilde{\mathcal{M}(\mathcal{D},\alpha_0)}=(\bigsqcup_{\alpha\in V\Delta} Mor_{\mathcal{M}(\mathcal{D})}(\alpha_0,\alpha)\times X_\alpha)/\approx_{\mathcal{M}}$$

where $\approx_{\mathcal{M}}$ is the equivalence relation on $\bigsqcup_{\alpha\in V\Delta} Mor_{\mathcal{M}(\mathcal{D})}(\alpha_0,\alpha)\times X_\alpha$ generated by the following relations:

$$\boxed{(1)_{\mathcal{M}}\qquad (\sigma g,x)\approx_{\mathcal{M}}(\sigma,x)}$$

where $\sigma\in Mor_{\mathcal{M}(\mathcal{D})}(\alpha_0,\alpha)$, $x\in X_\alpha$, $g\in G_x$,

$$\boxed{(2)_{\mathcal{M}}\qquad (\sigma\beta,x)\approx_{\mathcal{M}}(\sigma,\phi_\beta(x))}$$

where $\sigma\in Mor_{\mathcal{M}(\mathcal{D})}(\alpha_0,\alpha'')$, and $\beta\in E\Delta$ with $o(\beta)=\alpha''$ and $t(\beta)=\alpha'$, $x\in Y_\beta\subseteq X_{\alpha'}$, $\phi_\beta(x)\in\widetilde{Y_\beta}\subseteq X_{\alpha''}$.

(9) The group $\pi_1(\mathcal{M}(\mathcal{D}),\alpha_0)=Mor_{\mathcal{M}(\mathcal{D})}(\alpha_0,\alpha_0)$ acts on $\widetilde{\mathcal{M}(\mathcal{D},\alpha_0)}$:

$$\sigma\cdot(\sigma',x)_{\approx_{\mathcal{M}}}=(\sigma\sigma',x)_{\approx_{\mathcal{M}}},$$

where $\sigma\in Mor_{\mathcal{M}(\mathcal{D})}(\alpha_0,\alpha_0)$, $\sigma'\in Mor_{\mathcal{M}(\mathcal{D})}(\alpha_0,\alpha)$ and $x\in X_\alpha$.

(10) The space $\widetilde{\mathcal{M}(\mathcal{D},\alpha_0)}$ is called the *covering space* of $\mathcal{D}$ with respect to the basepoint α_0, and will be denoted $\mathcal{X}(\mathcal{D})$.

3. VERTEX GROUPS AS GROUPS OF SURFACE TRIPS.

(1) Let $(X, \mathcal{D})$ be a set with local data, where

$$\mathcal{D} = ((X_\alpha)_{\alpha \in V}, (\phi_\beta)_{\beta \in E}, (G_U)_{U \in \Phi}, (\rho_U^V), (\lambda_{\beta,U}), (g_{\gamma,Z})),$$

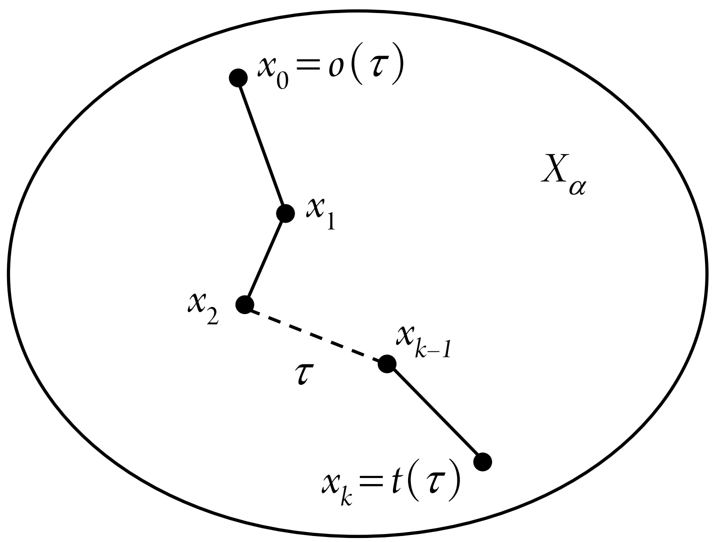
and let $\Delta = (V, E, o, t, -)$ be the corresponding graph. Let $\alpha \in V$. For each $x, y \in X_\alpha$ such that $\{x, y\} \in \Phi$, we associate the formal symbol t_{xy}.
(2) A *surface trip* τ in X_α from x_0 to x_k is a sequence of the form

$$\tau = g_0 t_{x_0 x_1} g_1 t_{x_1 x_2} \dots g_{k-1} t_{x_{k-1} x_k} g_k$$

where $x_0, x_1, \dots, x_k \in X_\alpha$, $g_i \in G_{x_i}$. When $k = 0$, g_0 is also viewed as a surface trip from x_0 to x_0. We set

$$\overline{\tau} = g_k^{-1} t_{x_k x_{k-1}} g_{k-1}^{-1} \dots t_{x_2 x_1} g_1^{-1} t_{x_1 x_0} g_0^{-1}$$

so that $\overline{\tau}$ is a surface trip from x_k to x_0.
(3) We call x_0 the *origin*, $o(\tau)$, of τ and x_k the *terminus*, $t(\tau)$, of τ.



Let

$$\tau = g_0 t_{x_0 x_1} g_1 t_{x_1 x_2} \dots g_{k-1} t_{x_{k-1} x_k} g_k$$

$$\tau' = g'_0 t_{x'_0 x'_1} g'_1 t_{x'_1 x'_2} \dots g'_{m-1} t_{x'_{m-1} x'_m} g'_m$$

be surface trips in X_α.
(4) The *juxtaposition* of surface trips τ and τ' in X_α is defined when

$$x_k = t(\tau) = o(\tau') = x'_0.$$

In this case we set

$$\tau \cdot \tau' = g_0 t_{x_0 x_1} g_1 \dots t_{x_{k-1} x_k} (g_k g'_0) t_{x'_0 x'_1} g'_1 \dots t_{x'_{m-1} x'_m} g'_m$$

where $(g_k g_0')$ is the product of g_k and g_0' in $G_{x_k} = G_{x_0'}$.

For each $x \in X$, we allow an empty surface trip $1_x \in G_x$, where $o(1_x) = x = t(1_x)$. Moreover, we can view t_{xy} as a surface trip $1_x t_{xy} 1_y$ from x to y, but we shall usually omit trivial group elements from our description of surface trips.

(5) Let '$\sim_s$' be the equivalence relation on surface trips, τ generated by the following relations:

(0) Identitity.

$$t_{xx} \sim_s 1_x$$

where $\{x\} \in \Phi$.

(I) Transitivity.

$$t_{xy}t_{yz} \sim_s t_{xz}$$

where $\{x, y\}, \{y, z\}, \{x, z\} \in \Phi$, and t_{xy}, t_{yz}, t_{xz} are viewed as paths from x to y, y to z, and x to z respectively.

(II) Fundamental Bass-Serre relation.

$$\rho_x^U(g)t_{xy} \sim_s t_{xy}\rho_y^U(g)$$

where $U \in \Phi$, $x, y \in U$, $g \in G_U$, and t_{xy} is viewed as a path from x to y.

We also require that $\sim_s$ is compatible with juxtaposition of surface trips; that is, if τ_1, τ_2, τ', τ'' are surface trips in X, and juxtapositions $\tau' \cdot \tau_1 \cdot \tau''$ and $\tau' \cdot \tau_2 \cdot \tau''$ are defined, then $\tau_1 \sim_s \tau_2$ if and only if

$$\tau' \cdot \tau_1 \cdot \tau'' \sim_s \tau' \cdot \tau_2 \cdot \tau''.$$

If τ is a surface trip in X, then $[\tau]_{\sim_s}$ denotes the $\sim_s$ - equivalence class of τ.

Choose a basepoint $x_0 \in X_\alpha$ for $\alpha \in V$, with $\{x_0\} \in \Phi$. The *group of surface trips* in X is defined to be

$$\mathcal{T}_\alpha(\mathcal{D}, x_0) = \{[\tau]_{\sim_s} \mid \tau \text{ is a surface trip in } X_\alpha \text{ from } x_0 \text{ to } x_0\}.$$

We observe that $\mathcal{T}_\alpha(\mathcal{D}, x_0)$ is a group with multiplication induced by juxtaposition of surface trips. The multiplication is naturally associative, and the existence of inverses follows from the fact that

$$\overline{\tau} \cdot \tau \sim_s \tau \cdot \overline{\tau} \sim_s 1_{x_0}$$

and for surface trips τ from x_0 to x_0 in X_α.

(6) We describe some natural conditions on Φ:

(Φ1) **One element subsets.**

If $U \in \Phi$ and $x \in U$, then $\{x\} \in \Phi$.

(Φ2) **Two element subsets.**

If $U \in \Phi$ and $x, y \in U$, then $\{x, y\} \in \Phi$.

(Φ3) **Connectivity.**

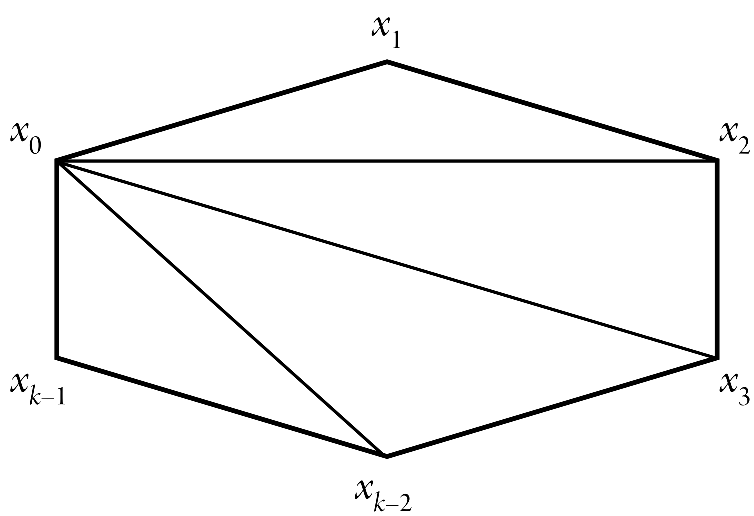
If $x, y \in X_\alpha$, and $\{x\}, \{y\} \in \Phi$ then there exists a sequence of the form

$$x = x_0,\ x_1,\ x_2,\ \ldots\ x_k = y$$

such that $\{x_i, x_{i+1}\} \in \Phi$ for $i = 0, \ldots, k = 1$.

(Φ4) **Convexity.**

Suppose $x_0,\ x_1,\ x_2,\ \ldots,\ x_k = x_0$ is a sequence in X_α such that $\{x_i, x_{i+1}\} \in \Phi$ for $i = 0, \ldots, k-1$ and $x_i \neq x_j$ for $0 \leq i, j \leq k-1$, $i \neq j$. Then $\{x_0, x_i\} \in \Phi$ for $i = 1, \ldots, k-1$.



Recall that X has local data

$$\mathcal{D} = ((X_\alpha)_{\alpha \in V}, (\phi_\beta)_{\beta \in E}, (G_U)_{U \in \Phi}, (\rho_U^V), (\lambda_{\beta,U}), (g_{\gamma,Z}))$$

and we have the direct limit of the multiplicity groups and restriction mappings:

$$\mathcal{G}(\alpha) = \varinjlim_{U \subseteq X_\alpha} (G_U, \rho_U^V)$$

as in 2.1.

For each $U \in \Phi$ there are canonical homomorphisms $\rho^U : G_U \longrightarrow G(\alpha)$ such that $\rho^U \cdot \rho_U^V = \rho^V$ whenever $U \subseteq V$; that is, the following diagram commutes:

$$\begin{array}{ccccc} & & G_V & & \\ & \overset{\rho_U^V}{\swarrow} & & \overset{\rho^V}{\searrow} & \\ G_U & & \overset{\rho^U}{\longrightarrow} & & \mathcal{G}(\alpha) \end{array}$$

(7) Proposition. *Let X be a set with local data $\mathcal{D}$. Suppose that Φ satisfies the conditions $\Phi 1$-$\Phi 4$ described above. Chose a basepoint $x_0 \in X_\alpha$ such that $\{x_0\} \in \Phi$ and let $\mathcal{T}_\alpha(\mathcal{D}, x_0)$ be the group of surface trips. Then there is a canonical isomorphism:*

$$\mathcal{G}(\alpha) = \varinjlim_{U \subseteq X_\alpha} (G_U, \rho_U^V) \cong \mathcal{T}_\alpha(\mathcal{D}, x_0).$$

Proof. To define $\Psi : \mathcal{G}(\alpha) \longrightarrow \mathcal{T}_\alpha(\mathcal{D}, x_0)$ we construct homomorphisms

$$\Psi^U : G_U \longrightarrow \mathcal{T}_\alpha(\mathcal{D}, x_0)$$

for each $U \in \Phi$ such that whenever $U \subseteq V$, $U, V \in \Phi$, the following diagram commutes:

$$\begin{array}{ccc} & G_V & \\ {\scriptstyle \rho_U^V}\swarrow & & \searrow{\scriptstyle \Psi^V} \\ G_U & \xrightarrow{\Psi^U} & \mathcal{T}_\alpha(\mathcal{D}, x_0) \end{array}$$

The universal property of direct limits will then imply the existence of a unique homomorphism

$$\Psi : \mathcal{G}(\alpha) \longrightarrow \mathcal{T}_\alpha(\mathcal{D}, x_0)$$

such that for each $U \in \Phi$, the following diagram commutes:

$$\begin{array}{ccc} & G_U & \\ {\scriptstyle \rho^U}\swarrow & & \searrow{\scriptstyle \Psi^U} \\ G(\alpha) & \xrightarrow{\Psi} & \mathcal{T}_\alpha(\mathcal{D}, x_0) \end{array}$$

To construct the maps Ψ^U, suppose $U \in \Phi$ and $g \in G_U$. Let $x \in U$. Then $\{x\} \in \Phi$ by condition ($\Phi 1$) and therefore $\rho_x^U(g) \in G_x$.

Then by conditions ($\Phi 1$) (one element subsets) and ($\Phi 3$) (connectivity) , there is a surface trip from the basepoint x_0 to x of the form

$$\tau = t_{x_0 x_1} t_{x_1 x_2} \ldots t_{x_{k-2} x_{k-1}} t_{x_{k-1} x}.$$

We set

$$\Psi^U(g) = [t_{x_0 x_1} \ldots t_{x_{k-1} x} \; \rho_x^U(g) \; t_{x x_{k-1}} \ldots t_{x_1 x_0}]_{\backsim_s}.$$

To show that $\Psi^U(g)$ is well defined, we need the following:

(8) Lemma. *Suppose the conditions of Proposition 3.7 are satisfied. Suppose*

$$\tau = t_{z_0 z_1} t_{z_1 z_2} \ldots t_{z_{m-1} z_0}$$

is a surface trip in X_α from z_0 to z_0. Then

$$t_{z_0 z_1} t_{z_1 z_2} \ldots t_{z_{m-1} z_0} \backsim_s 1_{z_0}.$$

Proof. We use induction on m. For $m = 1$,

$$t_{z_0 z_0} \backsim_s 1_{z_0}$$

by the identity axiom on paths.

For $m = 2$,

$$t_{z_0 z_1} t_{z_1 z_0} \backsim_s t_{z_0 z_0} \backsim_s 1_{z_0}$$

by the transitivity and identity axioms on paths.

Assume now that $m > 2$ and that for any $n < m$

$$t_{z'_0 z'_1} t_{z'_1 z'_2} \ldots t_{z'_{n-1} z'_0} \backsim_s 1_{z'_0}.$$

Let $\tau = t_{z_0 z_1} t_{z_1 z_2} \ldots t_{z_{m-1} z_0}$. There are two cases to consider. For the first case, assume that $z_i \neq z_j$ for $0 \leq i, j \leq m-1$, $i \neq j$.

Then by condition ($\Phi 4$) (convexity) we have $\{z_0, z_2\} \in \Phi$. By the transitivity axiom on paths we have $t_{z_0 z_1} t_{z_1 z_2} \backsim_s t_{z_0 z_2}$. Therefore,

$$\begin{aligned}\tau &= t_{z_0 z_1} t_{z_1 z_2} \ldots t_{z_{m-1} z_0} \\ &\backsim_s t_{z_0 z_2} t_{z_2 z_3} \ldots t_{z_{m-1} z_0} \\ &\backsim_s 1_{z_0} \text{ by the inductive hypothesis.}\end{aligned}$$

For the second case, suppose that $z_i = z_j$ for some $0 \leq i < j \leq m-1$. Then

$$t_{z_i z_{i+1}} \ldots t_{z_{j-1} z_j} \backsim_s 1_{z_i}$$

by the inductive hypothesis. So

$$\begin{aligned}\tau &= t_{z_0 z_1} \ldots t_{z_{i-1} z_i} t_{z_i z_{i+1}} \ldots t_{z_{j-1} z_j} t_{z_j z_{j+1}} \ldots t_{z_{m-1} z_0} \\ &\backsim_s t_{z_0 z_1} \ldots t_{z_{i-1} z_i} 1_{z_i} t_{z_i z_{j+1}} \ldots t_{z_{m-1} z_0} \\ &\backsim_s 1_{z_0} \text{ by the inductive hypothesis.}\end{aligned}$$

This completes the proof of Lemma 3.8.□

We continue the proof of Proposition 3.7. To show that $\Psi^U(g)$ is well defined, we need to check that the definition is independent of the choice of surface trip

$$t_{x_0 x_1} t_{x_1 x_2} \ldots t_{x_{k-2} x_{k-1}} t_{x_{k-1} x}$$

from the basepoint x_0 to x.

Let $t_{x_0 y_1} t_{y_1 y_2} \ldots t_{y_{m-1} x}$ be another surface trip from x_0 to x. We need to check that

$$[t_{x_0 x_1} \ldots t_{x_{k-1} x}\ \rho^U_x(g)\ t_{x x_{k-1}} \ldots t_{x_1 x_0}]_{\backsim_s} = [t_{x_0 y_1} \ldots t_{y_{m-1} x}\ \rho^U_x(g)\ t_{x y_{m-1}} \ldots t_{y_1 x_0}]_{\backsim_s}$$

or that

$$(9)\qquad [t_{xy_{m-1}} \dots t_{y_1x_0} t_{x_0x_1} \dots t_{x_{k-1}x}\ \rho_x^U(g)\ t_{xx_{k-1}} \dots t_{x_1x_0} t_{x_0y_1} \dots t_{y_{m-1}x}]_{\backsim_s} = [\rho_x^U(g)]_{\backsim_s}.$$

By Lemma 3.8,

$$t_{xy_{m-1}} \dots t_{y_1x_0} t_{x_0x_1} \dots t_{x_{k-1}x} \backsim_s 1_x,$$

$$t_{xx_{k-1}} \dots t_{x_1x_0} t_{x_0y_1} \dots t_{y_{m-1}x} \backsim_s 1_x,$$

which implies equation (9).

We also need to check that the definition of $\Psi^U(g)$ is independent of the choice of $x \in U$. Let $y \in U$. Then by condition ($\Phi 2$) (two element subsets) on Φ, we have $\{x, y\} \in \Phi$. Therefore $t_{x_0x_1} \dots t_{x_{k-1}x} t_{xy}$ is a surface trip from the basepoint x_0 to y.

Moreover

$$\begin{aligned} &[t_{x_0x_1} \dots t_{x_{k-1}x} t_{xy}\ \rho_y^U(g)\ t_{yx} t_{xx_{k-1}} \dots t_{x_1x_0}]_{\backsim_s} \\ = &[t_{x_0x_1} \dots t_{x_{k-1}x}\ \rho_x^U(g)\ t_{xy} t_{yx} t_{xx_{k-1}} \dots t_{x_1x_0}]_{\backsim_s} \\ &\text{by the fundamental Bass-Serre relation} \\ = &[t_{x_0x_1} \dots t_{x_{k-1}x}\ \rho_x^U(g)\ t_{xx_{k-1}} \dots t_{x_1x_0}]_{\backsim_s}, \end{aligned}$$

since $t_{xy} t_{yx} \backsim_s t_{xx} \backsim_s 1_x$.

So we have verified that $\Psi^U(g)$ is well defined. Thus we define

$$\Psi^U : G_U \longrightarrow \mathcal{T}_\alpha(\mathcal{D}, x_0)$$

by

$$g \longmapsto \Psi^U(g).$$

It is easy to see that Ψ^U is a homomorphism of groups.

We verify that the maps Ψ^U are compatible with restriction maps; that is, if $U \subseteq V$, $U, V \in \Phi$, then the following diagram commutes:

$$\begin{array}{ccc} & G_V & \\ {\scriptstyle \rho_U^V} \swarrow & & \searrow {\scriptstyle \Psi^V} \\ G_U & \overset{\Psi^U}{\longrightarrow} & \mathcal{T}_\alpha(\mathcal{D}, x_0) \end{array}$$

Let $g \in G_V$ and $x \in U$ so that $x \in V$. There is a surface trip

$$t_{x_0x_1} t_{x_1x_2} \dots t_{x_{k-2}x_{k-1}} t_{x_{k-1}x}$$

from x_0 to x. We have

$$\Psi^V(g) = [t_{x_0x_1} \dots t_{x_{k-1}x}\ \rho_x^V(g)\ t_{xx_{k-1}} \dots t_{x_1x_0}]_{\backsim_s}.$$

Moreover for $g' = \rho_U^V(g) \in G_U$, we have

$$\begin{aligned}\Psi^U(g') =&[t_{x_0x_1} \dots t_{x_{k-1}x}\ \rho_x^U(g')\ t_{xx_{k-1}} \dots t_{x_1x_0}]_{\backsim_s}\\ &[t_{x_0x_1} \dots t_{x_{k-1}x}\ \rho_x^U(\rho_U^V(g))\ t_{xx_{k-1}} \dots t_{x_1x_0}]_{\backsim_s}.\end{aligned}$$

By transitivity of restriction mappings, we have

$$\rho_x^U(\rho_U^V(g)) = \rho_x^V(g)$$

and therefore

$$\Psi^V(g) = \Psi^U(\rho_U^V(g))$$

for all $g \in G_V$.

