THE GEOMETRY OF RANK 2 HYPERBOLIC ROOT SYSTEMS

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ABSTRACT. Let Δ be a rank 2 hyperbolic root system. Then Δ has generalized Cartan matrix $H(a,b) = \left(\begin{smallmatrix} 2 & -b \\ -a & 2 \end{smallmatrix}\right)$ indexed by $a, b \in \mathbb{Z}$ with $ab \geq 5$. If $a \neq b$, then Δ is non-symmetric and is generated by one long simple root and one short simple root, whereas if $a = b$, Δ is symmetric and is generated by two long simple roots. We prove that if $a \neq b$, then Δ contains an infinite family of symmetric rank 2 hyperbolic root subsystems $H(k,k)$ for certain $k \geq 3$, generated by either two short or two long simple roots. We also prove that Δ contains non-symmetric rank 2 hyperbolic root subsystems $H(a', b')$, for certain $a', b' \in \mathbb{Z}$ with $a'b' \geq 5$.

1. INTRODUCTION

Let Δ be the root system of a rank 2 Kac–Moody algebra $\mathfrak{g}$. Then Δ has generalized Cartan matrix $H(a,b) := \left(\begin{smallmatrix} 2 & -b \\ -a & 2 \end{smallmatrix}\right)$ for some $a, b \in \mathbb{Z}$ with $ab \geq 1$. Let $S = \{\alpha_1, \alpha_2\}$ denote a basis of simple roots of Δ. If $a \neq b$, then Δ is non-symmetric and its base consists of a long simple root and a short simple root, whereas if $a = b$, Δ is symmetric and its base consists of two long simple roots.

We are primarily concerned with the case that $ab \geq 5$. In this case, Δ is a rank 2 hyperbolic root system. The real roots of Δ are of the form $w\alpha_i$ for some $w \in W$, where W is the Weyl group of the root system. In this case, $W \cong D_\infty$, is the infinite dihedral group. The additional 'imaginary roots' will not play a role in this work. The real roots are supported on the branches of a hyperbola in $\mathbb{R}^{(1,1)}$, with a pair of branches for each root length (Figures 3–6).

Motivated by the work of Morita ([Mor], [Mor2]) we wish to determine which pairs of real roots are such that their sum is a real root. This question was answered in arbitrary Kac–Moody root systems by Billig and Pianzola ([BP]). Here we consider only rank 2 Kac–Moody algebras and we adopt a different approach which will allow us, in a future project, to determine the non-trivial commutators and their structure constants in both the Kac–Moody algebra and Kac–Moody group associated to $H(a,b)$.

We obtain proofs of the results stated by Morita [Mor] that if a and b are both > 1, then no sum of real roots can be a real root. It follows that the prounipotent subgroup corresponding to the positive real roots on a single branch of the hyperbola is commutative. When a or $b = 1$ we prove, as stated in [Mor], that the prounipotent subgroup generated by all the positive real short root groups is metabelian and the prounipotent subgroup generated by all the positive real long root groups is commutative. Our results in Sections 3 and 4 also cover the affine cases $H(2,2)$ and $H(4,1)$.

In order to make our results precise, we use two different concepts of a subsystem generated by a subset Γ of real roots: namely a subsystem $\Phi(\Gamma)$, corresponding to a reflection subgroup of the Weyl group and consisting entirely of real roots; and $\Delta(\Gamma)$, a subsystem whose set of roots are all those that can be written as an integral linear combination of elements of Γ. Such a $\Delta(\Gamma)$ subsystem corresponds to a certain subalgebra of the Kac–Moody algebra.

This research made extensive use of the Magma computer algebra system.