Thus we have a collection of maps

$$(\Psi^U : G_U \longrightarrow \mathcal{T}_\alpha(\mathcal{D}, x_0))_{U\in\Phi}$$

that are compatible with restriction mappings. By the universal property of direct limits, there is a unique homomorphism

$$\Psi : \mathcal{G}(\alpha) = \varinjlim(G_U, \rho_U^V) \longrightarrow \mathcal{T}_\alpha(\mathcal{D}, x_0)$$

making the following diagram commute:

$$\begin{array}{ccc} & G_U & \\ \overset{\rho^U}{\swarrow} & & \overset{\Psi^U}{\searrow} \\ \mathcal{G}(\alpha) & \overset{\Psi}{\longrightarrow} & \mathcal{T}_\alpha(\mathcal{D}, x_0) \end{array}$$

We define

$$\eta : \mathcal{T}_\alpha(\mathcal{D}, x_0) \longrightarrow \mathcal{G}(\alpha)$$

by

$$\eta([g_0t_{x_0x_1}g_1 \dots g_{k-1}t_{x_{k-1}x_0}g_k]_{\backsim_s}) = \rho^{x_0}(g_0)\rho^{x_1}(g_1)\dots\rho^{x_{k-1}}(g_{k-1})\rho^{x_0}(g_k)$$

for a surface trip $g_0t_{x_0x_1}g_1 \dots g_{k-1}t_{x_{k-1}x_0}g_k$ from x_0 to x_0.

It is easy to see that η is well-defined; if τ_1 and τ_2 are surface trip that are $\backsim_s$-equivalent, then we can get from τ_1 to τ_2 by a finite sequence of the form

$$\tau'\rho_y^{yz}(g)t_{yz}\tau'' \backsim_s \tau' t_{yz}\rho_z^{yz}(g)\tau''$$

where $g \in G_{y,z}$.

We observe that

$$\rho^y(\rho_y^{yz}(g)) = \rho^{yz}(g) = \rho^z(\rho_z^{yz}(g)) \in \mathcal{G}(\alpha),$$

and therefore

$$\eta([\tau_1]_{\backsim_s}) = \eta([\tau_2]_{\backsim_s}).$$

It is also easy to see that $\eta : \mathcal{T}_\alpha(\mathcal{D}, x_0) \longrightarrow \mathcal{G}(\alpha)$ is a homomorphism of groups.

We now verify that

(i) $\Psi \circ \eta = Id_{\mathcal{T}_\alpha(\mathcal{D},x_0)}$
(ii) $\eta \circ \Psi = Id_{\mathcal{G}(\alpha)}$

For (i), let $\tau = g_0 t_{x_0x_1} g_1 \ldots g_{k-1} t_{x_{k-1}x_k} g_k$ be a surface trip from x_0 to $x_k = x_0$, with $g_i \in G_{x_i}$ for $i = 0, \ldots, k$. We observe that $t_{x_0x_1} t_{x_1x_2} \ldots t_{x_{i-1}x_i}$ is a surface trip from x_0 to x_i for $i = 0, \ldots, k$.

Therefore, by definition of Ψ;

$$\Psi(\rho^{x_i}(g_i)) = [t_{x_0x_1} \ldots t_{x_{i-1}x_i} g_i t_{x_ix_{i-1}} \ldots t_{x_1x_0}]_{\backsim_s}. \tag{10}$$

We also note that

$$t_{x_kx_{k-1}} t_{x_{k-1}x_{k-2}} \ldots t_{x_2x_1} t_{x_1x_0} \backsim_s 1_{x_0} \tag{11}$$

by Lemma 3.8, since $x_0 = x_k$.

We have

$$\begin{array}{c}
[\tau]_{\backsim_s} = [g_0 t_{x_0x_1} g_1 \ldots g_{k-1} t_{x_{k-1}x_k} g_k]_{\backsim_s} \\
\downarrow^{\eta} \\
\rho^{x_0}(g_0)\rho^{x_1}(g_1) \ldots \rho^{x_{k-1}}(g_{k-1})\rho^{x_0}(g_k) \\
\downarrow^{\Psi} \qquad \text{by } (10) \\
[g_0]_{\backsim_s}[t_{x_0x_1} g_1 t_{x_1x_0}]_{\backsim_s}[t_{x_0x_1} t_{x_1x_2} g_2 t_{x_2x_1} t_{x_1x_0}]_{\backsim_s} \ldots [t_{x_0x_1} \ldots t_{x_{k-1}x_k} g_k t_{x_kx_{k-1}} \ldots t_{x_1x_0}]_{\backsim_s} \\
\shortparallel \\
[g_0 t_{x_0x_1} g_1 \ldots g_{k-1} t_{x_{k-1}x_k} g_k t_{x_kx_{k-1}} \ldots t_{x_2x_1} t_{x_1x_0}]_{\backsim_s} \\
\shortparallel \qquad \text{by } (11) \\
[g_0 t_{x_0x_1} g_1 \ldots g_{k-1} t_{x_{k-1}x_k} g_k 1_{x_k}]_{\backsim_s} \\
\shortparallel \\
[g_0 t_{x_0x_1} g_1 \ldots g_{k-1} t_{x_{k-1}x_k} g_k]_{\backsim_s} = [\tau]_{\backsim_s},
\end{array}$$

thus we have verified that $\Psi \circ \eta = Id_{\mathcal{T}_\alpha(\mathcal{D}, x_0)}$.

For (ii), we observe that $\mathcal{G}(\alpha)$ is generated by $\bigcup\limits_{\{x\} \in \Phi} \rho^x(G_x)$ and so it suffices to verify that

$$\eta \circ \Psi \mid_{\rho^x(G_x)} = Id_{\rho^x(G_x)}$$

for each $\{x\} \in \Phi$.

Let $\{x\} \in \Phi$, $g \in G_x$ and let $t_{x_0x_1} \ldots t_{x_{k-1}x}$ be a surface trip from the basepoint x_0 to x. Then

$$\begin{aligned}
\Psi(\rho^x(g)) &= [t_{x_0x_1} \ldots t_{x_{k-1}x} \rho^x_x(g) t_{xx_{k-1}} \ldots t_{x_1x_0}]_{\backsim_s} \\
&= [t_{x_0x_1} \ldots t_{x_{k-1}x} g t_{xx_{k-1}} \ldots t_{x_1x_0}]_{\backsim_s}.
\end{aligned}$$

Thus $\eta(\Psi(\rho^x(g))) = \rho^x(g)$ and so we have verified (ii), and this completes the proof of Proposition 3.7.□

4. EMBEDDING THEOREMS.

As we have seen, an action of a group on a set naturally gives rise to a local pseudogroup of partial isometries. In this section, we consider the following question:

(1) Question. *Given a local pseudogroup of partial isometries* (Ψ, Y) *consisting of a family of isometries* ϕ_β *between subsets* Y_β *and* $\widetilde{Y_\beta}$ *of* Y*, when does it arise from an action* $H \times Z \longrightarrow Z$ *of a group* H *on a set* Z*?*

Let $H \times Z \longrightarrow Z$ be an action of a group H on a set Z. We recall that the action is *free* if no $1 \neq h \in H$ has a fixed point.

(2) Definition. *Give a local pseudogroup* (Ψ, Y) *of partial isometries, we say that* (Ψ, Y) *embeds in a free action if there is a free action* $H \times Z \longrightarrow Z$ *of a group* H *on a set* Z *and mappings* (λ, μ) *such that the diagram*

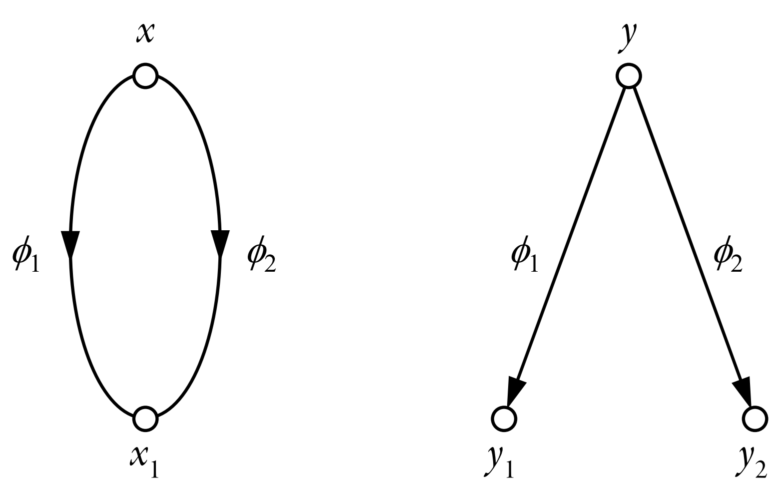
$$\begin{array}{ccccc} \Psi & \times & Y & \longrightarrow & Y \\ \downarrow_\lambda & & \downarrow_\mu & & \downarrow_\mu \\ H & \times & Z & \longrightarrow & Z \end{array}$$

commutes, and μ *is injective.*

We shall give necessary and sufficient conditions for embedding a local pseudogroup (Ψ, Y) of partial isometries in a free action.
(3) In general, we seek $H \times Z \longrightarrow Z$ and (λ, μ) such that the map λ takes a composition $\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k$ of partial mappings in Ψ to a group product $\lambda(\phi_1) \cdot \lambda(\phi_2) \cdot \ldots \cdot \lambda(\phi_k)$ in H. Moreover, the map μ should have the property that for each $\phi_\gamma = \phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k \in \Psi$, and for every $y \in Y$, we have:

$$\lambda((\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k))(\mu(y)) \quad = \quad \mu((\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k)(y)).$$

The following example indicates that for a general free action, we cannot always embed a local pseudogroup of partial isometries.



(4) Example.

Let (Ψ, Y) be the following local pseudogroup with $y_1 \neq y_2$:

Assume that we have a free action $H \times Z \longrightarrow Z$ and mappings (λ, μ) such that the following diagram commutes:

$$\begin{array}{ccccc} \Psi & \times & Y & \longrightarrow & Y \\ \downarrow_{\lambda} & & \downarrow_{\mu} & & \downarrow_{\mu} \\ H & \times & Z & \longrightarrow & Z \end{array}$$

For any mappings (λ, μ) we have:

$$\begin{array}{ccccc} \lambda(\phi_1)(\mu(x)) & = & \mu(\phi_1(x)) & = & \mu(x_1), \\ \lambda(\phi_2)(\mu(x)) & = & \mu(\phi_2(x)) & = & \mu(x_1) \end{array}$$

and since $H \times Z \longrightarrow Z$ is free, we must have:

$$\lambda(\phi_1) \quad = \quad \lambda(\phi_2).$$

Therefore

$$\begin{array}{ccccc} \lambda(\phi_1)(\mu(y)) & = & \mu(\phi_1(y)) & = & \mu(y_1), \\ \lambda(\phi_2)(\mu(y)) & = & \mu(\phi_2(y)) & = & \mu(y_2) \end{array}$$

implies

$$\mu(y_1) \quad = \quad \mu(y_2),$$

and so μ is not injective. □

Example 6.4 suggests the following necessary condition for embedding a local pseudogroup (Ψ, Y) in a free action $H \times Z \longrightarrow Z$:

(5) Condition (C1). *Let*

$$\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k \in \Psi.$$

If

$$\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k(x) \quad = \quad x, \quad \textit{for some } x \in Y,$$

and $\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k(y)$ *is defined for* $y \in Y$*, then*

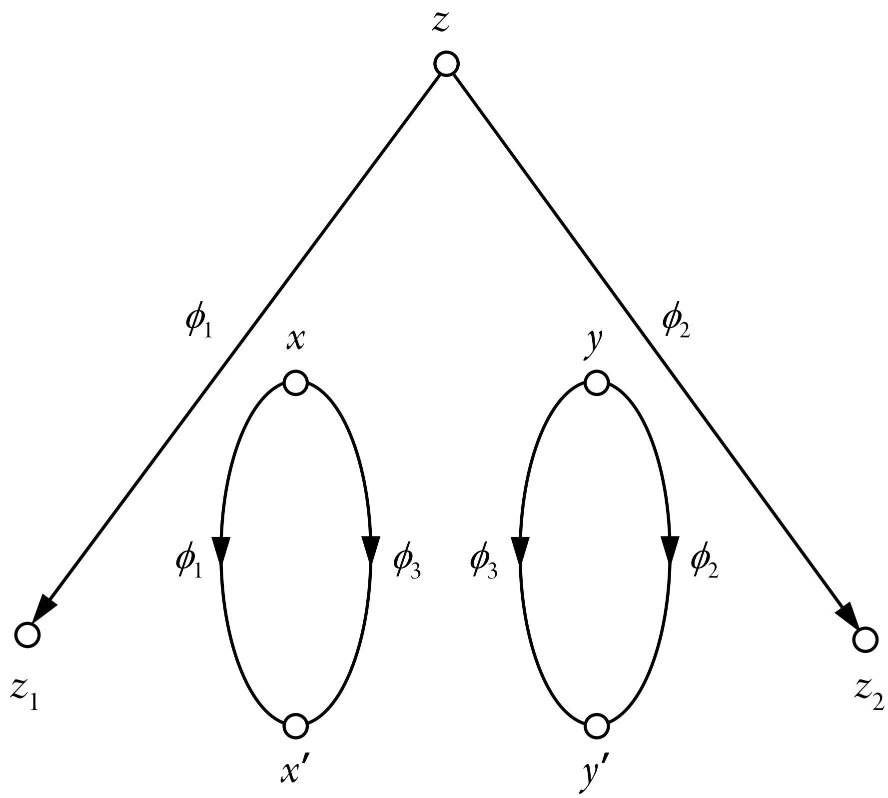
$$\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k(y) \quad = \quad y.$$

Condition C1 states that if a composition $\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k$ of partial mappings has a fixed point, then $\phi_1 \cdot \phi_2 \cdot \ldots \cdot \phi_k$ should act as the identity on its domain of definition.

The following example demonstrates that while C1 is necessary, it is not sufficient for embedding a local pseudogroup (Ψ, Y) in a free action $H \times Z \longrightarrow Z$:

(6) Example.

Let (Ψ, Y) be the following local pseudogroup with $z_1 \neq z_2$:



Then condition C1 is automatically satisfied. Suppose that we have an embedding

$$\begin{array}{ccccc} \Psi & \times & Y & \longrightarrow & Y \\ \downarrow_{\lambda} & & \downarrow_{\mu} & & \downarrow_{\mu} \\ H & \times & Z & \longrightarrow & Z \end{array}$$

of (Ψ, Y) into a free action $H \times Z \longrightarrow Z$. Since $H \times Z \longrightarrow Z$ is free, for any mappings (λ, μ) we have:

$$\lambda(\phi_1) \quad = \quad \lambda(\phi_2) = \quad \lambda(\phi_3).$$

Thus

$$\begin{aligned} \lambda(\phi_1)(\mu(z)) &= \mu((z_1)), \\ \lambda(\phi_2)(\mu(z)) &= \mu((z_2)) \end{aligned}$$

implies

$$\mu(z_1) \quad = \quad \mu(z_2),$$

and so μ is not injective. □

(6) We modify condition (C1) to give a condition that is both necessary and sufficient. Let $\mathcal{D}$ be data on a set X with a local pseudogroup of partial isometries

$$(\Psi, X) \quad = \quad \{\phi_\beta : Y_\beta \longrightarrow \widetilde{Y_\beta}\}$$

and assume that (Ψ, X) contains

$$\{\phi_\beta^{-1} : \widetilde{Y_\beta} \longrightarrow Y_\beta\}.$$

Assume that all multiplicity groups of $\mathcal{D}$ are trivial. Then $\mathcal{D}$ essentially consists of (Ψ, X) alone. Assume also that the graph Δ associated to $\mathcal{D}$ has a single vertex α and that $X = X_\alpha$. Let $\mathcal{G}(\mathcal{D})$ be the corresponding group. Then the presentation of $\mathcal{G}(\mathcal{D})$ takes the following form:

$$\mathcal{G}(\mathcal{D}) = \langle (f_\beta)_{\beta\in I} \mid f_\gamma := f_{\beta_1}^{\epsilon_1} f_{\beta_2}^{\epsilon_2} \dots f_{\beta_k}^{\epsilon_k} = 1 \text{ whenever } \phi_\gamma(x) := \phi_{\beta_1}^{\epsilon_1}\phi_{\beta_2}^{\epsilon_2}\dots\phi_{\beta_k}^{\epsilon_k}(x) = x, \\ \text{for some } x \in X,\ \epsilon_i = \pm 1\rangle.$$

(7) Condition (C2). *We say that $\mathcal{D}$ satisfies condition (C2) if for any $f_\gamma := f_{\beta_1}^{\epsilon_1} f_{\beta_2}^{\epsilon_2} \dots f_{\beta_\ell}^{\epsilon_\ell}$ such that $f_\gamma =_{\mathcal{G}(\mathcal{D})} 1$, $\phi_\gamma(x) := \phi_{\beta_1}^{\epsilon_1}\phi_{\beta_2}^{\epsilon_2}\dots\phi_{\beta_\ell}^{\epsilon_\ell}(x)$ defined at $x \in X$ implies that $\phi_\gamma(x) = x$.*

(8) Lemma. *Let $\mathcal{D}$ be data on a set X with a local pseudogroup of partial isometries (Ψ, X) and trivial multiplicity groups. Assume that the graph associated with $\mathcal{D}$ has a single vertex denoted α and that $X = X_\alpha$. Let $\mathcal{G}(\mathcal{D})$ be the group and $\mathcal{X}(\mathcal{D})$ the space corresponding to $\mathcal{D}$. Let*

$$\begin{aligned} \rho : X \quad &\longrightarrow \quad \mathcal{X}(\mathcal{D}) \\ x \quad &\mapsto \quad (1_{\mathcal{G}(\mathcal{D})}, x)_{\approx_\mathcal{M}} \end{aligned}$$

be the canoncial map. If condition (C2) is satisfied for $\mathcal{D}$ then the map ρ is injective.

Proof. Suppose that $\rho(x) = \rho(y)$ for some $x, y \in X$, that is

$$(1_{\mathcal{G}(\mathcal{D})}, x)_{\approx_\mathcal{M}} \quad = \quad (1_{\mathcal{G}(\mathcal{D})}, y)_{\approx_\mathcal{M}}.$$

By the definition of $\mathcal{X}(\mathcal{D})$ and since all multiplicity groups of $\mathcal{D}$ are trivial, only axiom $(2)_\mathcal{M}$ of Section (2.4) generates $\approx_\mathcal{M}$. Hence there exists a composition $\phi_\gamma = \phi_{\beta_1}^{\epsilon_1}\phi_{\beta_2}^{\epsilon_2}\dots\phi_{\beta_k}^{\epsilon_k} \in \Psi$ such that $\phi_\gamma(x) = y$ and the element f_γ corresponding to ϕ_γ equals 1 in $\mathcal{G}(\mathcal{D})$:

$$f_\gamma \quad := \quad f_{\beta_1}^{\epsilon_1} f_{\beta_2}^{\epsilon_2} \dots f_{\beta_k}^{\epsilon_k} \quad =_{\mathcal{G}(\mathcal{D})} \quad 1.$$

Since condition (C2) is satisfied, it follows that ϕ_γ is the identity on its domain of definition. Hence $x = y$. $\square$

Suppose now that we are in the setting of Lemma (4.8) above. For $n = 1, 2, \dots,$ let B_n denote the ball of radius n in $\mathcal{G}(\mathcal{D})$:

$$B_n \quad := \quad \{f_\gamma := f_{\beta_1}^{\epsilon_1} f_{\beta_2}^{\epsilon_2} \dots f_{\beta_k}^{\epsilon_k} \in \mathcal{G}(\mathcal{D}) \mid k \leq n\}.$$

Then B_n is a finite subset of $\mathcal{G}(\mathcal{D})$ since Δ is locally finite. Let $X_n \subseteq \mathcal{X}(\mathcal{D})$ be defined as follows:

$$X_n \quad = \quad \bigcup_{g \in B_n, x \in X} (g, x)_{\approx}.$$

Then $(1, X)_{\approx} \subseteq X_n$. Moreover, by Lemma (4.8) we can identify $(1, X)_{\approx}$ with itself. The ball B_n has a partial action on X_n.

(9) We define data $\mathcal{D}_n$ on X_n. For fixed n, we define all the multiplicity groups of $\mathcal{D}_n$ to be trivial. We define the graph Δ_n associated to $\mathcal{D}_n$ to have a single vertex denoted α_n, and we set $X_n = X_{\alpha_n}$. We define a local pseudogroup ψ_n of $\mathcal{D}_n$ as follows:

$$\psi_n \quad := \quad \{\phi_{\beta,n} : Y_{\beta,n} \longrightarrow \widetilde{Y_{\beta,n}}\} \quad \cup \quad \{\phi_{\beta,n}^{-1} : \widetilde{Y_{\beta,n}} \longrightarrow Y_{\beta,n}\},$$

where

$$\widetilde{Y_{\beta,n}} \quad := \quad X_n \cap f_\beta X_n \quad \subseteq \quad X_n$$
$$Y_{\beta,n} \quad := \quad (f_\beta^{-1})\widetilde{Y_{\beta,n}} \quad \subseteq \quad X_n$$

and $f_\beta \in \mathcal{G}(\mathcal{D})$ acts on $\mathcal{X}(\mathcal{D})$. Then

$$\phi_{\beta,n} \quad := \quad f_\beta \mid_{Y_{\beta,n}} .$$

Let $\mathcal{G}(\mathcal{D}_n)$ be the group and $\mathcal{X}(\mathcal{D}_n)$ the space corresponding to the data $\mathcal{D}_n$.

(10) Lemma. *For each $n = 1, 2, \dots$ we have*

1. *Condition (C2) is satisfied for $\mathcal{D}_n$.*
2. *We have group isomorphisms*

   $$\mu : \mathcal{G}(\mathcal{D}) \longrightarrow \mathcal{G}(\mathcal{D}_n)$$

   given by

   $$f_\beta \mapsto f_{\beta,n}$$

   where $(f_\beta)_{\beta \in I}$ generates $\mathcal{G}(\mathcal{D})$ and $(f_{\beta,n})_{\beta \in I}$ generates $\mathcal{G}(\mathcal{D}_n)$.
3. *We have canonical bijections*

   $$\nu : \mathcal{X}(\mathcal{D}) \longrightarrow \mathcal{X}(\mathcal{D}_n)$$

   given by

   $$(g, x)_{\approx} \mapsto (\mu(g), x)_{\approx}$$

   where $g \in \mathcal{G}(\mathcal{D})$, $x \in X \subseteq X_n$ and $\mu(g) \in \mathcal{G}(\mathcal{D}_n)$.
4. *The following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{G}(\mathcal{D}) & \times & \mathcal{X}(\mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{D}) \\ \downarrow\mu & & \downarrow\nu & & \downarrow\nu \\ \mathcal{G}(\mathcal{D}_n) & \times & \mathcal{X}(\mathcal{D}_n) & \longrightarrow & \mathcal{X}(\mathcal{D}_n) \end{array}$$

We remark that when $n = 1$, $\mathcal{D}_1 = \mathcal{D}$ by Lemma (4.8) above.

(11) Theorem. *Let $\mathcal{D}$ be data on a set X with a local pseudogroup of partial isometries (Ψ, X) and trivial multiplicity groups. Assume that the graph associated with $\mathcal{D}$ has a single vertex denoted α and that $X = X_\alpha$. Then there exists an embedding of (Ψ, X) in a free action $H \times Z \longrightarrow Z$ if and only if condition (C2) is satisfied for $\mathcal{D}$.*

Proof. It is easy to see that condition (C2) is necessary for embedding(Ψ, X) in a free action. To see that condition (C2) is sufficient, assume that (C2) is satisfied for $\mathcal{D}$. Let $\mathcal{G}(\mathcal{D})$ be the group and $\mathcal{X}(\mathcal{D})$ the space corresponding to $\mathcal{D}$. By Lemma (4.8) above we have an embedding

$$X \quad \hookrightarrow \quad \mathcal{X}(\mathcal{D}).$$

It suffices to show that the action of $\mathcal{G}(\mathcal{D})$ on $\mathcal{X}(\mathcal{D})$ is free. Suppose conversely that the action is not free. Then there exists $x \in X$ such that $(1, x)_\approx$ has non-trivial stabilizers in $\mathcal{G}(\mathcal{D})$. Hence there exists

$$f_\gamma \quad := \quad f_{\beta_1}^{\epsilon_1} f_{\beta_2}^{\epsilon_2} \dots f_{\beta_{n_0}}^{\epsilon_{n_0}} \quad \in \quad \mathcal{G}(\mathcal{D})$$

such that $f_\gamma \neq_{\mathcal{G}(\mathcal{D})} 1$ and

(12) $$f_\gamma \cdot (1, x)_\approx \quad = \quad (1, x)_\approx.$$

Then

(13) $$f_\gamma \cdot (1, x)_\approx \quad = \quad (f_\gamma, x)_\approx \quad = \quad (1, x)_\approx.$$

It follows that in $\mathcal{D}$ the composition

(14) $$\phi_\gamma \quad := \quad \phi_{\beta_1}^{\epsilon_1} \phi_{\beta_2}^{\epsilon_2} \dots \phi_{\beta_{n_0}}^{\epsilon_{n_0}}$$

has non-trivial domain of definition containing $x \in (1, X)_\approx$. Then

(15) $$(1, x)_\approx \quad = \quad (f_\gamma, x)_\approx \quad = \quad (f_{\gamma,n}, x)_\approx \quad = \quad (1, \phi_{\gamma,n}(x))_\approx,$$

where $f_{\gamma,n} \in \mathcal{G}(\mathcal{D}_n)$ and $\phi_{\gamma,n} \in \mathcal{X}(\mathcal{D}_n)$. By Lemma (4.10) (C2) is satisfied for $\mathcal{D}_n$ and

$$X_n \quad \hookrightarrow \quad \mathcal{X}(\mathcal{D}_n)$$

is injective, and hence

$$X \quad \hookrightarrow \quad \mathcal{X}(\mathcal{D}_n)$$
$$x \mapsto (1, x)_\approx$$

is injective. Hence $(1, \phi_{\gamma,n}(x))_\approx = (1, x)_\approx$ implies that $x = \phi_{\gamma,n}(x)$. Hence $\phi_{\gamma,n}$ admits monodromy at the point $x \in X \subseteq X_n$ and hence by definition of $\mathcal{G}(\mathcal{D}_n) \cong \mathcal{G}(\mathcal{D})$ we have

(16) $$f_{\gamma,n} \quad =_{\mathcal{G}(\mathcal{D}_n)} \quad 1,$$

and hence

(17) $$f_\gamma \quad =_{\mathcal{G}(\mathcal{D})} \quad 1$$

which is a contradiction. □

Section II.
APPLICATIONS OF THE STRUCTURE THEOREMS

5. RECONSTRUCTING GROUP ACTIONS.

5.1 Approximating group actions.

We recall the setting of 1.5 where a group G acts non-trivially on a set X.
(1) Given a family $(X_\alpha)_{\alpha\in V}$ of subsets of X, and a family $(g_k)_{k\in K}$ of elements of G, we described a natural local data

$$\mathcal{D} = ((X_\alpha)_{\alpha\in V}, (\phi_\beta)_{\beta\in E}, (G_Z)_{Z\in\Phi}, (\rho_Z^{Z'}), (\lambda_{\beta,Z}), (g_{\gamma,Z}))$$

on $\bigsqcup_{\alpha\in V} X_\alpha$ arising from this group action. If we fix a basepoint $\alpha_0 \in V$, we can construct the monodromy groupoid $\mathcal{M}(\mathcal{D})$ and corresponding covering space $\widetilde{\mathcal{M}(\mathcal{D},\alpha_0)}$ with natural action

$$\pi_1(\mathcal{M}(\mathcal{D}),\alpha_0) \times \widetilde{\mathcal{M}(\mathcal{D},\alpha_0)} \longrightarrow \widetilde{\mathcal{M}(\mathcal{D},\alpha_0)}$$

as in 2.3 and 2.4. In this section, we show that this action approximates the original action of G on X.

(2) Theorem. *Let G be a group acting non-trivially on a set X. Let $\mathcal{D}$ be any choice of local data for the action of G on X (in the sense of Section (1.5)). There is a canonical homomorphism*

$$\mu : \pi_1(\mathcal{M}(\mathcal{D}),\alpha_0) \longrightarrow G$$

and canonical set map

$$\nu : \widetilde{\mathcal{M}(\mathcal{D},\alpha_0)} \longrightarrow X$$

such that the following diagram commutes:

$$\begin{array}{ccccc} \pi_1(\mathcal{M}(\mathcal{D}),\alpha_0) & \times & \widetilde{\mathcal{M}(\mathcal{D},\alpha_0)} & \longrightarrow & \widetilde{\mathcal{M}(\mathcal{D},\alpha_0)} \\ \downarrow\mu & & \downarrow\nu & & \downarrow\nu \\ G & \times & X & \longrightarrow & X \end{array}$$

that is;

$$\nu(\sigma \cdot y) = \mu(\sigma)\cdot\nu(y)$$

for $\sigma \in Mor_{\mathcal{M}}(\alpha_0,\alpha_0)$, $y \in \widetilde{\mathcal{M}(\mathcal{D},\alpha_0)}$.

Proof. We will define a homomorphism:

$$\mu^* : \mathcal{M}(\mathcal{D},\alpha_0) \longrightarrow G$$

and then take μ to be the restriction of μ^* to $\pi_1(\mathcal{M}(\mathcal{D}),\alpha_0)$.

(3) Define μ^* on generators of

$$\mathcal{M}(\mathcal{D}) = \langle (\mathcal{G}(\alpha))_{\alpha \in V\Delta}, (\beta)_{\beta \in E\Delta} \mid \beta \cdot w_\beta(g) \cdot \beta^{-1} = \omega_{\overline{\beta}}(g) \text{ for every } g \in \mathcal{G}(\beta), \\ \beta\overline{\beta} = 1_{o(\beta)}, \overline{\beta}\beta = 1_{t(\beta)}, \gamma = \overline{g_{\gamma,Z}} \text{ for every } \gamma \text{ with } \phi_{\gamma,Z} = Id_Z \rangle$$

as follows. For each $\alpha \in V\Delta$, we have

$$\mathcal{G}(\alpha) = \varinjlim_{Z \subseteq X_\alpha} (G_Z, \rho_Z^{Z'}).$$

Moreover, for each $Z \subseteq X_\alpha$, we have $G_Z \leq G$, and whenever $Z \subseteq Z' \subseteq X_\alpha$, there is a natural inclusion

$$G_{Z'} \leq G_Z \leq G.$$

(4) Therefore, there exists a canonical homomorphism:

$$\Psi : \mathcal{G}(\alpha) = \varinjlim_{Z \subseteq X_\alpha} (G_Z, \rho_Z^{Z'}) \longrightarrow G$$

such that for every $g \in G_Z$ and for every $Z \subseteq X_\alpha$, the homomorphism Ψ takes the image $\overline{g}$ of g in $\mathcal{G}(\alpha)$ to g; that is, $\Psi(\overline{g}) = g$.
(5) We define μ^* on the generators $\mathcal{G}(\alpha)$ of $\mathcal{M}(\mathcal{D})$ by:

$$\mu^* \mid_{\mathcal{G}(\alpha)} = \Psi.$$

For each $\beta = (\alpha, k, \alpha') \in E^+\Delta$, we set:

$$\mu^*(\beta) = g_k \text{ and } \mu^*(\overline{\beta}) = g_k^{-1}.$$

(6) We check that μ^* preserves the relations of $\mathcal{M}(\mathcal{D})$. For $\beta = (\alpha, k, \alpha') \in E^+\Delta$, $Z \subseteq Y_\beta \subseteq X_\alpha$ and $g \in G_Z$, we have the the following equality in $\mathcal{M}(\mathcal{D})$:

$$\beta \cdot \overline{g} \cdot \beta^{-1} = \overline{\lambda_{\beta,Z}(g)} = \overline{g_k \cdot g \cdot g_k^{-1}}.$$

(7) Therefore

$$\begin{aligned} \mu^*(\beta \cdot \overline{g} \cdot \beta^{-1}) &= \mu^*(g_k)\mu^*(g)\mu^*(g_k^{-1}) \\ &= g_k \cdot g \cdot g_k^{-1} \\ &= \mu^*(\overline{g_k \cdot g \cdot g_k^{-1}}). \end{aligned}$$

It is also clear that for $\beta \in E^+\Delta$, the relations $\beta\overline{\beta} = 1_{o(\beta)}$, $\overline{\beta}\beta = 1_{t(\beta)}$ of $\mathcal{M}(\mathcal{D})$ are preserved by μ^*.
(8) Suppose now that for some Z, there is a composition of partial mappings:

$$\phi_\gamma = \phi_{\beta_1} \circ \cdots \circ \phi_{\beta_t}$$

that acts identically on Z: that is; $\phi_{\beta_1} \circ \cdots \circ \phi_{\beta_t} |_Z = Id_Z$, where $\beta_i = (\alpha_i, k_i, \alpha_{i-1})$, $i = 1, \ldots, t$ with $\alpha_0 = \alpha_t$. In this situation, the monodromy element $g_{\gamma,Z} \in G_Z$ is the product $g_1 \ldots g_t \in G_Z$.
(9) In $\mathcal{M}(\mathcal{D})$, we have the following monodromy relation:

$$\beta_1 \ldots \beta_t = \overline{g_1 \ldots g_t}.$$

We have to verify that μ^* preserves this relation. We have:

$$\mu^*(\beta_1 \ldots \beta_t) = g_1 \ldots g_t = \mu^*(\overline{g_1 \ldots g_t}).$$

Thus, we have verified that $\mu^* : \mathcal{M}(\mathcal{D}, \alpha_0) \longrightarrow G$ is a homomorphism.
(10) We set $\mu : \pi_1(\mathcal{M}(\mathcal{D}), \alpha_0) \longrightarrow G$ to be $\mu^* |_{\pi_1(\mathcal{M}(\mathcal{D}), \alpha_0)}$.
We define the map

$$\nu : \widetilde{\mathcal{M}(\mathcal{D}, \alpha_0)} \longrightarrow X$$

as follows:

$$\nu : (\sigma, x)_{\approx_\mathcal{M}} \mapsto \mu(\sigma) \cdot x$$

where $\sigma \in Mor_{\mathcal{M}(\mathcal{D})}(\alpha_0, \alpha)$ and $x \in X_\alpha$.
(11) To check that ν is well-defined, we need to verify that ν is independent of the choice of (σ, x) in $(\sigma, x)_{\approx_\mathcal{M}}$; that is, we need to verify

(1) $\mu(\sigma) \cdot x = \mu(\sigma \cdot g) \cdot x$, for $g \in G_x$
(2) $\mu(\sigma\beta) \cdot x = \mu(\sigma) \cdot \phi_\beta(x)$, for $\beta = (\alpha, k, \alpha') \in E^+\Delta$.

(12) We have that

$$\begin{aligned} \mu(\sigma \cdot g) \cdot x &= \mu(\sigma) \cdot \mu(g) \cdot x \\ &= \mu(\sigma) \cdot g \cdot x \\ &= \mu(\sigma) \cdot x \end{aligned}$$

and

$$\begin{aligned} \mu(\sigma \cdot \beta) \cdot x &= \mu(\sigma) \cdot \mu(\beta) \cdot x \\ &= \mu(\sigma) \cdot g_k \cdot x \\ &= \mu(\sigma) \cdot \phi_\beta(x), \end{aligned}$$

so we have verified (1) and (2).
(13) Finally, we observe that for $\sigma' \in \pi_1(\mathcal{M}(\mathcal{D}), \alpha_0)$ and $(\sigma, x)_{\approx_\mathcal{M}} \in \widetilde{\mathcal{M}(\mathcal{D}, \alpha_0)}$:

$$\begin{aligned} \nu(\sigma' \cdot (\sigma, x)_{\approx_\mathcal{M}}) &= \nu((\sigma' \cdot \sigma, x)_{\approx_\mathcal{M}}) \\ &= \mu(\sigma' \cdot \sigma) \cdot x \\ &= \mu(\sigma')\mu(\sigma) \cdot x \\ &= \mu(\sigma') \cdot \nu((\sigma, x)_{\approx_\mathcal{M}}), \end{aligned}$$

that is; the diagram

$$\begin{array}{ccccc} \pi_1(\mathcal{M}(\mathcal{D}), \alpha_0) & \times & \widetilde{\mathcal{M}(\mathcal{D}, \alpha_0)} & \longrightarrow & \widetilde{\mathcal{M}(\mathcal{D}, \alpha_0)} \\ \downarrow_\mu & & \downarrow_\nu & & \downarrow_\nu \\ G & \times & X & \longrightarrow & X \end{array}$$

commutes. □

(14) Remarks.

(1) If further $\mathcal{D}$ in Theorem (5.1.2) is 'complete' in the sense of Subsection (5.2), then the map μ is a group isomorphism, and ν is a set bijection. Moreover a complete set of local data $\mathcal{D}$ always exists for the action of G on X (Subsection (5.2)).

(2) If $\mathcal{D}$ is local data from a non-trivial action of a group G on a set X, then there are canonical maps

$$\begin{array}{ccc} X_\alpha & \longrightarrow & \mathcal{X}(\mathcal{D}) \\ x & \mapsto & (1,x)_\approx \end{array}$$

$$G_Z \longrightarrow \mathcal{G}(\mathcal{D})$$

which are embeddings.

5.2 Existence and uniqueness of data from a group action.

(1) Let G be a group acting non-trivially on a set X. Let $\mathcal{D}$ be local data for the action of G on X (in the sense of Section (1.5)). Let $\mathcal{G}(\mathcal{D})$ be the group and $\mathcal{X}(\mathcal{D})$ the space for $\mathcal{D}$ respectively as in Section 4. We say that $\mathcal{D}$ is *complete* for $G \times X \longrightarrow X$ if
(i) the map $\mu : \mathcal{G}(\mathcal{D}) \longrightarrow G$ is a group isomorphism,
(ii) the map $\nu : \mathcal{X}(\mathcal{D}) \longrightarrow X$ is a set bijection, and
(iii) the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{G}(\mathcal{D}) & \times & \mathcal{X}(\mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{D}) \\ \downarrow_\mu & & \downarrow_\nu & & \downarrow_\nu \\ G & \times & X & \longrightarrow & X \end{array}$$

Let $\mathcal{D}$ be a complete set of data for an action $G \times X \longrightarrow X$. Then

$$G \cdot W \quad = \quad X$$

where

$$W \quad = \quad \bigsqcup_{\alpha \in V} X_\alpha.$$

In this way, we can view a complete set of data as containing an approximation to a fundamental domain for a group action $G \times X \longrightarrow X$.
(2) Let $\mathcal{D}'$ be another set of local data on $G \times X \longrightarrow X$. We say that $\mathcal{D}'$ is *equivalent* to $\mathcal{D}$ if
(i) there is a group isomorphism $\mu : \mathcal{G}(\mathcal{D}) \longrightarrow \mathcal{G}(\mathcal{D}')$,
(ii) there is a set bijection $\nu : \mathcal{X}(\mathcal{D}) \longrightarrow \mathcal{X}(\mathcal{D}')$, and
(iii) the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{G}(\mathcal{D}) & \times & \mathcal{X}(\mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{D}) \\ \downarrow_\mu & & \downarrow_\nu & & \downarrow_\nu \\ \mathcal{G}(\mathcal{D}') & \times & \mathcal{X}(\mathcal{D}') & \longrightarrow & \mathcal{X}(\mathcal{D}') \end{array}$$

(3) Let $\mathcal{D}$ be local data from a group action, where

$$\mathcal{D} = ((X_\alpha)_{\alpha\in V}, (\phi_\beta)_{\beta\in E}, (G_Z)_{Z\in\Phi}, (\rho_Z^{Z'}), (\lambda_{\beta,Z}), (g_{\gamma,Z}))$$

Assume that the index sets V and E are *finite*. (This occurs in particular if G is finitely generated and if the local pseudogroup generators ϕ_β imitate the action of the generators of G). We say that

$$\mathcal{D}' = ((X'_{\alpha'})_{\alpha'\in V'}, (\phi'_{\beta'})_{\beta'\in E'}, (G'_{Z'})_{Z'\in\Phi'}, ({\rho'_{Z'}}^{Z''}), (\lambda'_{\beta',Z'}), (g'_{\gamma',Z'}))$$

is an *enlargement* of $\mathcal{D}$ if $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a sequence of moves of the following type:
E1) Let $(\phi_\gamma)_{\gamma\in\Omega}$ be a family of elements of the local pseudogroup Ψ, where $\gamma = \beta_1\beta_2\dots\beta_k$ is a path in the graph Δ and the corresponding map ϕ_γ has non-empty domain. We set

$$E' \quad = \quad E \cup \Omega.$$

We take $V' = V$, $\Phi' = \Phi$, $G'_{Z'} = G_Z$, ${\rho'_{Z'}}^{Z''} = \rho_Z^{Z'}$. (Since $\Phi' = \Phi$ this reduces to $G'_Z = G_Z$, ${\rho'_Z}^{Z'} = \rho_Z^{Z'}$.) We define $\lambda'_{\beta',Z'}$ and $g'_{\gamma',Z'}$ as follows. Suppose that $\gamma' = \beta'_1\beta'_2\dots\beta'_k$ is a path in Δ' such that

$$\phi_{\gamma'}\mid_Z \quad = \quad Id_Z.$$

We rewrite γ' as follows. If $\beta'_j \in E$ then we make no change to γ'. If $\beta'_j \in \Omega$ then we replace β'_j by its corresponding path in Δ. Then γ' also corresponds to a path γ in Δ, and

$$\phi_\gamma\mid_Z \quad = \quad Id_Z.$$

We take $g'_{\gamma',Z'} = g_{\gamma,Z}$. Suppose now that $\beta' \in E'$. If $\beta' \in E$ then $\lambda'_{\beta',Z'} = \lambda_{\beta',Z}$. If $\beta' \in \Omega$ then $\beta' = \beta_1\beta_2\dots\beta_k$ is a path in Δ and

$$\lambda'_{\beta'} \quad = \quad \lambda_{\beta_1}\cdot\lambda_{\beta_2}\cdot\ldots\cdot\lambda_{\beta_k}.$$

E2) Let $\beta \in E$ and $g_\beta \in G$ be such that $\phi_\beta = g_\beta\mid_{Y_\beta}$, where

$$\phi_\beta : Y_\beta \subseteq X_{t(\beta)} \quad\longrightarrow\quad \widetilde{Y_\beta} \subseteq X_{o(\beta)}.$$

Set $X'_{o(\beta)} = X_{o(\beta)} \cup g_\beta(W_\beta)$ where

$$Y_\beta \quad\subseteq\quad W_\beta \quad\subseteq\quad X_{t(\beta)}.$$

This is an enlargement of the domain of definition of β. Otherwise choose $\mathcal{D}$ as in (1.5).