We have completely classified both kinds of subsystem inside a rank 2 hyperbolic root system, and found that the two concepts of subsystem are equivalent in almost all cases:

Theorem 1.1. *Let Δ is a hyperbolic rank 2 root system and let Γ be a subset of real roots in Δ. If a or $b = 1$ and $\Phi(\Gamma)$ is the set of all* short *real roots in Δ, then $\Delta(\Gamma) = \Delta$. In all other cases, the set of real roots in $\Delta(\Gamma)$ is $\Phi(\Gamma)$.*

Our classification also gives us the following result, which holds for either concept of subsystem:

Theorem 1.2. *If Δ is a rank 2 hyperbolic root system, then Δ contains symmetric rank 2 hyperbolic root subsystems of type $H(k, k)$ for infinitely many distinct $k \geq 3$. If Δ is non-symmetric of type $H(a, b)$, then it also contains non-symmetric rank 2 hyperbolic root subsystems of type $H(a\ell, b\ell)$ for infinitely many distinct $\ell \geq 2$.*

We also classify the rank 2 Φ-subsystems as finite, affine or hyperbolic systems:

Theorem 1.3. *Let Δ be a rank 2 root system and let Γ be a nonempty set of real roots in Δ.*

(i) *If Δ is finite, then $\Phi(\Gamma)$ is finite.*
(ii) *If Δ is affine of type $\widetilde{A}_1$, then $\Phi(\Gamma)$ has finite type A_1 or affine type $\widetilde{A}_1$.*
(iii) *If Δ is affine of type $\widetilde{A}_2^{(2)}$, then $\Phi(\Gamma)$ has finite type A_1, or affine type $\widetilde{A}_1$ or $\widetilde{A}_2^{(2)}$.*
(iv) *If Δ is hyperbolic, then $\Phi(\Gamma)$ has finite type A_1 or hyperbolic type.*

We mention the following related works: This work was inspired by the papers [Mor] and [Mor2] where the results of interest were stated without proof. Feingold and Nicolai ([FN], Theorem 3.1) gave a method for generating a subalgebra corresponding to a $\Delta(\Gamma)$–type root subsystem for a certain choice of real roots in any Kac–Moody algebra.

Tumarkin [T] gave a classification the sublattices of hyperbolic root lattices of the same rank. However, he requires conditions on the possible angles between roots that exclude all but a finite number of rank 2 hyperbolic root systems. In contrast, for our intended application to Kac-Moody groups, we require the explicit construction of the embedding of the simple roots of a subsystems into the ambient system, rather than just describing its root lattice.

The authors are very grateful to Chuck Weibel for his careful reading of the MSc thesis ([Sr]) of the third author. This research was greatly facilitated by experiments carried out in the computational algebra systems Magma [BCFS] and Maple [M].

2. Real Roots

Let $A = H(a, b)$[1] be the 2×2 generalized Cartan matrix

$$A = H(a, b) = (a_{ij})_{i,j=1,2} = \begin{pmatrix} 2 & -b \\ -a & 2 \end{pmatrix}$$

for positive integers a, b, with Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$, root system $\Delta = \Delta(A)$, and Weyl group $W = W(A)$. When $ab < 4$, A is positive definite and so Δ is finite. When $ab = 4$, A is positive semi-definite but not positive definite and so Δ is affine. When $ab > 4$, A is indefinite but every proper generalized Cartan submatrix is positive definite, and so A is hyperbolic. Without loss of generality, we assume that $a \geq b$.

Let $S = \{\alpha_1, \alpha_2\}$ denote a basis of simple roots of Δ. We have the simple root reflections

$$w_j(\alpha_i) = \alpha_i - a_{ij}\alpha_j$$

[1] This is the transpose of the generalized Cartan matrix A in [ACP].

for $i = 1, 2$ with matrices with respect to S

$$[w_1]_S = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}, \qquad [w_2]_S = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}.$$

The Weyl group $W = W(A) = \langle w_1, w_2 \rangle$.

Let

$$B = B(a,b) = \begin{pmatrix} 2a/b & -a \\ -a & 2 \end{pmatrix}$$

be a symmetrization of A. This defines the symmetric bilinear form $(u, v) = [u]_S^T B[v]_S$ and quadratic form $||u||^2 = (u, u)$, which are preserved under the action of W. So the set of real roots is

$$\Delta^{\text{re}} = W\alpha_1 \cup W\alpha_2.$$

Since Δ is finite, affine, or hyperbolic, the set of imaginary roots is

$$\Delta^{\text{im}} = \{\alpha \in \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 \mid \alpha \neq 0 \text{ and } ||\alpha||^2 \leq 0\}.$$

The finite root systems are given in Figure 1 and the affine roots systems are given in Figure 2. Examples of hyperbolic roots systems are given in Figures 3–6. These diagrams were created using Maple [M]. As usual we define the positive roots Δ_+ to be all roots with positive coefficients in terms of the simple roots, $\Delta_- := -\Delta_+$, $\Delta^{\text{re}}_\pm := \Delta^{\text{re}}_\pm$, and $\Delta^{\text{im}}_\pm := \Delta^{\text{im}}_\pm$.

Now $||\alpha_1||^2 = 2a/b$ and $||\alpha_2||^2 = 2$. So all real roots $x\alpha_1 + y\alpha_2$ in the orbit $W\alpha_1$ satisfy

$$ax^2 - abxy + by^2 = b,$$

and all real roots $x\alpha_1 + y\alpha_2$ in the orbit $W\alpha_2$ satisfy

$$ax^2 - abxy + by^2 = a.$$

These curves are displayed in Figures 1–6 as blue (resp. red) dotted lines. These curves are elliptical for finite systems, straight lines for affine systems, and hyperbolas for hyperbolic systems. If Δ is nonsymmetric ($a > b$) the roots in $W\alpha_1$ are called *long* and the roots in $W\alpha_2$ are called *short*. If Δ is symmetric ($a = b$) then all roots are considered to be *long*. Note that (with the exception of A_2), the real roots fall into two distinct orbits under the action of W. With have used red for the orbit of α_1, blue for the orbit of α_2, and black for the imaginary roots. The horizontal lines indicate the action of w_1 while the vertical lines indicate the action of w_2.

In case $a = b$, we may designate one simple root to be long and the other simple root to be short. For example, in A_2, all roots lie in the same orbit under the action of the associated Weyl group ([CCFP], [Hu1]) so all roots are of equal squared length 2, and each root can be thought to be long as well as short.

Referring to Figures 3-6, we see that the real roots lie on the branches of the hyperbolae

$$ax^2 - abxy + by^2 = b \text{ and } ax^2 - abxy + by^2 = a$$

with two branches above and two below the diagonal. The branches closest to the diagonal support the short roots and the branches furthest from the diagonal support the long roots. For a real root α, we will use the notation

$$\alpha^{LL},\ \alpha^{LU},\ \alpha^{SU},\ \alpha^{SL}$$

to signify that α is a long root on the lower branch, a long root on the upper branch, a short root on the upper branch or a short root on the lower branch respectively.

Redefine this notation...we need 2 different fonts for long and lower

j	-1	0	1
α_j^{LL}	$-\alpha_1 - a\alpha_2$	α_1	$(ab-1)\alpha_1 + a\alpha_2$
α_j^{LU}	$-\alpha_1$	$\alpha_1 + a\alpha_2$	$(ab-1)\alpha_1 + a(ab-2)\alpha_2$
α_j^{SU}	$-b\alpha_1 - \alpha_2$	α_2	$b\alpha_1 + (ab-1)\alpha_2$
α_j^{SL}	$-\alpha_2$	$b\alpha_1 + \alpha_2$	$b(ab-2)\alpha_1 + (ab-1)\alpha_2$

TABLE 1. Examples of real roots

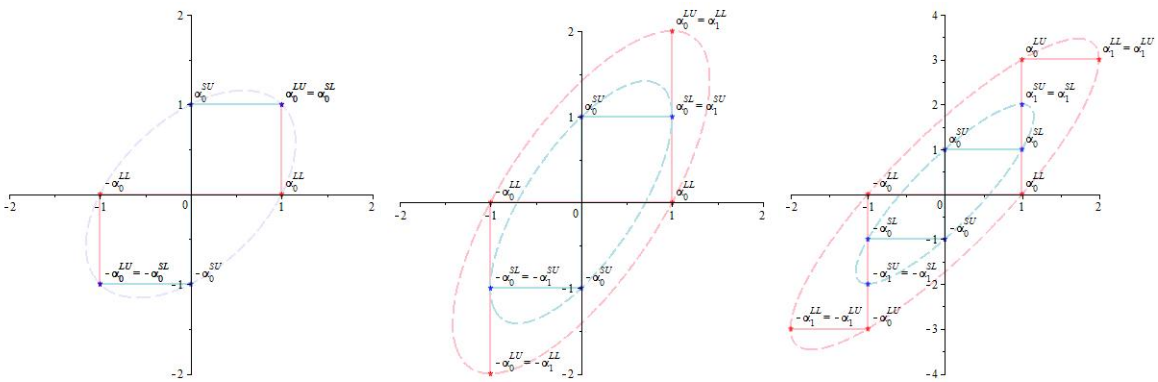

FIGURE 1. Root systems of types $H(1,1) = A_2$, $H(2,1) = B_2$, and $H(3,1) = G_2$