(5.2.1) Proposition. *Let G be a group acting non-trivially on a set X. Let $\mathcal{D}$ be local data for the action of G on X. If $\mathcal{D}'$ is an enlargement of $\mathcal{D}$ then $\mathcal{D}'$ is equivalent to $\mathcal{D}$.*

Proof. Let $\mathcal{G}(\mathcal{D})$ and $\mathcal{G}(\mathcal{D}')$ be the groups associated to $\mathcal{D}$ and $\mathcal{D}'$ respectively. We can assume that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a single move of type (E1) or of type (E2). Since $\mathcal{D}'$ is an enlargement of $\mathcal{D}$ ($\mathcal{D} \subseteq \mathcal{D}'$), we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{G}(\mathcal{D}) & \times & \mathcal{X}(\mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{D}) \\ \downarrow_\mu & & \downarrow_\nu & & \downarrow_\nu \\ \mathcal{G}(\mathcal{D}') & \times & \mathcal{X}(\mathcal{D}') & \longrightarrow & \mathcal{X}(\mathcal{D}') \end{array}$$

with $\mu : \mathcal{G}(\mathcal{D}) \longrightarrow \mathcal{G}(\mathcal{D}')$ a group homomorphism and $\nu : \mathcal{X}(\mathcal{D}) \longrightarrow \mathcal{X}(\mathcal{D}')$ a set map.

It is easy to see from the definitions of $\mathcal{G}(\mathcal{D})$ and $\mathcal{G}(\mathcal{D}')$ and the fact that $\mathcal{D}$ and $\mathcal{D}'$ arise from an action of a group on a set that μ is an *isomorphism* of groups corresponding to Tietze transformations of the presentation of $\mathcal{G}(\mathcal{D})$. Similarly one can check that ν is a set bijection. □

(5.2.2) Corollary. *Let G be a group acting non-trivially on a set X. Let $\mathcal{D}$ be local data for the action of G on X. Let $\mathcal{D}'$ be an enlargement of $\mathcal{D}$. Then $\mathcal{D}'$ is complete if and only if $\mathcal{D}$ is complete.* □

The following proposition establishes the existence of a complete set of data for any non-trivial action of a group G on a set X. We use the notation and setting of Section 1.5.

(5.2.3) Proposition. *Let $G \times X \longrightarrow X$ be a non-trivial action of a group G on a set X. Then there exists local data $\mathcal{D}$ that is complete for $G \times X \longrightarrow X$. If further G is finitely generated, then there exists complete local data $\mathcal{D}$ with a locally finite graph Δ.*

Proof. Using the notation of Section 1.5, if G is not finitely generated, we set $X = X_\alpha$ and take the graph Δ to have a single vertex $V\Delta = X = X_\alpha$. The edges $E\Delta$ are then maps

$$\beta_g : X \longrightarrow gX,\ g \in G$$

which reflect the action of g on X. Thus we also have $Y_{\beta_g} = X$ for each $g \in G$. Note that Δ is locally infinite. The vertex group of the graph of groups $G(V, E)$ is

$$G(X) = \varinjlim_{Z \subseteq X} (G_Z, \rho_Z^{Z'}),$$

and we have canonical isomorphisms of direct limits $\lambda_g : G_X \longrightarrow G_{gX}$, where

$$G_X = \{g \in G \mid gx = x \text{ for every } x \in X\}.$$

Since the action of G on X is non-trivial, $G_x \lneq G$ for each $x \in X$ hence $G_X \lneq G$. We have

$$G(\beta_g) \cong G(o(\beta_g))$$

for all edges β_g, that is, the edge group on the edge β_g is isomorphic to the vertex group at $o(\beta_g)$ since $Y_{\beta_g} = X$ for each $g \in G$. Thus each edge homomorphism

$$\omega_{\beta_g} : G(\beta_g) \longrightarrow G(t(\beta_g))$$

is an isomorphism.

The vertex group $G(X)$ occurs in the generating set of $\mathcal{G}(D)$, as do elements g_β for each edge β and hence for each $g \in G$. So $\mathcal{G}(\mathcal{D}) \longrightarrow G$ is a group isomorphism since every generator of G and every defining relation of G is automatically represented in $\mathcal{G}(\mathcal{D})$. The map $\mathcal{X}(\mathcal{D}) \longrightarrow X$ is automatically a set bijection

$$\begin{array}{rcl} X_{\alpha,x} & \longrightarrow & \mathcal{X}(\mathcal{D}) \\ x & \leftrightarrow & (1,x)_{\approx} \end{array}$$

and the action $\mathcal{G}(\mathcal{D}) \times \mathcal{X}(\mathcal{D}) \longrightarrow \mathcal{X}(\mathcal{D})$ commutes with the action $G \times X \longrightarrow X$. Hence this choice of local data is complete.

If G is finitely generated, let $S = \langle s_1, s_2, \ldots, s_k \rangle$ be any generating set for G. Set $X = X_\alpha$ with a single α and

$$\phi_{\beta_i} \quad = \quad \text{action of } s_i \text{ on } X, \quad i = 1, \ldots, k.$$

Set $Y_{\beta_i} = X_\alpha$ for each i. Thus

$$\phi_{\beta_i} : Y_{\beta_i} = X \longrightarrow \widetilde{Y}_{\beta_i} = s_i X.$$

The graph of groups $G(V,E)$ has a single vertex $V = X$ the edges are maps

$$\beta_i : X \longrightarrow s_i X, \ i = 1, \ldots, k.$$

The vertex group is

$$G(X) = \varinjlim_{Z \subseteq X} (G_Z, \rho_Z^{Z'}),$$

and we have isomorphisms of direct limits $\lambda_i : G_X \longrightarrow G_{s_i X}$, where

$$G_X = \{g \in G \mid gx = x \text{ for every } x \in X\}.$$

Since the action of G on X is non-trivial, $G_x \lneqq G$ for each $x \in X$ hence $G_X \lneqq G$. Since $Y_{\beta_i} = X_\alpha$ for each i,

$$G(\beta_i) \cong G(o(\beta_i))$$

for all edges β_i, that is vertex and edge groups are isomorphic, and so each edge homomorphism

$$\omega_{\beta_i} : G_X \longrightarrow G_{s_i X}$$

is an isomorphism. If the action of G is fixed point free, then G_X and $G_{s_i X}$ are trivial, the vertex group $G(X)$ is trivial, and λ_i, ω_i are isomorphisms of trivial groups.

The relation

$$\beta w_{\beta_i}(g) \beta^{-1} = \overline{w}_{\beta_i}(g)$$

says that $G_{s_i X}$ is conjugate to $G_{s_i^{-1} X}$ by the group element β_i corresponding to the edge $X \longrightarrow s_i X$.

Thus $G(V,E)$ has a single vertex and finitely many edges emanating from each vertex, indexed over the generators of G, so $G(V,E)$ is finite.

We can enlarge this data by moves of type (E1) to give data $\mathcal{D}'$ with $\phi_\beta = g_\beta$ for each group element $g_\beta \in G$. Since by the above argument, the enlargement $\mathcal{D}'$ is complete, by Corollary (5.2.2) $\mathcal{D}$ is complete for $G \times X \longrightarrow X$. $\square$

We note that there is significant redundancy in this choice of complete data, particularly when G is not finitely generated and each group element appears as a generator in the presentation of $\mathcal{G}(\mathcal{D})$.

We now give some examples of group actions with complete data.

Example - An infinitely but countably generated free group $G = \langle s_1, s_2, \dots \rangle$ acting on itself by left multiplication. We set $X = X_\alpha = G$ and take the graph Δ to have a single vertex $V\Delta = X = X_\alpha$. The edges $E\Delta$ are then maps

$$\beta_i : X \longrightarrow s_i X,\ i = 1, 2, \dots$$

Note that Δ is locally infinite. Since the action of G is fixed point free, the multiplicity groups G_x and $G_{s_i x}$ are trivial for all $x \in G$, λ_β and ω_β are isomorphisms of trivial groups and there is no monodromy. As in 2.3 (5), any closed path in Δ can be viewed as a morphism from α to α. There is thus a morphism β_i corresponding to every generator s_i. Hence the group $\mathcal{G}(\mathcal{D})$ is generated by a set indexed over the generators s_i and has no relations. Thus $\mathcal{G}(\mathcal{D})$ is isomorphic to G.

Example - $G = \mathbb{Z}/2\mathbb{Z}$ acting on itself by left multiplication. Let $G = \langle a \mid a^2 = 1 \rangle = \{1,\ a\}$. We take $X = X_\alpha = G = \{1,\ a\}$. The graph Δ has a single vertex $V\Delta = X = X_\alpha$ and the edges are maps $\beta_a : g \longrightarrow ag$ for $g \in G$. We have

$$\beta_1 : \{1, a\} \longrightarrow \{1, a\},$$

$$\beta_a : \{1, a\} \longrightarrow \{a, 1\}.$$

All multiplicity groups are trivial. As in 2.3 (5) any closed path in Δ with can be viewed as a morphism from α to α. In particular, the closed path $\beta_a^2 : g \longrightarrow ag \longrightarrow a^2 g = g$ is a morphism from α to α, so $\beta_a^2 = 1$ is a monodromy relation in $\mathcal{G}(\mathcal{D})$. The group $\mathcal{G}(\mathcal{D})$ thus has presentation

$$\mathcal{G}(\mathcal{D}) = \langle \beta_a \mid \beta_a^2 = 1 \rangle \cong G.$$

Example - A finitely generated free group G acting on itself by left multiplication. Let $S = \langle s_1, s_2, \dots, s_k \rangle$ be any generating set for G. Set $X = X_\alpha$ and

$$\phi_{\beta_i} \quad = \quad \text{action of } s_i \text{ on } X, \quad i = 1, \dots, k.$$

The graph Δ has a single vertex $V = X = X_\alpha$ and the edges are maps

$$\beta_i : X \longrightarrow s_i X,\ i = 1, \dots, k.$$

We have $Y_{\beta_i} = X_\alpha$ for each i. Since the action of G is fixed point free, the multiplicity groups G_X and $G_{s_i X}$ are trivial, the vertex group $\mathcal{G}(X)$ is trivial, λ_i and ω_i are isomorphisms of trivial groups for each i and there is no monodromy. Hence the group $\mathcal{G}(\mathcal{D})$ is generated by a set indexed over the edge set β_i for $i = 1, \dots, k$ and has no relations. Thus $\mathcal{G}(\mathcal{D})$ is isomorphic to G.

(4) Let $\mathcal{D}$ and $\mathcal{D}$' be two complete sets of data for an action $G \times X \longrightarrow X$. We say that $\mathcal{D}$ and $\mathcal{D}$' are *compatible* if there exist finite subsets F_1 and F_2 of G such that

$$W \subseteq F_1 \cdot W'$$

$$W' \subseteq F_2 \cdot W,$$

where

$$W = \bigsqcup_{\alpha \in V} X_\alpha, \qquad W' = \bigsqcup_{\alpha' \in V'} X_{\alpha'}.$$

For example if X is a proper metric space and if G acts on X by isometries properly discontinuously and cocompactly, then any $\mathcal{D}$ and $\mathcal{D}$' with *compact* W and W' are compatible.

Proposition (5.2.4). *Let G be a finitely generated group acting non-trivially on a set X. Let $\mathcal{D}$ and $\mathcal{D}$' be two complete compatible sets of data for $G \times X \longrightarrow X$. If $\mathcal{D}$ and $\mathcal{D}$' have finite graphs Δ and Δ', then $\mathcal{D}$ and $\mathcal{D}$' have a common enlargement $\mathcal{D}''$ also with a finite graph Δ''.*

5.3 Examples.

In this section, we give some examples to demonstrate how our machinery may be used to reconstruct group actions; that is, given an action $G \times X \longrightarrow X$ of a group on a set, we shall use (appropriately chosen) local data

$$\mathcal{D} = ((X_\alpha)_{\alpha \in V}, (\phi_\beta)_{\beta \in E}, (G_Z)_{Z \in \Phi}, (\rho_Z^{Z'}), (\lambda_{\beta,Z}), (g_{\gamma,Z}))$$

arising from this group action (as in 1.5) to build a group $\mathcal{G}(\mathcal{D})$ and a space $\mathcal{X}(\mathcal{D})$ such that $\mathcal{G}(\mathcal{D})$ acts on $\mathcal{X}(\mathcal{D})$, and the diagram

(1)

$$\begin{array}{ccccc} \mathcal{G}(\mathcal{D}) & \times & \mathcal{X}(\mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{D}) \\ \downarrow\mu & & \downarrow\nu & & \downarrow\nu \\ G & \times & X & \longrightarrow & X \end{array}$$

commutes (cf 3.1).

The group $\mathcal{G}(\mathcal{D})$ will be the fundamental group $\pi_1(\mathcal{M}(\mathcal{D}, \alpha_0))$ of the monodromy groupoid $\mathcal{M}(\mathcal{D}, \alpha_0)$. The space $\mathcal{X}(\mathcal{D})$ will be the covering space $\widetilde{\mathcal{M}(\mathcal{D}, \alpha_0)}$ as in 2.4. In many of the following examples, the maps μ and ν in (1) will be isomorphisms.

Example 1. *A free action on $\mathbb{R}^2$ by translations.*

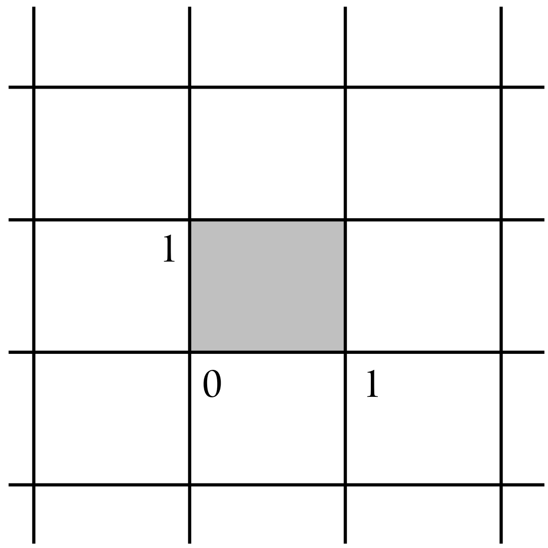
The group $G = \mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle$ acts on the plane $X = \mathbb{R}^2$ by translations:

$$G \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

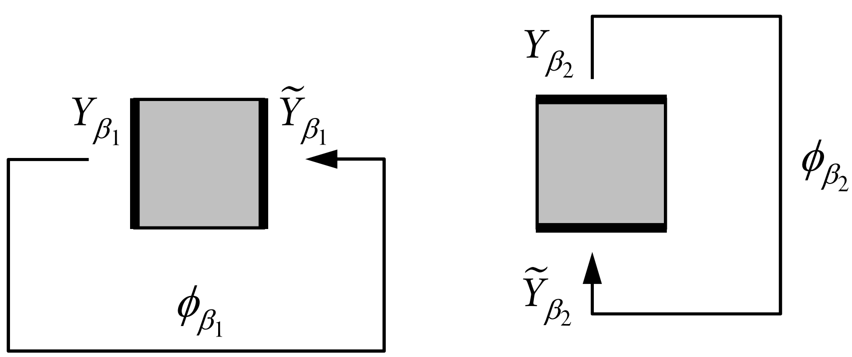
$$a : (x, y) \mapsto (x + 1, y)$$

$$b : (x, y) \mapsto (x, y + 1)$$

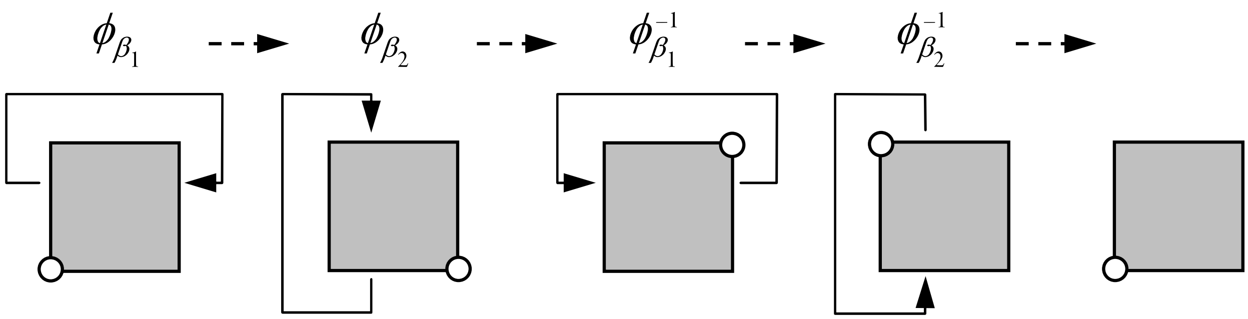
The subset $Y \subseteq X$, where Y is a unit square in the plane, is a fundamental domain for the action of G on X.



The local pseudogroup structure on X is generated by partial isometries (in the usual metric on $\mathbb{R}^2$) $\phi_{\beta_1} : Y_{\beta_1} \longrightarrow \widetilde{Y_{\beta_1}}$ and $\phi_{\beta_2} : Y_{\beta_2} \longrightarrow \widetilde{Y_{\beta_2}}$, where Y_{β_1} is the closed interval on the y-axis from $y = 0$ to $y = 1$, $\widetilde{Y_{\beta_1}}$ is the closed interval on the line $x = 1$ from $y = 0$ to $y = 1$, Y_{β_2} is the closed interval on the line $y = 1$ from $x = 0$ to $x = 1$ and $\widetilde{Y_{\beta_2}}$ is the closed interval on the x-axis from $x = 0$ to $x = 1$:



The multiplicity structure on X is trivial since G acts freely. There is monodromy in the corners of the unit square; consider the orbit of the point $(0, 0)$ under the sequence of partial isometries $\phi_{\beta_2}^{-1}\phi_{\beta_1}^{-1}\phi_{\beta_2}\phi_{\beta_1}$:

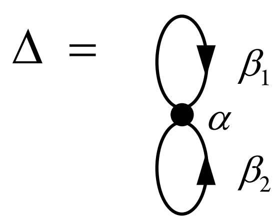


The monodromy element corresponding to this fixed point of the local pseudogroup is the trivial element since all multiplicity groups are trivial. This gives rise to a monodromy relation:

$$\beta_2{}^{-1}\beta_1{}^{-1}\beta_2\beta_1 = 1.$$

All other monodromy relations are a consequence of this relation.

The graph Δ corresponding to $\mathcal{D}$ is the graph:

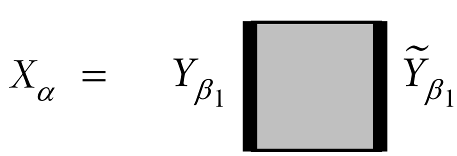


The group $\mathcal{G}(\mathcal{D}) = \pi_1(\mathcal{M}(\mathcal{D},\alpha))$ has the presentation

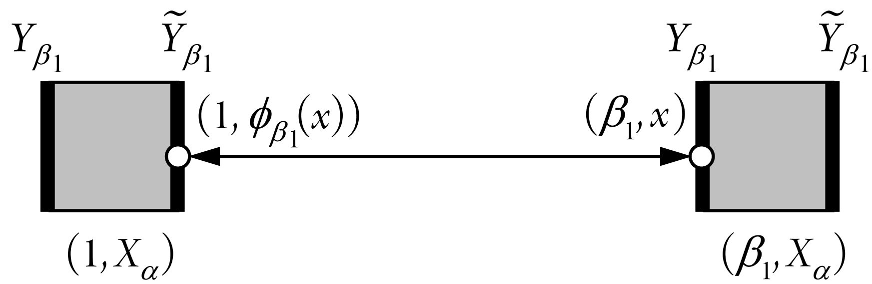
$$\mathcal{G}(\mathcal{D}) = \langle \beta_1, \beta_2 \mid \beta_2^{-1}\beta_1^{-1}\beta_2\beta_1 = 1\rangle$$

which is isomorphic to $G = \mathbb{Z}\times\mathbb{Z} = \langle a, b \mid ab = ba\rangle$.