j	γ_j	η_j
0	0	1
1	1	$ab-1$
2	$ab-2$	$a^2b^2 - 3ab + 1$
3	$a^2b^2 - 4ab + 3$	$a^3b^3 - 5a^2b^2 + 6ab - 1$
4	$a^3b^3 - 6a^2b^2 + 10ab - 4$	$a^4b^4 - 7a^3b^3 + 15a^2b^2 - 10ab + 1$
5	$a^4b^4 - 8a^3b^3 + 21a^2b^2 - 20ab + 5$	$a^5b^5 - 9a^4b^4 + 28a^3b^3 - 35a^2b^2 + 15ab - 1$
6	$a^5b^5 - 10a^4b^4 + 36a^3b^3 - 56a^2b^2 + 35ab - 6$	$a^6b^6 - 11a^5b^5 + 45a^4b^4 - 84a^3b^3 + 70a^2b^2 - 21ab + 1$
7	$a^6b^6 - 12a^5b^5 + 55a^4b^4 - 120a^3b^3 + 126a^2b^2 - 56ab + 7$	$a^7b^7 - 13a^6b^6 + 66a^5b^5 - 165a^4b^4 + 210a^3b^3 - 126a^2b^2 + 28ab - 1$
8	$a^7b^7 - 14a^6b^6 + 78a^5b^5 - 220a^4b^4 + 330a^3b^3 - 252a^2b^2 + 84ab - 8$	$a^8b^8 - 15a^7b^7 + 91a^6b^6 - 286a^5b^5 + 495a^4b^4 - 462a^3b^3 + 210a^2b^2 - 36ab + 1$

TABLE 2. Values of η_j and γ_j for small j

For $j \in \mathbb{Z}$, we define

$$\alpha_j^{LL} := (w_1w_2)^j\alpha_1, \qquad \alpha_j^{LU} := (w_2w_1)^j w_2\alpha_1$$
$$\alpha_j^{SU} := (w_2w_1)^j\alpha_2, \qquad \alpha_j^{SL} := (w_1w_2)^j w_1\alpha_2.$$

All real roots are given by these four sequences. If $ab \geq 4$, then these are all distinct, and a root is positive if and only if $j \geq 0$. The roots for $j = -1, 0, 1$ are given in Table 1

The following lemma characterizes the real roots in terms of recursive sequences η_j and γ_j. Values of these sequences for small j are given in Table 2.

Lemma 2.1. *[ACP, Lemmas 3.2 and 3.3] For all integers j,*

$$\alpha_j^{LL} = \eta_j\alpha_1 + a\gamma_j\alpha_2, \qquad \alpha_j^{LU} = \eta_j\alpha_1 + a\gamma_{j+1}\alpha_2,$$
$$\alpha_j^{SU} = b\gamma_j\alpha_1 + \eta_j\alpha_2, \qquad \alpha_j^{SL} = b\gamma_{j+1}\alpha_1 + \eta_j\alpha_2,$$

where

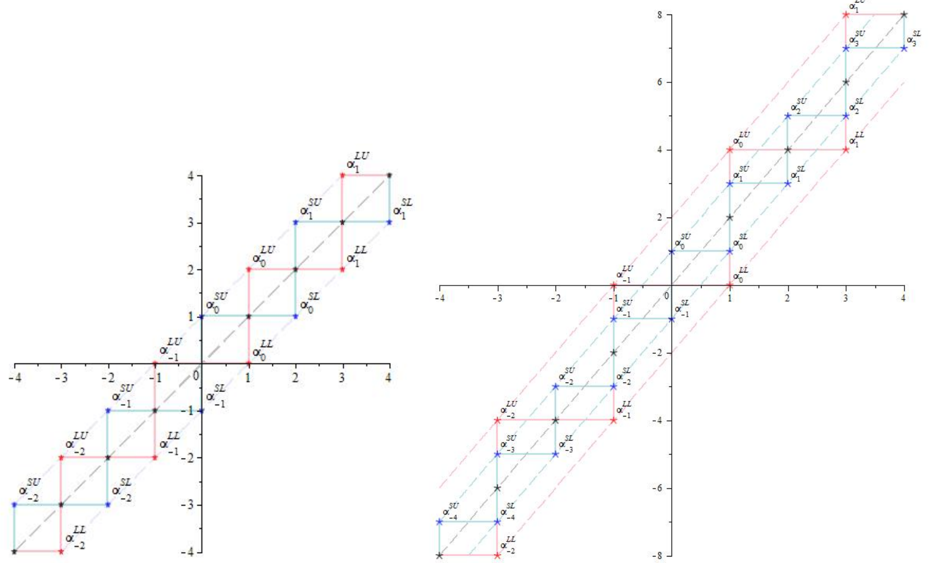

FIGURE 2. Root systems of types $H(2,2) = \widetilde{A}_1$ and $H(4,1) = \widetilde{A}_2^{(2)}$

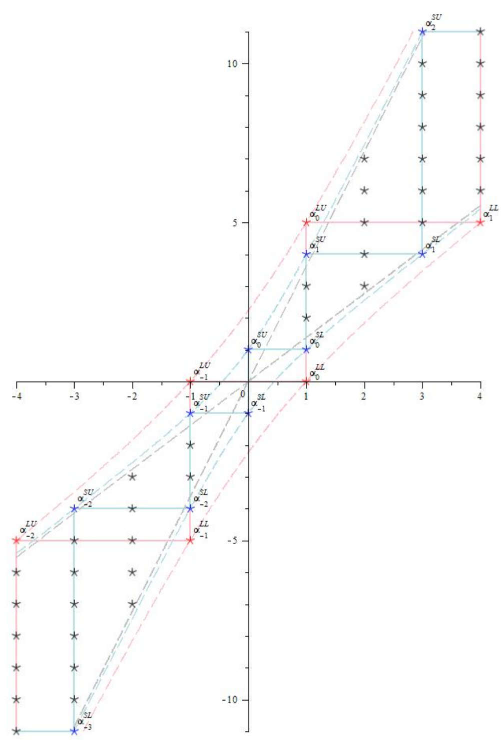

FIGURE 3. Root system of type $H(5,1)$

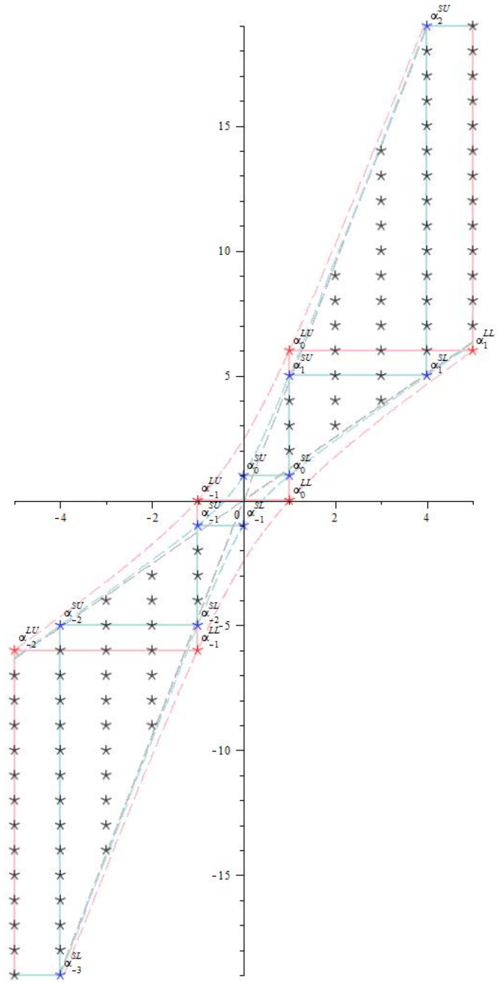

FIGURE 4. Root system of type $H(6,1)$

(i) $\gamma_0 = 0$, $\gamma_1 = 1$, $\eta_0 = 1$, $\eta_1 = ab - 1$;
(ii) $\eta_j = ab\gamma_j - \eta_{j-1}$;
(iii) $\gamma_j = \eta_{j-1} - \gamma_{j-1}$;
(iv) *both sequences* $X_j = \eta_j$ *and* γ_j *satisfy the recurrence relation*