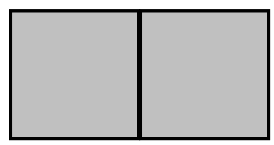
The space $\mathcal{X}(\mathcal{D}) = \widetilde{\mathcal{M}(\mathcal{D},\alpha)}$ is obtained in the following way: let



Then $(1, X_\alpha) \subseteq \widetilde{\mathcal{M}(\mathcal{D},\alpha)}$. Similarly, $(\phi_{\beta_1}, X_\alpha) \subseteq \widetilde{\mathcal{M}(\mathcal{D},\alpha)}$. By the relation $(2)_\mathcal{M}$ in 2.4, we identify points $(1\cdot\beta_1, x) \approx_\mathcal{M} (1, \phi_{\beta_1}(x))$, where $x \in Y_{\beta_1}$. That is; we glue copies of X_α as follows:



resulting in:



In a similar way, we glue together all other translates of X_α by elements of $\mathcal{G}(\mathcal{D}) = \mathbb{Z} \times \mathbb{Z}$ to obtain the space $\mathcal{X}(\mathcal{D})$ as the plane $\mathbb{R}^2$ tessellated by unit squares, where we give $\mathcal{X}(\mathcal{D})$ the metric induced by giving Y_{β_1}, $\widetilde{Y_{\beta_1}}$, Y_{β_2}, $\widetilde{Y_{\beta_2}}$ length 1. We observe also that $\mathcal{G}(\mathcal{D})$ acts on $\mathcal{X}(\mathcal{D})$ by translations.

In this example, we have reconstructed the group G up to isomorphism, the space X up to isometry, and the action $G \times X \longrightarrow X$ up to equivariant isometry.

Example 2. *A non discrete free action on* $\mathbb{R}$*.*

The group $G = \langle a, b \mid ab = ba \rangle$ acts freely on $X = \mathbb{R}$ by translations:

$$a : x \mapsto x + 1, \; b : x \mapsto x + \sqrt{2}.$$

The orbits for this action are dense and the action is not properly discontinuous. There is no fundamental domain for this action in the usual sense, but our machinery permits us to reconstruct the original action by using the local data to imitate a fundamental domain.

We take Φ to be the closed subsets of $\mathbb{R}$ and we take

$$X_\alpha = [0, 1 + \sqrt{2}] \subset \mathbb{R}$$

$$Y_{\beta_1} = [0, \sqrt{2}]$$

$$\widetilde{Y_{\beta_1}} = [1, 1 + \sqrt{2}]$$

$$Y_{\beta_2} = [0, 1]$$

$$\widetilde{Y_{\beta_2}} = [\sqrt{2}, 1 + \sqrt{2}]$$

and we have partial mappings

$$\phi_{\beta_1} : Y_{\beta_1} \longrightarrow \widetilde{Y_{\beta_1}}$$

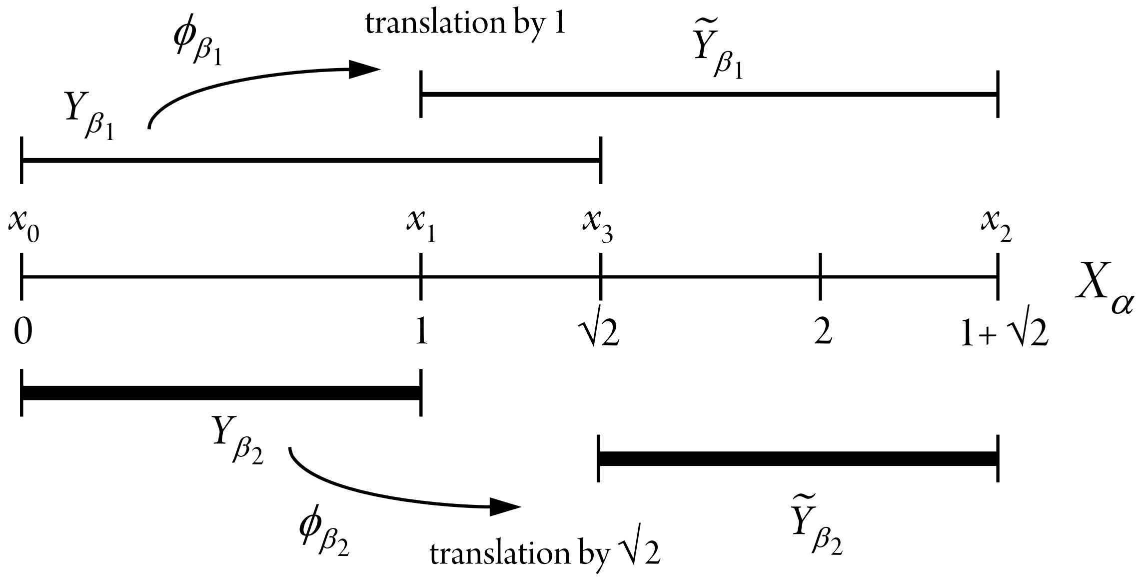
$$x \mapsto x + 1$$

and

$$\phi_{\beta_2} : Y_{\beta_2} \longrightarrow \widetilde{Y_{\beta_2}}$$

$$x \mapsto x + \sqrt{2}$$

which are restrictions of the action of the generators a and b to X_α:



The local pseudogroup structure on X is generated by the partial mappings ϕ_1 and ϕ_2; the multiplicity structure is trivial since the action is free.

The local pseudogroup admits monodromy; all elements of the pseudogroup which have fixed points are consequences of the following composition of partial mappings:

$$x_0 \xrightarrow{\phi_{\beta_1}} x_1 \xrightarrow{\phi_{\beta_2}} x_2 \xrightarrow{\phi_{\beta_1}^{-1}} x_3 \xrightarrow{\phi_{\beta_2}^{-1}} x_0$$

where $x_0 = 0$, $x_1 = 1$, $x_2 = 1 + \sqrt{2}$, $x_3 = \sqrt{2}$, therefore we have a single monodromy relation:

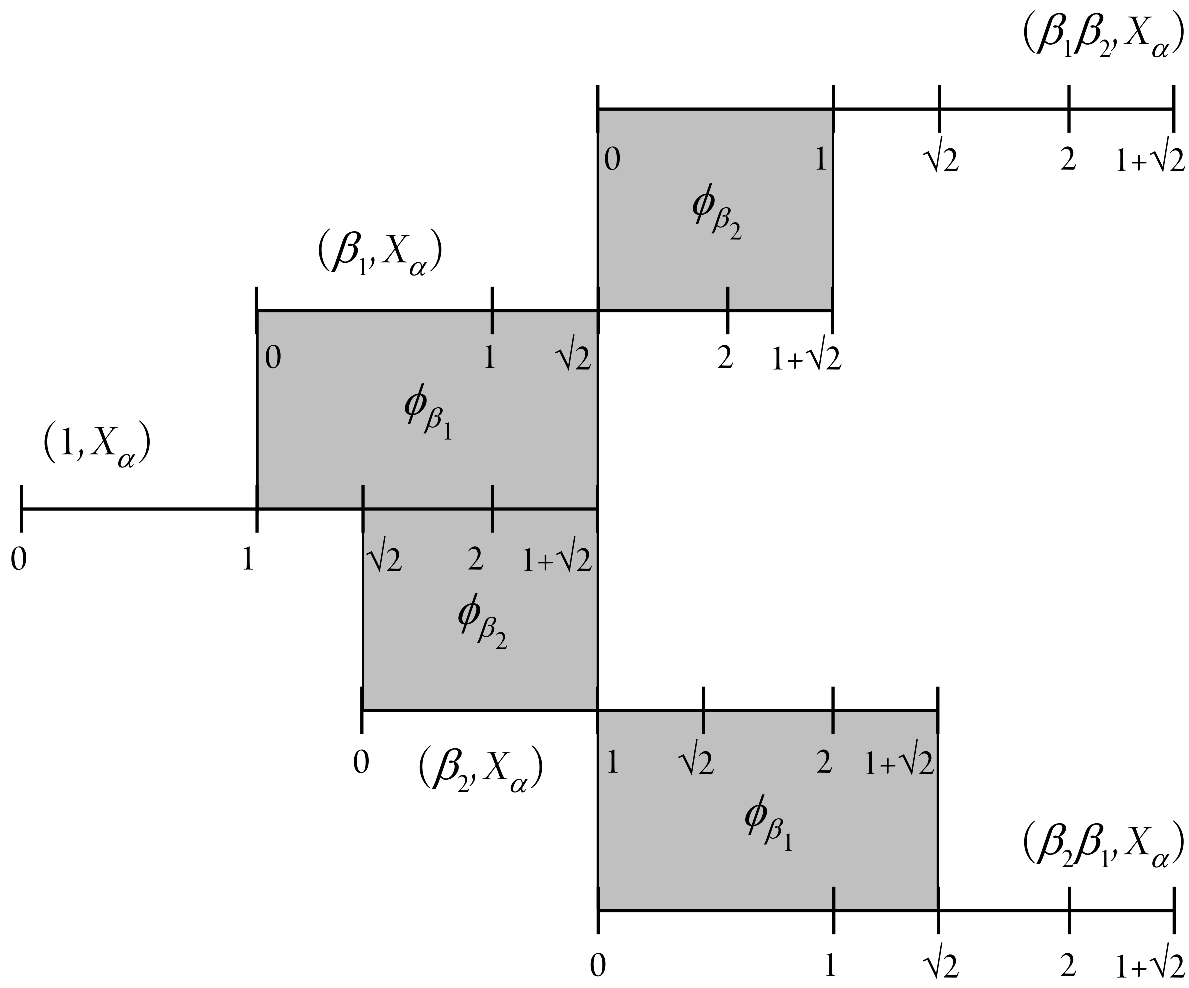
$$\beta_2^{-1}\beta_1^{-1}\beta_2\beta_1 = 1.$$

This local pseudogroup and its monodromy (with trivial multiplicity structure) defines local data $\mathcal{D}$ on X. We can therefore construct a group $\mathcal{G}(\mathcal{D})$ and a space $\mathcal{X}(\mathcal{D})$ with an action $\mathcal{G}(\mathcal{D}) \times \mathcal{X}(\mathcal{D}) \longrightarrow \mathcal{X}(\mathcal{D})$ imitating the original action $G \times X \longrightarrow X$.

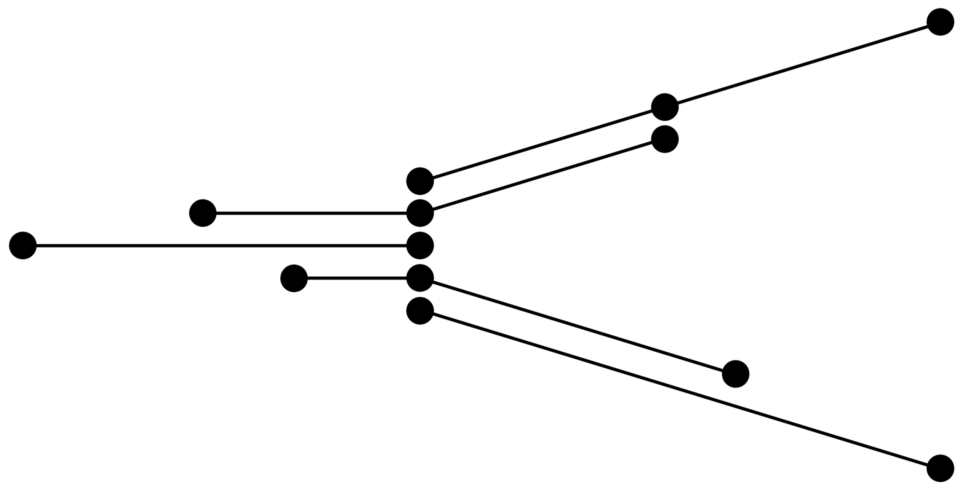
Since we have a single monodromy relation, and all multiplicity groups are trivial, the group $\mathcal{G}(\mathcal{D})$ is given by the presentation:

$$\mathcal{G}(\mathcal{D}) = \langle \beta_1,\ \beta_2 \mid \beta_2^{-1}\beta_1^{-1}\beta_2\beta_1 = 1\rangle \ \cong G.$$

We build the space $\mathcal{X}(\mathcal{D})$ as follows:

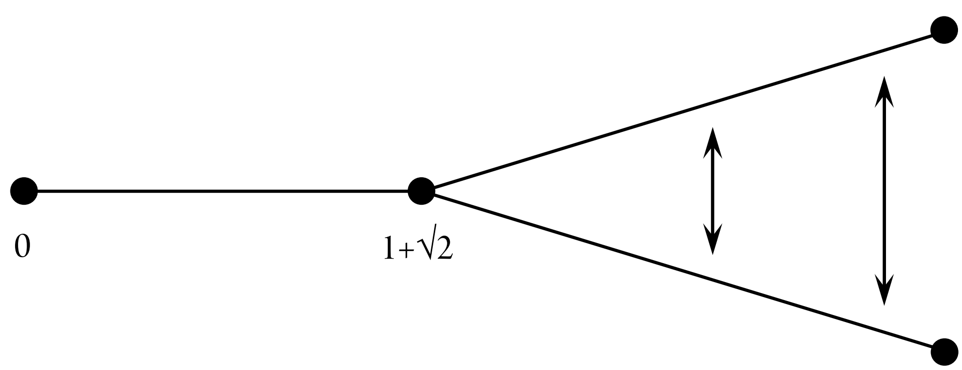


Schematically, this picture can be represented:

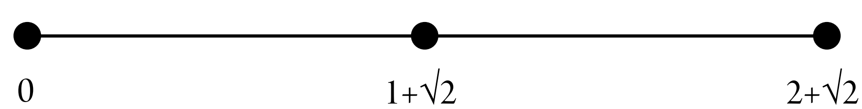


We give $\mathcal{X}(\mathcal{D})$ the metric induced by giving Y_{β_1}, $\widetilde{Y_{\beta_1}}$ length 1 and Y_{β_2}, $\widetilde{Y_{\beta_2}}$ length $\sqrt{2}$.

We have the equality $\beta_2\beta_1 = \beta_1\beta_2$ in $\mathcal{G}(\mathcal{D})$, therefore, in $\mathcal{X}(\mathcal{D})$, we identify the $\beta_2\beta_1$-copy of X_α with the $\beta_1\beta_2$-copy of X_α:



yielding:

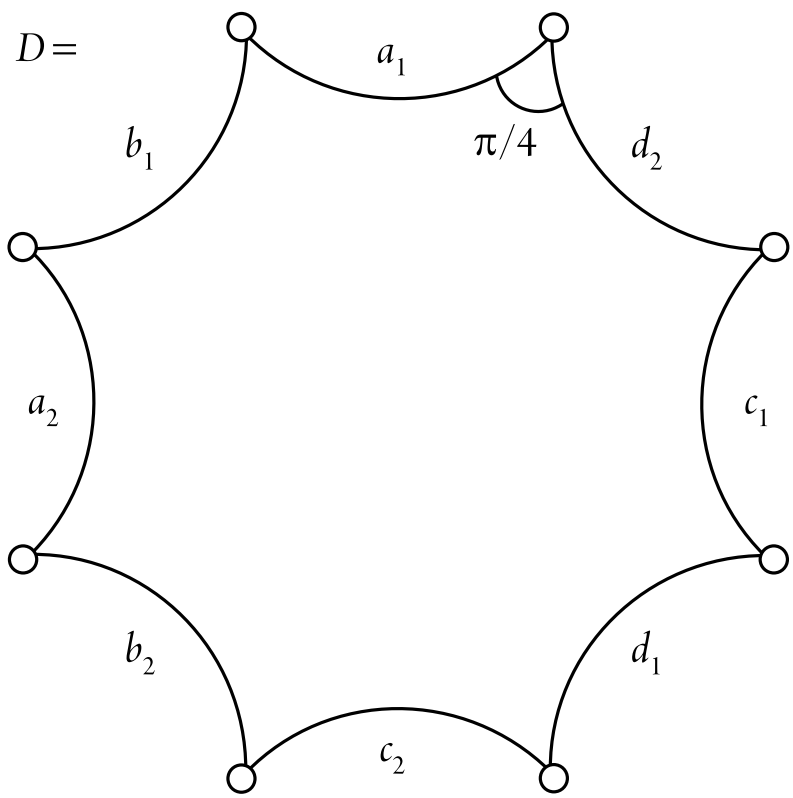


By iterating such gluings, we construct the space $\mathcal{X}(\mathcal{D})$ which is isometric to $\mathbb{R}$. We have reconstructed the group G up to isomorphism, the space X up to isometry, and the action

$G \times X \longrightarrow X$ up to equivariant isometry, even though the action of G on X is not properly discontinuous.

Example 3. *A free action of a surface group on the Poincaré disk.*

Let $X = \mathbb{H}^2$ in the Poincaré disk model. We construct a Fuchsian group Γ acting discretely on X by isometries in the following way. Choose a regular octagon D in $\mathbb{H}^2$ with all angles equal to $\pi/4$, with side labels:



There are unique isometries g_a, g_b, g_c and g_d in $PSL_2(\mathbb{R})$, the group of orientation preserving isometries of $\mathbb{H}^2$, such that:

$$\begin{aligned} g_a(a_1) &= a_2 \\ g_b(b_1) &= b_2 \\ g_c(c_1) &= c_2 \\ g_d(d_1) &= d_2. \end{aligned}$$

It follows that the group $\Gamma = \langle g_a, g_b, g_c, g_d \rangle$, generated by g_a, g_b, g_c and g_d is a discrete subgroup of $PSL_2(\mathbb{R})$ acting on $\mathbb{H}^2$ with fundamental domain D, and presentation:

$$\Gamma = \langle g_a, g_b, g_c, g_d \mid [g_c, g_d][g_a, g_b] = 1 \rangle,$$

where

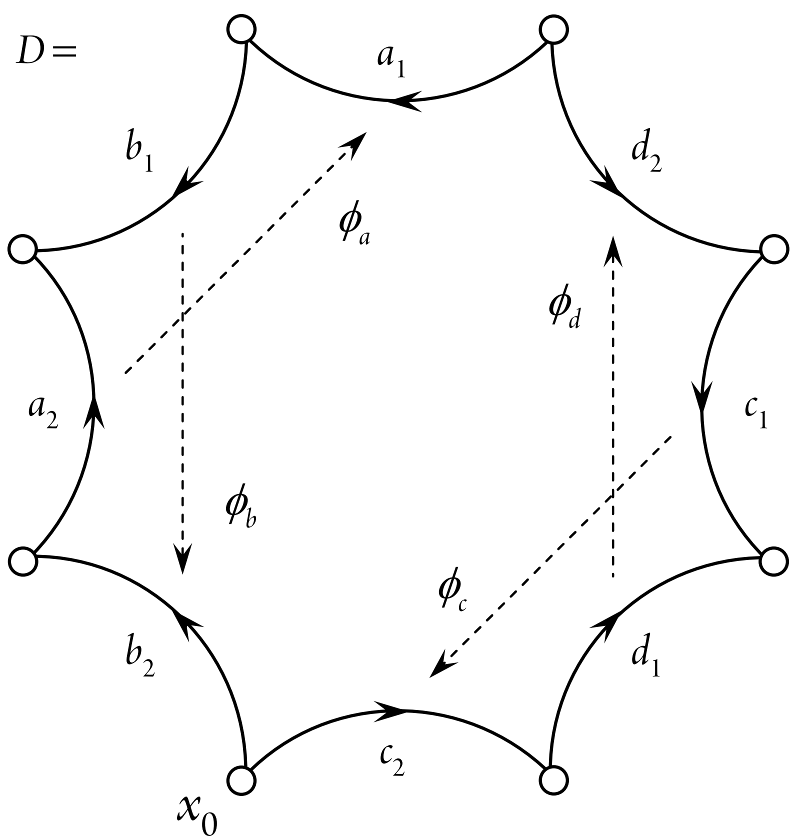
$$
\begin{aligned}
[g_a, g_b] &= g_a g_b g_a^{-1} g_b^{-1} \\
[g_c, g_d] &= g_c g_d g_c^{-1} g_d^{-1}.
\end{aligned}
$$

We now choose local data on $X_\alpha = D$, induced by the action of Γ on $\mathbb{H}^2$ as in 1.5. We take:

$$X_\alpha \quad = \quad D \quad \subseteq \quad \mathbb{H}^2$$

$$
\begin{aligned}
Y_a = a_1, &\qquad \widetilde{Y}_a = a_2 \\
Y_b = b_1, &\qquad \widetilde{Y}_b = b_2 \\
Y_c = c_1, &\qquad \widetilde{Y}_c = c_2 \\
Y_d = d_1, &\qquad \widetilde{Y}_d = d_2
\end{aligned}
$$

We choose orientations of the geodesic segments $a_i, b_i, c_i, d_i,\ i = 1, 2$ as indicated in the diagram:



We define maps:

$$\begin{aligned}
\phi_a : Y_a &\longrightarrow \widetilde{Y}_a \\
a_1 &\mapsto a_2 \\
\phi_b : Y_b &\longrightarrow \widetilde{Y}_b \\
b_1 &\mapsto b_2 \\
\phi_c : Y_c &\longrightarrow \widetilde{Y}_c \\
c_1 &\mapsto c_2 \\
\phi_d : Y_d &\longrightarrow \widetilde{Y}_d \\
d_1 &\mapsto d_2
\end{aligned}$$

to be orientation preserving isometries from the geodesic segments a_1, b_1, c_1, d_1 to a_2, b_2, c_2, d_2 respectively.

Since the action of Γ on $\mathbb{H}^2$ is free, we choose all multiplicity groups to be trivial. We have a single monodromy relation:

$$x_0 \quad = \quad [\phi_c, \phi_d][\phi_a, \phi_b](x_0),$$

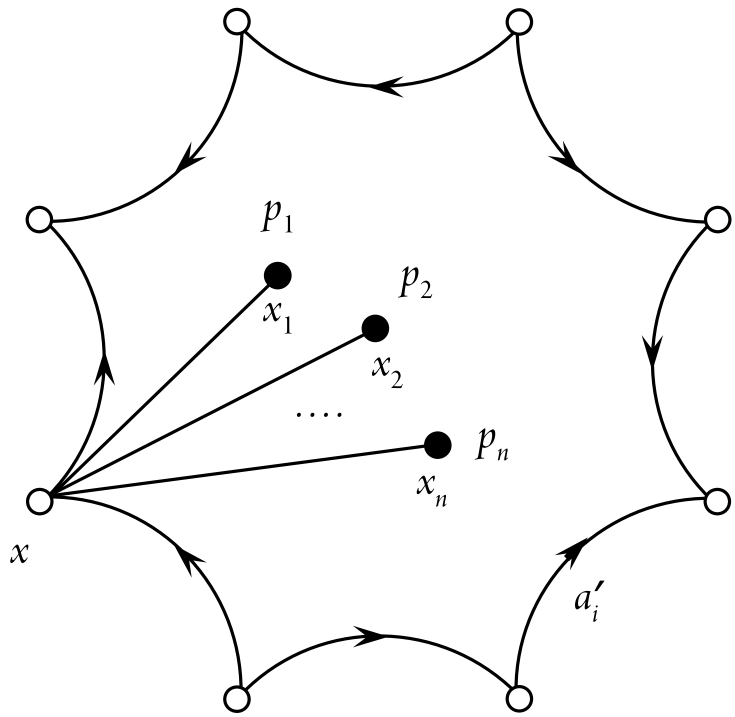
Thus $\mathcal{G}(\mathcal{D})$ is given by the presentation:

$$\langle \phi_a, \phi_b, \phi_c, \phi_d \mid [\phi_c, \phi_d][\phi_a, \phi_b] = 1 \rangle \cong \Gamma.$$

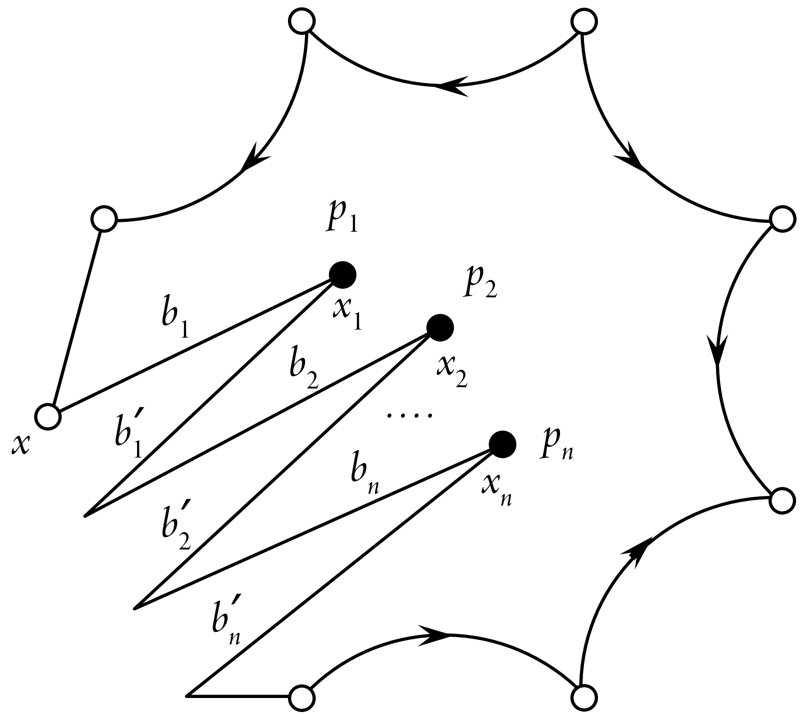
In Example 3 we showed that our local data as in Section (1.5) allows us to reconstruct a free action of a surface group on the Poincaré disk. In the next example we show that there is local data from a polygon associated with a two-dimensional orbifold $\mathcal{O}$ such that we can reconstruct the fundamental group of $\mathcal{O}$ as the group of the data.

Example 4. *Let $\mathcal{O}$ be a two-dimensional orbifold whose universal covering $\widetilde{\mathcal{O}}$ is a manifold. Let $G = \pi_1(\mathcal{O})$ and let $p : \widetilde{\mathcal{O}} \longrightarrow \mathcal{O}$ be the quotient projection. Then there is local data $\mathcal{D}$ for which $\mathcal{G}(\mathcal{D}) \quad \cong \quad G$.*

Proof: We cut $\mathcal{O}$ into a $\mathcal{P}$ polygon with an even number of sides labelled $a_1, a_2, \ldots, a_k$ with side identifications. We can always arrange so that the interior of $\mathcal{P}$ contains cone points $x_1, x_2, \ldots, x_n$ which are labelled with the orders $p_1, p_2, \ldots, p_n$ of cyclic groups from $\mathcal{O}$:



We choose a vertex x of $\mathcal{P}$ and make a cut from x to each cone point. Let $\mathcal{P}'$ denote the resulting polygon:



Let $X_\alpha = \mathcal{P}'$. For each side identification

$$a_i \longrightarrow a_i'$$

let ϕ_{β_i} be the bijection

$$\phi_{\beta_i} : a_i \longrightarrow a_i'.$$

For each cone point x_i let

$$\phi_{b_i} : b_i \longrightarrow b_i'$$

be a bijection identifying the cuts from x to x_i. We introduce multiplicity groups C_{p_i} at points x_i:

$$C_{p_i} \quad = \quad \langle c_i \mid c_i^{p_i} = 1 \rangle,$$

and trivial multiplicity elsewhere. There are 2 types of monodromy relations for this choice of data. At the points x_i,

$$\phi_{b_i}(x_i) \quad = \quad x_i$$

with corresponding monodromy element c_i.

(a) **Oriented case** Suppose that $\mathcal{O}$ has genus g and n cone points as above. Then the number of sides of the polygon $\mathcal{P}$ is divisible by 4. At the point x we have

$$\phi_{b_n} \dots \phi_{b_2} \phi_{b_1} [\phi_{\beta_g} \phi_{\beta_{g-1}}] \dots [\phi_{\beta_4} \phi_{\beta_3}][\phi_{\beta_2} \phi_{\beta_1}](x) \quad = \quad x.$$

(b) **Non-oriented case** Suppose that $\mathcal{O}$ has genus g and n cone points as above. At the point x we have

$$\phi_{b_n} \dots \phi_{b_2}\phi_{b_1}\phi^2_{\beta_g} \dots \phi^2_{\beta_2}\phi^2_{\beta_1}(x) \quad = \quad x.$$

For this choice of data $\mathcal{D}$, we have

Oriented case

$$\mathcal{G}(\mathcal{D}) = \langle \beta_1, \dots, \beta_g, c_1, \dots c_n \mid [\phi_{\beta_g}\phi_{\beta_{g-1}}] \dots [\phi_{\beta_4}\phi_{\beta_3}][\phi_{\beta_2}\phi_{\beta_1}] \;=\; c_n \dots c_2 c_1,\; c_i^{p_i} = 1\rangle,$$

which is isomorphic to the fundamental group of $\mathcal{O}$.

Non-oriented case

$$\mathcal{G}(\mathcal{D}) = \langle \beta_1, \dots, \beta_g, c_1, \dots c_n \mid \phi^2_{\beta_g} \dots \phi^2_{\beta_3}\phi^2_{\beta_2}\phi^2_{\beta_1} \;=\; c_n \dots c_2 c_1,\; c_i^{p_i} = 1\rangle,$$

which is isomorphic to the fundamental group of $\mathcal{O}$.

6. RECOVERING THE BASS-SERRE THEORY.

In this section we show that when we choose local data (as in 1.5) from a graph of groups $\mathbb{A}$, our results coincide with the classical results of the theory of Bass-Serre for reconstructing group actions on simplicial trees ([B], [S]). We will prove the following:

(1) Theorem. *Let $\mathbb{A}$ be a graph of groups with underlying graph A. Let $a \in VA$ and let $G = \pi_1(\mathbb{A}, a)$. Let $X = \widetilde{(\mathbb{A}, a)}$ so that G acts on X without inversions. Then there is local data $\mathcal{D}$ from which we construct a group $\mathcal{B}(\mathcal{D})$ and a space $\mathcal{X}(\mathcal{D})$ with an action of $\mathcal{B}(\mathcal{D})$ on $\mathcal{X}(\mathcal{D})$, giving a canonical homomorphism $\mu : \mathcal{B}(\mathcal{D}) \longrightarrow G$ and canonical set map $\nu : \mathcal{X}(\mathcal{D}) \longrightarrow X$ that makes the following diagram commute:*

$$\begin{array}{ccccc} \mathcal{B}(\mathcal{D}) & \times & \mathcal{X}(\mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{D}) \\ \downarrow_{\mu} & & \downarrow_{\nu} & & \downarrow_{\nu} \\ G & \times & X & \longrightarrow & X \end{array}$$

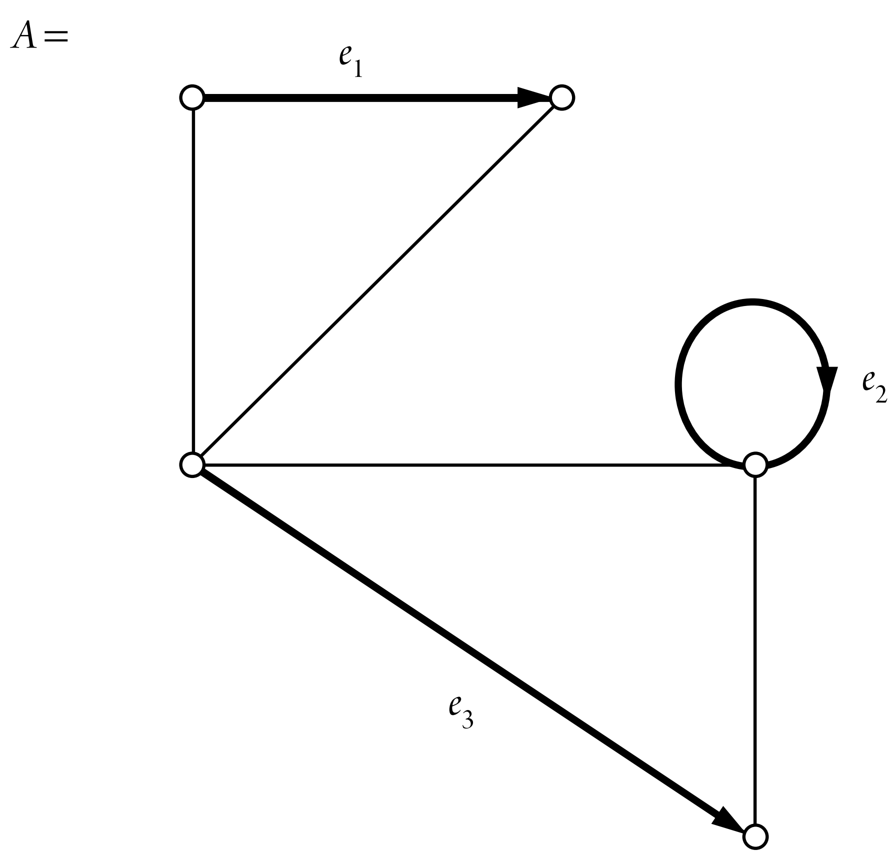
The map μ is an isomorphism of groups. The map ν is a μ-equivariant isomorphism of covering trees.

(2) This theorem will follow from Theorem 6.28, and 6.33-6.35. This result shows that when our local data is chosen from a graph of groups, we recover the Bass-Serre correspondence between actions (without inversions) on trees and quotient graphs of groups.

(3) Let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups with underlying graph A, where $\mathcal{A}$ assigns vertex groups $\mathcal{A}_v$ for each $v \in VA$ and edge groups $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$ for each $e \in EA$. The origin and terminus of an edge e will be denoted by $o(e)$ and $t(e)$ respectively. For each $e \in EA$ we denote the boundary monomorphism from $\mathcal{A}_e$ to $\mathcal{A}_{o(e)}$ by α_e.

We choose a maximal tree $T \subseteq A$ and an orientation on $EA - ET$ which we denote by $(EA - ET)^+$.

(4) Example.



e_1, e_2, e_3 in $EA - ET$

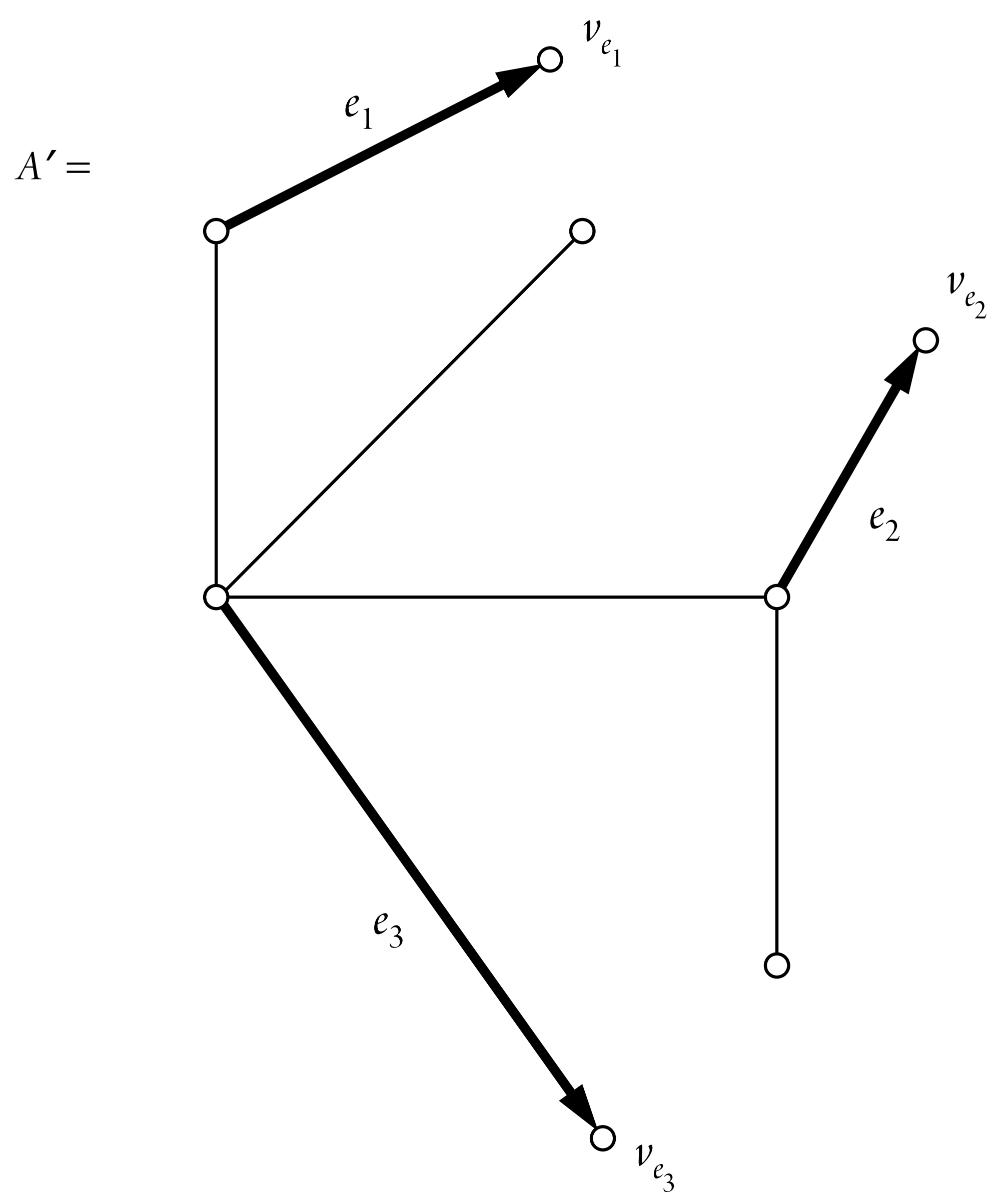
(5) We define a new graph A' by 'opening' each edge $e \in (EA - ET)^+$ at its terminal point.
(6) For each $e \in (EA - ET)^+$ we introduce a new vertex denoted v_e. Then the graph A' is defined as follows:

$$
\begin{aligned}
VA' &= VA \cup \{v_e \mid e \in (EA - ET)^+\}, \\
EA' &= EA, \\
o_{A'}(e) &= o_A(e), \quad e \in (EA - ET)^+, \\
t_{A'}(e) &= v_e, \quad e \in (EA - ET)^+, \\
o_{A'}(e) &= o_A(e), \quad e \in (ET), \\
t_{A'}(e) &= t_A(e), \quad e \in (ET),
\end{aligned}
$$

where $o_{A'}$, $t_{A'}$ and o_A, t_A denote the origin and terminal vertices in A' and A respectively. We observe that the graph A' is in fact a tree.

(7) Example.

For A as above, we have:



(8) We define a new graph of groups $\mathbb{A}'$ with underlying graph A'. For each $e \in (EA - ET)^+$ we let $\widetilde{\mathcal{A}}_{t_A(e)}$ be an isomorphic copy of $\mathcal{A}_{t_A(e)}$, where

$$\begin{aligned} \widetilde{\rho}_e : \mathcal{A}_{t_A(e)} &\xrightarrow{\cong} \widetilde{\mathcal{A}}_{t_A(e)} \\ a &\mapsto \widetilde{a}, \quad a \in \mathcal{A}_{t_A(e)}. \end{aligned}$$

(9) For each $e \in (EA - ET)^+$ we set $\mathcal{A}'_{v_e} = \widetilde{\mathcal{A}}_{t_A(e)}$. For $v \in VA = VT$, we set $\mathcal{A}'_v = \mathcal{A}_v$. For $e \in ET$, we set $\mathcal{A}'_e = \mathcal{A}_e$. For $e \in (EA - ET)$, we set $\mathcal{A}'_e = \mathcal{A}_e$. For edges of A' inside T, the boundary monomorphisms are defined as for A.
(10) For $e \in (EA - ET)^+$ we define

$$\alpha'_e : \mathcal{A}'_e \hookrightarrow \mathcal{A}'_{o_{A'}(e)}$$

to be

$$\alpha_e : \mathcal{A}_e \hookrightarrow \mathcal{A}_{o_{A'}(e)},$$

and

$$\alpha'_{\overline{e}} : \mathcal{A}'_e \hookrightarrow \mathcal{A}'_{v_e} = \widetilde{\mathcal{A}}_{t_A(e)}$$

to be

$$\alpha'_{\overline{e}} = (\widetilde{\rho}_e \cdot \alpha_{\overline{e}}) \quad : \quad \mathcal{A}_e \hookrightarrow \mathcal{A}'_{v_e} = \widetilde{\mathcal{A}}_{t_A(e)}.$$

(11) From now on, we identify the graph A' with its geometric realization, which will also be denoted A'; that is, we view edges of A' as *intervals*, isometric to the closed unit interval $[0,1] \subset \mathbb{R}$. This gives A' the structure of a metric space.
(12) We now define a local pseudogroup structure on A'. We define Φ to be the set of all closed intervals of A', including single point sets.
(13) A single point set $\{x\} \subseteq A'$ will be viewed as a degenerate interval $[x,x] \subseteq A'$.
(14) For each $e \in (EA - ET)^+$ we define a map

$$\phi_e : \{t_A(e)\} \longrightarrow \{v_e\}$$
$$t_A(e) \mapsto v_e.$$

The collection of maps $\{\phi\}_{e \in (EA-ET)^+}$ defines a local pseudogroup Γ on A' with graph Δ consisting of a single vertex, denoted α, and a loop denoted β_e for each $e \in (EA-ET)^+$ with $o(\beta_e) = \alpha = t(\beta_e)$.
(15) We assume hereafter (by an obvious modification of $\mathbb{A}'$) that for each $e \in EA'$, we have $\mathcal{A}'_e = \mathcal{A}'_{o_{A'}(e)} \cap \mathcal{A}_{t_{A'}(e)}$, and that the boundary monomorphisms α'_e and $\alpha'_{\overline{e}}$ are inclusion maps:

$$\alpha'_e : \mathcal{A}'_e \hookrightarrow \mathcal{A}'_{o_{A'}(e)}$$

$$\alpha'_{\overline{e}} : \mathcal{A}'_{\overline{e}} \hookrightarrow \mathcal{A}'_{t_{A'}(e)}.$$

(16) We now define a multiplicity structure on A'. For an interval $[a,b] \subseteq A'$, the multiplicity group for $[a,b]$ will be denoted $G_{[a,b]}$.
(17) For a vertex $v \in VA'$ we set:

$$G_v \quad = \quad \mathcal{A}'_v.$$

Let $e \in EA'$, and let $v_0 = o_{A'}(e)$, $v_1 = t_{A'}(e)$. For $[a,b] \subseteq e$ with $[a,b] \neq \{v_0\}$ and $[a,b] \neq \{v_1\}$ we set:

$$G_{[a,b]} \quad = \quad \mathcal{A}'_e.$$

(18) For closed subintervals $[a,b] \subseteq e$ we define restriction maps as follows: the map

$$\rho^{[v_0,b]}_{[v_0,v_0]} : G_{[v_0,b]} \hookrightarrow G_{[v_0,v_0]},$$

for $b \in [v_0, v_1]$ is defined to be:

$$\rho^{[v_0,b]}_{[v_0,v_0]} = \alpha'_e : \mathcal{A}'_e \hookrightarrow \mathcal{A}'_{v_0}.$$

The map

$$\rho^{[a,v_1]}_{[v_1,v_1]} : G_{[a,v_1]} \hookrightarrow G_{[v_1,v_1]},$$

for $a \in [v_0, v_1]$ is defined to be:

$$\rho^{[a,v_1]}_{[v_1,v_1]} = \alpha'_{\overline{e}} : \mathcal{A}'_e \hookrightarrow \mathcal{A}'_{v_1}.$$

(19) Let $[a,b]$ be a closed interval in (the metric tree) A'. Then $[a,b]$ can be described as a union of intervals:

$$[a, v_1] \cup [v_1, v_2] \cup \cdots \cup [v_{n-1}, v_n] \cup [v_n, b],$$

for vertices $v_0, v_1, v_2, \ldots, v_{n+1} \in VA'$ such that (v_i, v_{i+1}) are adjacent vertices, $i = 0, \ldots, n$. The path with vertex sequence $(v_0, v_1, v_2, \ldots, v_{n+1})$ is then a reduced path, and $a \in [v_0, v_1]$, $b \in [v_n, v_{n+1}]$ with $[a,b] \subseteq [v_0, v_{n+1}]$.

(20) We define the multiplicity group $G_{[a,b]}$ to be the group:

$$G_{[a,v_1]} \cap G_{[v_1,v_2]} \cap \cdots \cap G_{[v_{n-1},v_n]} \cap G_{[v_n,b]}.$$

(21) Let $[c,d] \subseteq [a,b]$ be a closed subinterval. It is easy to see that $G_{[c,d]} \subseteq G_{[a,b]}$, and we can therefore define

$$\rho^{[c,d]}_{[a,b]} : G_{[c,d]} \longrightarrow G_{[a,b]}$$

to be the inclusion map $G_{[c,d]} \hookrightarrow G_{[a,b]}$.

(22) We recall that $\{\phi\}_{e \in (EA-ET)^+}$ defines a local pseudogroup structure on A', where

$$\phi_e : \{t_A(e)\} \longrightarrow \{v_e\}$$
$$t_A(e) \mapsto v_e,$$

and for each $v_e \in A'$, $e \in (EA - ET)^+$, there is a canonical isomorphism:

$$\widetilde{\rho}_e : \mathcal{A}_{t_A(e)} \xrightarrow{\cong} \widetilde{\mathcal{A}}_{t_A(e)}.$$

(23) We define isomorphisms of multiplicity groups:

$$\lambda_{e,t_{A'}(e)} := \widetilde{\rho}_e : G_{t_{A'}(e)} \longrightarrow G_{v_e},$$

where $G_{t_{A'}(e)} = \mathcal{A}'_{t_{A'}(e)}$, $G_{v_e} = \widetilde{\mathcal{A}'}_{t_{A'}(e)}$. By 5.4, this defines a multiplicity structure on A', and the relevant axioms (compatibility with restrictions as in 1.2) are trivially satisfied since the pseudogroup maps are defined on single point sets.
(24) We observe that the local pseudogroup structure on A' admits no monodromy.
(25) The local pseudogroup structure and multiplicity structure described above define data $\mathcal{D}$ on $A'(= X_\alpha)$ in the sense of 1.4. We can therefore associate a graph of groups (as in 2.1) and a Bass-Serre groupoid (as in 2.3) to $\mathcal{D}$.
(26) The graph $\Delta = (V, E, o, t, -)$ associated to $\mathcal{D}$ (as in 2.1) consists of a single vertex, denoted α, and a loop denoted β_e for each $e \in (EA - ET)^+$, with $o(\beta_e) = \alpha = t(\beta_e)$. Thus Δ is a bouquet of circles, with a circle for each $e \in (EA - ET)^+$.

We build a graph of groups $\mathcal{G}(V, E)$ on Δ as follows: for the vertex α, we take the vertex group $\mathcal{G}(\alpha)$ to be the direct limit of the multiplicity groups and restriction mappings (as in 2.1.3):

$$\mathcal{G}(\alpha) = \varinjlim_{Z \subseteq X_\alpha} (G_Z, \rho_Z^{Z'}).$$

Since $\mathbb{A}'$ is a tree of groups, it is easy to see that $\mathcal{G}(\alpha) = \pi_1(\mathbb{A}')$, the fundamental group of the graph of groups $\mathbb{A}'$. This gives a natural inclusion $\mathcal{A}'_v \hookrightarrow \mathcal{G}(\alpha)$ for each $v \in VA'$.
(27) For each $e \in (EA - ET)^+$ we set:

$$\mathcal{G}(\beta_e) \quad := \quad \mathcal{A}_{t_A(e)}.$$

We define boundary monomorphisms as in 2.1.9 as follows:

$$\omega_{\beta_e} : \mathcal{A}_{t_A(e)} \hookrightarrow \mathcal{G}(\alpha) = \pi_1(\mathbb{A}'),$$

where ω_{β_e} is the canonical inclusion,

$$\omega_{\beta_{\overline{e}}} : \mathcal{A}_{t_A(e)} \hookrightarrow \mathcal{G}(\alpha) = \pi_1(\mathbb{A}') \\ a \mapsto \widetilde{a},$$

for each $a \in \mathcal{A}_{t_A(e)}$.

(28) Theorem. *With the notations above, there is a canonical isomorphism*

$$\pi_1(\mathbb{A}, T) \quad \cong \quad \mathcal{B}(\mathcal{D}),$$

where $\mathcal{B}(\mathcal{D})$ is the Bass-Serre groupoid of $\mathcal{D}$.

Proof. The fundamental group of the graph of groups $\mathbb{A}$ with respect to T is given as follows ([B]):

$$\begin{aligned}\pi_1(\mathbb{A}, T) =& \langle (\mathcal{A}_v)_{v \in VA},\ EA^+ \mid e\alpha_{\overline{e}}(g)e^{-1} = \alpha_e(g)\ \forall g \in \mathcal{A}_e,\ e \in EA^+,\ e = 1\ \forall e \in T \rangle \\ & \langle (\mathcal{A}_v)_{v \in VA},\ E(A - T)^+ \mid e\alpha_{\overline{e}}(g)e^{-1} = \alpha_e(g)\ \forall g \in \mathcal{A}_e,\ e \in E(A - T)^+, \\ & \alpha_{\overline{e}}(g) = \alpha_e(g)\ \forall g \in \mathcal{A}_e,\ e \in ET^+ \rangle.\end{aligned}$$

(29) We have $\mathcal{G}(\alpha) = \pi_1(\mathbb{A}')$, where

$$\begin{aligned}\mathcal{G}(\alpha) =&\langle (\mathcal{A}_v)_{v\in VA}, (\widetilde{\mathcal{A}}_{t_A(e)})_{e\in E(A-T)^+} \mid \alpha_e(g) = \alpha_{\overline{e}}(g)\ \forall g \in \mathcal{A}_e,\ e\in ET^+,\\ &\alpha_e(g) = \widetilde{\alpha_{\overline{e}}(g)}\ \forall g\in \mathcal{A}_e,\ e \in E(A-T)^+\rangle.\end{aligned}$$

(30) By 2.3.3 the Bass-Serre groupoid $\mathcal{B}(\mathcal{D})$ has the following presentation:

$$\begin{aligned}\mathcal{B}(\mathcal{D}) &= \langle \mathcal{G}(\alpha), (\beta_e)_{e\in E(A-T)^+} \mid \beta_e\omega_{\beta_e}(g)\beta_e^{-1} = \omega_{\overline{\beta}_e}(g),\ e\in E(A-T)^+,\ g\in\mathcal{A}_{t_A(e)}\rangle\\ &= \langle \mathcal{G}(\alpha), (\beta_e)_{e\in E(A-T)^+} \mid \beta_e g\beta_e^{-1} = \widetilde{g},\ e\in E(A-T)^+,\ g\in\mathcal{A}_{t_A(e)}\rangle,\end{aligned}$$

by 6.27.
(31) Using the presentation 6.29 for $\mathcal{G}(\alpha)$, we obtain:

$$\begin{aligned}\mathcal{B}(\mathcal{D}) = \langle& (\mathcal{A}_v)_{v\in VA}, (\widetilde{\mathcal{A}}_{t_A(e)})_{e\in E(A-T)^+}, (\beta_e)_{e\in E(A-T)^+} \mid \alpha_e(g) = \alpha_{\overline{e}}(g)\ \forall g\in\mathcal{A}_e,\ e\in ET^+,\\ &\alpha_e(g) = \widetilde{\alpha_{\overline{e}}(g)}\ \forall g\in\mathcal{A}_e,\ e\in E(A-T)^+,\ \beta_e g\beta_e^{-1} = \widetilde{g},\ e\in E(A-T)^+,\ g\in\mathcal{A}_{t_A(e)}\rangle.\end{aligned}$$

(32) By a Tietze transformation we can modify the presentation for $\mathcal{B}(\mathcal{D})$ as follows:

$$\begin{aligned}\mathcal{B}(\mathcal{D}) &= \langle (\mathcal{A}_v)_{v\in VA}, (\widetilde{\mathcal{A}}_{t_A(e)})_{e\in E(A-T)^+}, (\beta_e)_{e\in E(A-T)^+} \mid \alpha_e(g) = \alpha_{\overline{e}}(g),\ \forall g\in\mathcal{A}_e,\ e\in ET^+,\\ &\qquad \alpha_e(g) = \beta_e\alpha_{\overline{e}}(g)\beta_e^{-1}\ \forall g\in\mathcal{A}_e,\ e\in E(A-T)^+,\\ &\qquad \beta_e g\beta_e^{-1} = \widetilde{g},\ e\in E(A-T)^+,\ g\in\mathcal{A}_{t_A(e)}\rangle\\ &= \langle (\mathcal{A}_v)_{v\in VA}, (\beta_e)_{e\in E(A-T)^+} \mid \alpha_e(g) = \alpha_{\overline{e}}(g),\ \forall g\in\mathcal{A}_e,\ e\in ET^+,\\ &\qquad \alpha_e(g) = \beta_e\alpha_{\overline{e}}(g)\beta_e^{-1},\ \forall g\in\mathcal{A}_e,\ e\in E(A-T)^+\rangle\\ &= \pi_1(\mathbb{A}, T) \text{ by 6.28. } \square\end{aligned}$$

(33) Our remaining objective in this section is to compare the classical description of the Bass-Serre covering tree of a graph of groups with the fibered product space $\mathcal{X}(\mathcal{D})$ associated to our data $\mathcal{D}$ chosen from a graph of groups as in 6.3-6.24. Let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups as in 6.3. Following ([B], 1.16) and ([S], 5.3) $\mathbb{A}$ has a universal covering tree X which can be constructed as follows.
(34) Let T be a maximal tree in A and let $G = \pi_1(\mathbb{A}, T)$. Then the vertices of $X = \widetilde{(\mathbb{A}, T)}$ are:

$$VX = \coprod_{v\in VA} \pi_1(\mathbb{A}, T)/\mathcal{A}_v,$$

the edges of X are:

$$EX = \coprod_{e\in EA} \pi_1(\mathbb{A}, T)/\alpha_e\mathcal{A}_e,$$

and for $s = g\alpha_e\mathcal{A}_e \in EX$ where $g\in G$, we have:

$$\begin{aligned}o_X(s) &= g\mathcal{A}_{o_A(e)}\\ t_X(s) &= ge\mathcal{A}_{t_A(e)},\end{aligned}$$

where $o_X(s)$ and $t_X(s)$ denote the origin and terminus in X of s respectively. We observe that if $e \in ET$, then $e =_G 1$, and therefore

$$t_X(s) \quad = \quad g\mathcal{A}_{t_A(e)}.$$

(35) We recall from that the fibered product space

$$\mathcal{X}(\mathcal{D}) \quad = \quad \mathcal{B}(\mathcal{D}) \times A'/\approx,$$

where $\mathcal{B}(\mathcal{D}) = \pi_1(\mathbb{A}, T)$ by 6.28, A' is as in 6.5, and $\approx$ is an equivalence relation generated by:

$$(gh, v) \quad \approx \quad (g, v),$$

where $h \in \mathcal{A}_v$, $g \in G$, $v \in VA$,

$$(ge, t_A(e)) \quad \approx \quad (g, v_e),$$

where $g \in G$, $v \in VA$, $e \in E(A-T)^+$, and

$$(g, e) \quad \approx \quad (g', e')$$

if

$$(g, o_{A'}(e)) \quad \approx \quad (g', o_{A'}(e'))$$

and

$$(g, t_{A'}(e)) \quad \approx \quad (g', t_{A'}(e')).$$

The space $\mathcal{X}(\mathcal{D})$ naturally has the structure of a combinatorial graph on which $\mathcal{B}(\mathcal{D}) = \pi_1(\mathbb{A}, T)$ acts as:

$$\begin{aligned} g(g_1, v)_\approx &= (gg_1, v)_\approx \\ g(g_1, e)_\approx &= (gg_1, e)_\approx, \end{aligned}$$

for $g, g_1 \in \pi_1(\mathbb{A}, T)$, $v \in VA'$, $e \in EA'$.

It is a routine check to verify that the graph $\mathcal{X}(\mathcal{D})$ coincides with the Bass-Serre covering tree X of $\mathbb{A}$, and that the actions

$$\pi_1(\mathbb{A}, T) \times X \longrightarrow X$$

and

$$\pi_1(\mathbb{A}, T) \times \mathcal{X}(\mathcal{D}) \longrightarrow \mathcal{X}(\mathcal{D})$$

correspond up to equivariant isomorphism. This together with Theorem 6.28 completes the proof of Theorem 6.1. □

7. RECONSTRUCTING NON PROPERLY DISCONTINUOUS GROUP ACTIONS ON THE UPPER HALF PLANE.

When a group acts freely and discretely on a space X, significant information about the action can be determined from the quotient space. When the action is not free and not discrete, a different approach is needed. In many such cases, the machinery described in Sections 1-3 can be used to approximate a fundamental domain for the action.

In this section we examine a non discrete free action of a group G on $\mathbb{H}^2$ and we show that we can reconstruct the action of G on $\mathbb{H}^2$ by considering only local information.

Let G be the group with presentation

$$G = \langle x, y \mid yxy^{-1} = x^2 \rangle.$$

(1) By the normal form theorem for HNN-extensions, it is easy to see that every element $1 \neq g \in G$, or its inverse, is represented by a word of the form $y^{-m}x^k y^n$, $m \geq 0$, $n \geq 0$, $k \in \mathbb{Z}$, which is conjugate in G to a word of the form $x^k y^\ell$, where $k \in \mathbb{Z}$, $\ell \in \mathbb{Z}$, and k and ℓ are not both zero.

(2) The group G acts on the hyperbolic plane $\mathbb{H}^2$. In the upper half plane model, this action can be described as follows. Let $\sigma : G \longrightarrow PSL_2(\mathbb{R})$ be defined on generators:

$$\sigma : x \mapsto \sigma_x,$$

$$\sigma : y \mapsto \sigma_y,$$

where, for $z \in \mathbb{H}^2$, σ_x is the translation:

$$\sigma_x : z \mapsto z + 1,$$

and σ_y is the homothety:

$$\sigma_y : z \mapsto 2z.$$

(3) We observe that for $z \in \mathbb{H}^2$:

$$\begin{aligned} \sigma_y \cdot \sigma_x \cdot \sigma_y^{-1}(z) &= \sigma_y \cdot \sigma_x(z/2) \\ &= \sigma_y(1 + z/2) \\ &= 2 + z \\ &= \sigma_x^2(z). \end{aligned}$$

Therefore σ extends to a homomorphism from G to $PSL_2(\mathbb{R})$ which will also be denoted σ. We have the following:

(4) Proposition. *The action of G on $\mathbb{H}^2$ is faithful, free and non discrete.*

Proof. By 5.1.1 every element $1 \neq g \in G$, or its inverse, is represented by a word which is conjugate in G to a word of the form $x^k y^\ell$, where $k \in \mathbb{Z}$, $\ell \in \mathbb{Z}$, and k and ℓ are not both zero. The image of such a word under σ is the following:

$$\sigma : x^k y^\ell \mapsto \sigma_{k,\ell},$$

where $\sigma_{k,\ell} = \sigma_x^k \sigma_y^\ell$, so that

$$\sigma_{k,\ell} : z \mapsto 2^\ell z + k.$$

If $\ell \neq 0$, then $2^\ell \neq 1$ and so

$$\sigma_{k,\ell}(z) \quad \neq \quad z, \quad \text{for every } z \in \mathbb{H}^2.$$

If $\ell = 0$ but $k \neq 0$, then

$$\sigma_{k,\ell} : z \mapsto z + k,$$

so $\sigma_{k,\ell}(z) \neq z$ for every $z \in \mathbb{H}^2$. Therefore the action of G on $\mathbb{H}^2$ via σ is faithful and free. To observe that the action of G on $\mathbb{H}^2$ via σ is not discrete, consider the image of the word $y^{-s}xy^s$, $s \in \mathbb{Z}$ under σ:

$$\sigma : y^{-s}xy^s \mapsto \sigma_s = \sigma(y^{-s}xy^s),$$

where

$$\sigma_s : z \stackrel{(\sigma_y)^s}{\longrightarrow} 2^s z \stackrel{(\sigma_x)}{\longrightarrow} 2^s z + 1 \stackrel{(\sigma_y)^{-s}}{\longrightarrow} z + 1/2^s.$$

By choosing s arbitrarily large, for fixed $z \in \mathbb{H}^2$ $d(z, \sigma_s(z))$ is arbitrarily small in the hyperbolic metric. $\square$

We will prove the following:

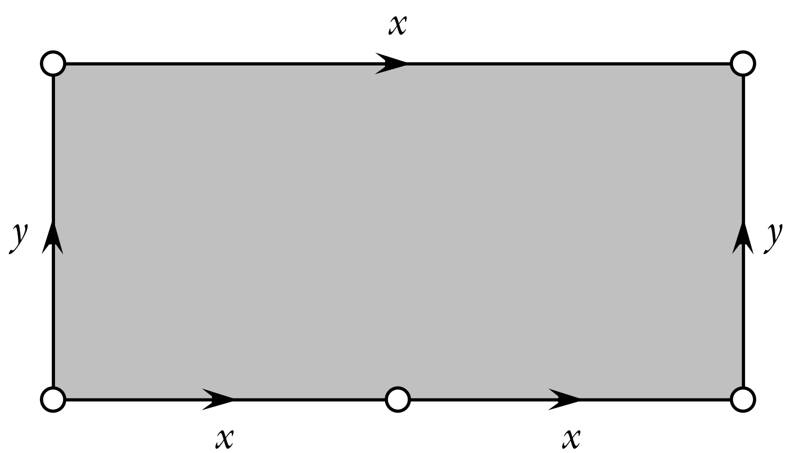
(5) Theorem. *Let $G = \langle x, y \mid yxy^{-1} = x^2 \rangle$ with its non discrete action on the upper half plane $\mathbb{H}^2$ by translation $\sigma_x : z \mapsto z + 1$ and homothety $\sigma_y : z \mapsto 2z$. Then there is local data $\mathcal{D}$ from which we can construct a group $\mathcal{G}(\mathcal{D})$ and a space $\mathcal{X}(\mathcal{D})$ with a natural action $\mathcal{G}(\mathcal{D}) \times \mathcal{X}(\mathcal{D}) \longrightarrow \mathcal{X}(\mathcal{D})$ that commutes with the action of G on $\mathbb{H}^2$:*

$$\begin{array}{ccccc} \mathcal{G}(\mathcal{D}) & \times & \mathcal{X}(\mathcal{D}) & \longrightarrow & \mathcal{X}(\mathcal{D}) \\ \downarrow_\mu & & \downarrow_\nu & & \downarrow_\nu \\ G & \times & X & \longrightarrow & X \end{array}$$

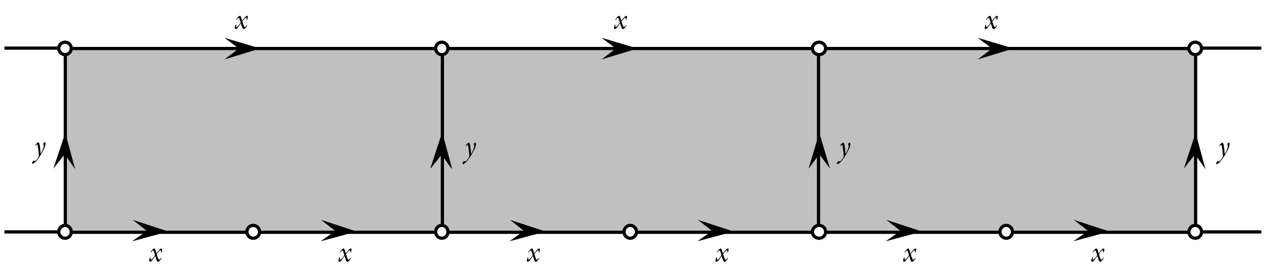
The action of $\mathcal{G}(\mathcal{D})$ on $\mathcal{X}(\mathcal{D})$ is discrete and cocompact. The map $\mu : \mathcal{G}(\mathcal{D}) \longrightarrow G$ is an isomorphism of groups. The map $\nu : \mathcal{X}(\mathcal{D}) \longrightarrow X$ is a $\mathcal{G}(\mathcal{D})$-equivariant map from $\mathcal{X}(\mathcal{D})$ to the hyperbolic plane $\mathbb{H}^2$.

(6) In order to motivate our choice of local data for Theorem 5.1.5, we build a $K(\Pi, 1)$ space (as in ([Ep], pp 154-160)) denoted K, with fundamental group $G = \langle x, y \mid yxy^{-1} = x^2 \rangle$ by taking loops for the generators x and y, and adding a two cell for the relator $yxy^{-1}x^{-2} = 1$. The 1-skeleton Γ of the universal covering $\widetilde{K}$ is the Cayley graph of G.

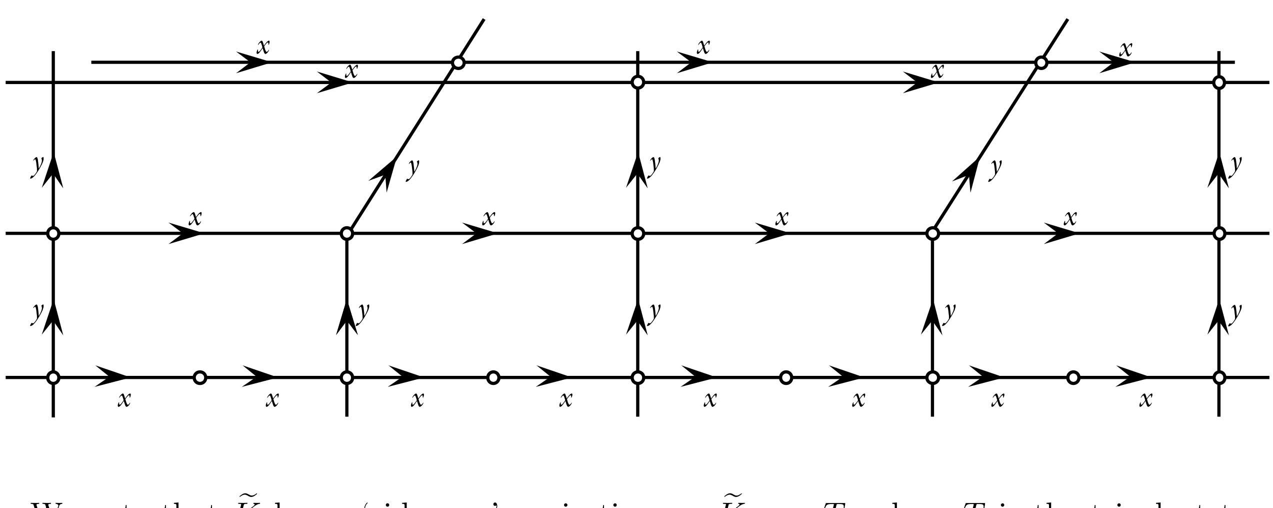
(7) To construct the universal covering $\widetilde{K}$, we start with the defining relation $yxy^{-1}x^{-2} = 1$ drawn as a filled in rectangle:



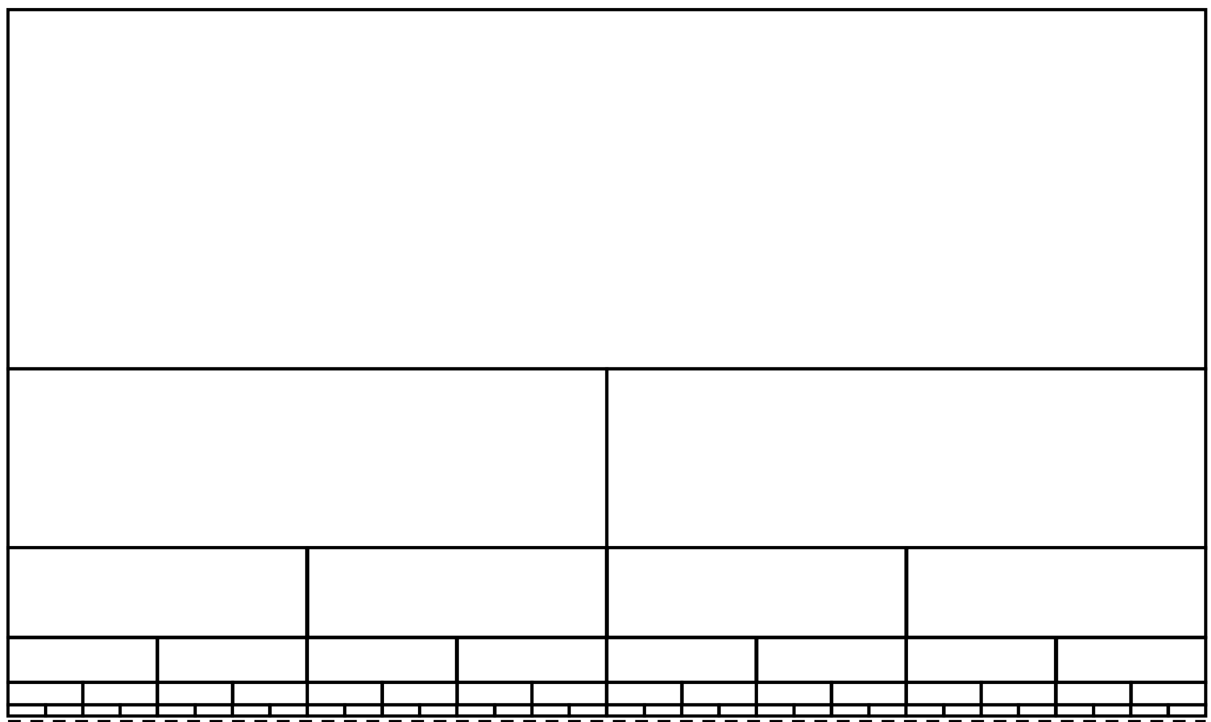
We glue copies of the rectangle along vertical sides to get an infinite horizontal strip:



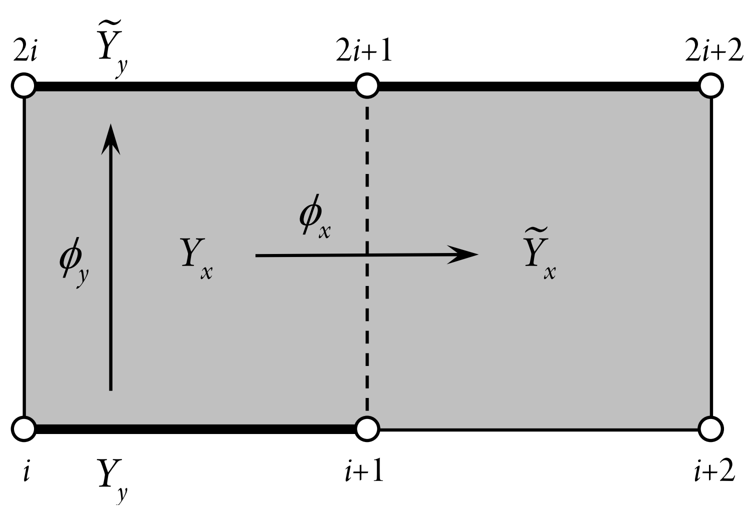
and we observe that every vertex has both incoming and outgoing edges labelled 'x'. To complete the universal covering $\widetilde{K}$, every vertex must have both incoming and outgoing edges labelled 'y' (drawn without shading):



We note that $\widetilde{K}$ has a 'sideways' projection $p : \widetilde{K} \longrightarrow T$, where T is the trivalent tree. Choosing a bi-infinite path γ in T, the inverse image of γ in $\widetilde{K}$, denoted S_γ, is homeomorphic to $\mathbb{R}^2$ and can be mapped to the upper half plane in such a way that the homothety $\sigma_y : z \mapsto 2z$ maps each rectangular horizontal strip to the one above it. The following gives the image of S_γ in the upper half plane, showing a tessellation $\mathbb{H}^2$ by horocycles and radial lines:



(8) Proof of Theorem 7.1.5. We begin by choosing local data for the action of $G = \langle x, y \mid yxy^{-1} = x^2 \rangle$ on $\mathbb{H}^2$. First we construct a local pseudogroup of partial isometries. Let X_α be the rectangle in the upper half plane with corners i, $2i$, $2i+2$, $i+2$:



Let Y_x be the square including the corners i, $2i$, $2i+1$, $i+1$. Let $\widetilde{Y_x}$ be the square including the corners $i+1$, $2i+1$, $2i+2$, $i+2$. Let ϕ_x be the map sending Y_x to $\widetilde{Y_x}$. Let Y_y be the piece of the horocycle of height i from 0 to $i+1$. Let $\widetilde{Y_y}$ be the piece of the horocycle of height $2i$ from 0 to $2i+2$. Let ϕ_y be the map sending Y_y to $\widetilde{Y_y}$.

It is easy to see that the maps ϕ_x and ϕ_y are restrictions of the action of the generators σ_x and σ_y of G on $\mathbb{H}^2$ to the rectangle X_α. Moreover $(\phi_x : Y_x \longrightarrow \widetilde{Y_x}, \phi_y : Y_y \longrightarrow \widetilde{Y_y})$ forms a local pseudogroup of partial isometries.

By Proposition 7.1.4 the action of $G = \langle x, y \mid yxy^{-1} = x^2 \rangle$ on $\mathbb{H}^2$ is free and it follows that the multiplicity groups of our data are trivial. There is non-trivial monodromy at the point i:

$$\begin{aligned} \phi_y^{-1}\phi_x^{-2}\phi_y\phi_x(i) &= \phi_y^{-1}\phi_x^{-2}\phi_y(i+1) \\ &= \phi_y^{-1}\phi_x^{-2}(2i+2) \\ &= \phi_y^{-1}(2i) \\ &= i, \end{aligned}$$

and it is easy to see that all other monodromy relations follow from this one. This completes the description of our local data $\mathcal{D}$. It follows that the group $\mathcal{G}(\mathcal{D})$ (as in 2.4.10) is given by the presentation:

$$\mathcal{G}(\mathcal{D}) = \langle \beta_x, \beta_y \mid \beta_y\beta_x\beta_y^{-1} = \beta_x^2 \rangle,$$

so $\mathcal{G}(\mathcal{D}) \cong G = \langle x, y \mid yxy^{-1} = x^2 \rangle$.

Our next task is to construct the space $\mathcal{X}(\mathcal{D})$ (as in 2.4.10). The space $\mathcal{X}(\mathcal{D})$ is formed as follows:

$$\mathcal{X}(\mathcal{D}) = (\mathcal{G}(\mathcal{D}) \times X_\alpha)/\approx_{\mathcal{M}},$$

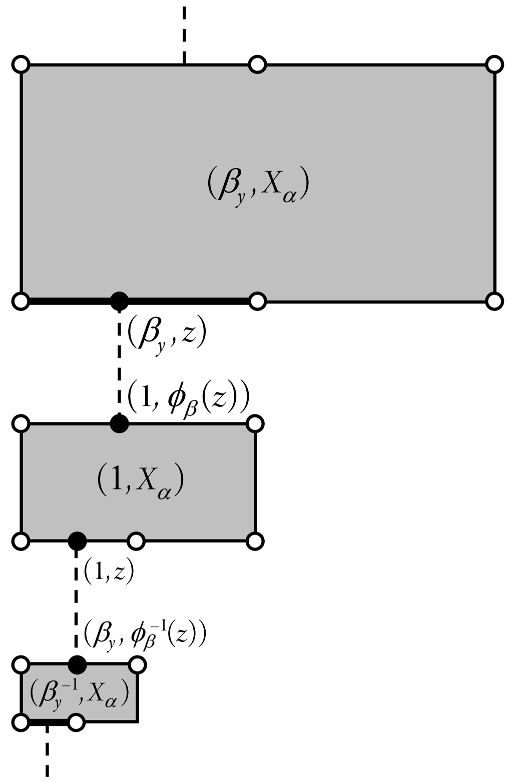
where $\approx_{\mathcal{M}}$ is the equivalence relation on $\mathcal{G}(\mathcal{D}) \times X_\alpha$ generated by relations (as in 2.4.10):

$$(g\beta_x, z) \approx_{\mathcal{M}} (g, \phi_x(z)) \tag{9}$$

$$(g\beta_y, z) \approx_{\mathcal{M}} (g, \phi_y(z)), \tag{10}$$

where $g \in \mathcal{G}(\mathcal{D})$, $z \in X_\alpha$.

The relation 7.1.10 induces the following identifications:



which gives:

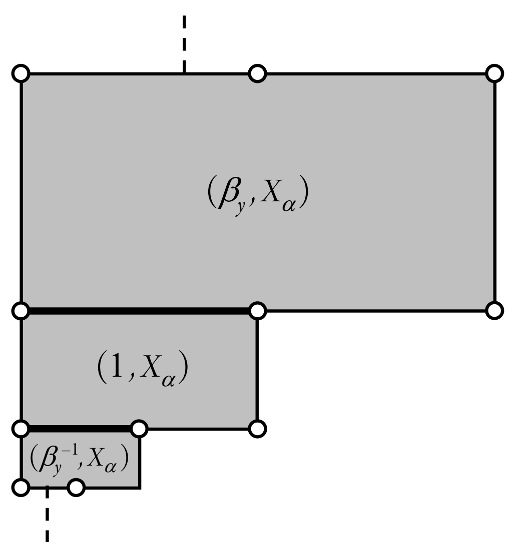
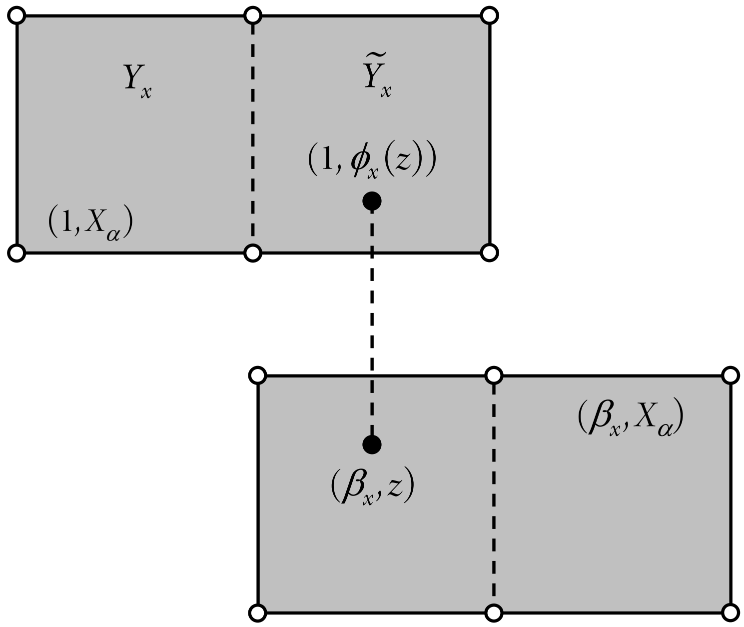
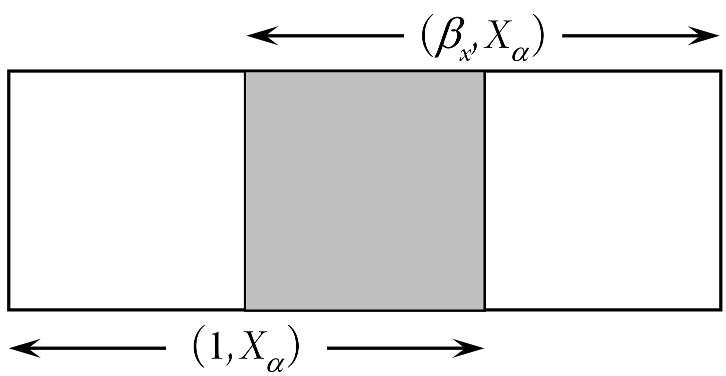


fig 7.1.11

The relation 7.1.9 induces identifications:

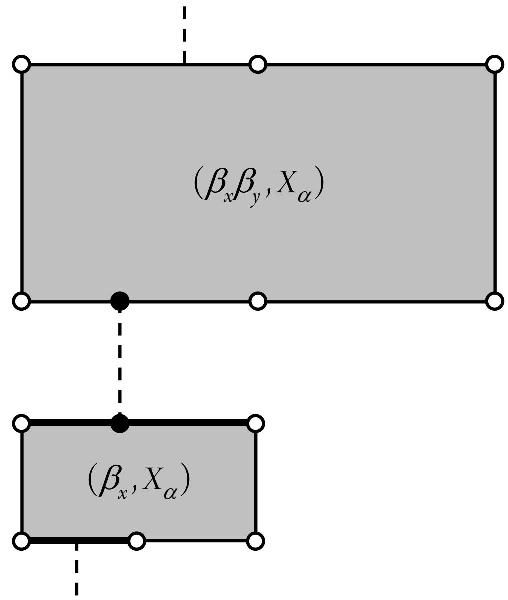


which gives

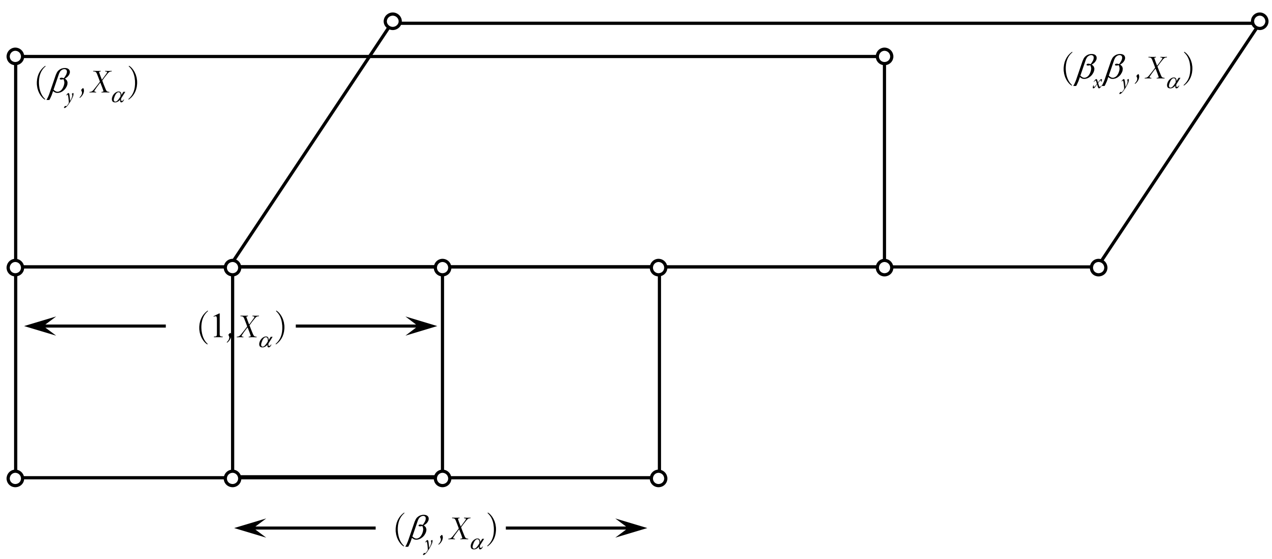


Iterating such gluings allows us to fill in each horizontal strip in *fig* 7.1.11 to complete the constructon of the space $\mathcal{X}(\mathcal{D})$, giving a copy of the upper half plane tessellated by horocycles and radial lines.

Combining identifications 7.1.9 and 7.1.10 gives rise to additional identifications of the form:



This in turn gives rise to branchings in the space $\mathcal{X}(\mathcal{D})$ of the form:



One can check that $(\beta_x\beta_y, X_\alpha)$ and $(\beta_y\beta_x, X_\alpha)$ are identified in $\mathcal{X}(\mathcal{D})$. It follows that $\mathcal{X}(\mathcal{D})$ is the filled in Cayley graph $\widetilde{K}$ of the group $G = \langle x, y \mid yxy^{-1} = x^2 \rangle$ as in 5.1.6, and the action of $\mathcal{G}(\mathcal{D})$ on $\mathcal{X}(\mathcal{D})$ is discrete and cocompact. $\square$

There is an analogous choice of local data that allows us to reconstruct the non discrete actions of the Baumslag-Solitar groups

$$B(1, n) = \langle x, y \mid yxy^{-1} = x^n \rangle$$

on $\mathbb{H}^2$, for $n \geq 2$.

(12) Remark.

As we have observed, for non properly discontinuous group actions our local data as in Section (1.5) allows us to imitate a fundamental domain, quotient space and universal covering for the quotient. For non properly discontinuous action of a group G on proper quasi-geodesic metric space X by isometries, we may also use our local data and a result of L. Mosher ([Mos]) to build a space on which the group acts properly discontinuously and cocompactly. Let $\mathcal{D}$ be a complete set of data for the action of G on X. If we assume that the orbit map for $G \times X \longrightarrow X$ has a quasi-isometric section, then by Lee Mosher's theorem ([Mos]) the action of G on X is 'laminable'. In this case there exists a 'transversal' τ and 'product data' $\mathcal{D}\times\tau$ such that we can reconstruct G up to isomorphism, a new space X' and a properly discontinuous and cocompact action $G \times X' \longrightarrow X'$.

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