$$X_j = (ab-2)X_{j-1} - X_{j-2}.$$

Note that these are both generalized Fibonacci sequences provided that $ab > 4$. In particular, γ_j is the Lucas sequence with parameters $P = ab - 2, Q = 1$.

As with all generalized Fibonacci sequences, we can also find closed-form equations:

Lemma 2.2. *[ACP, Proposition 4.3] For each* $j \in \mathbb{Z}$,

$$\eta_j = \frac{\psi_+^{j+1}}{\psi_+ - 1} + \frac{\psi_-^{j+1}}{\psi_- - 1} \quad \textit{and} \quad \gamma_j = \frac{\psi_+^j - \psi_-^j}{\sqrt{ab(ab-4)}},$$

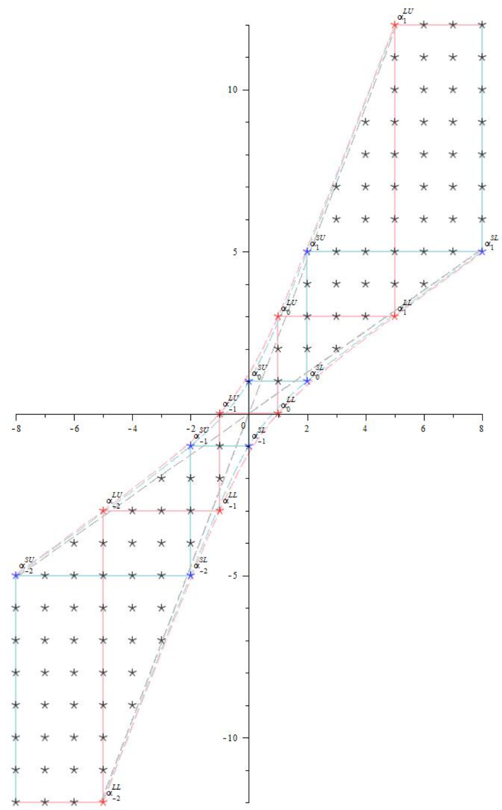

FIGURE 5. Root system of type $H(3,2)$

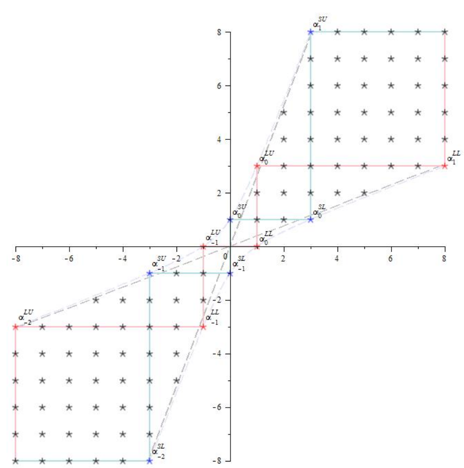

FIGURE 6. Root system of type $H(3,3)$

where $\psi_\pm := \frac{1}{2}\left((ab-2) \pm \sqrt{ab(ab-4)}\right)$.

Finally we state a useful and easy to prove lemma giving negatives of roots:

Lemma 2.3. *For all* $j \in \mathbb{Z}$*,* $\gamma_{-j} = -\gamma_j$ *and* $\eta_{-j} = -\eta_{j-1}$*. Also*

$$-\alpha_j^{LL} = \alpha_{-j-1}^{LU}, \quad -\alpha_j^{LU} = \alpha_{-j-1}^{LL}, \quad -\alpha_j^{SU} = \alpha_{-j-1}^{SL}, \quad -\alpha_j^{SL} = \alpha_{-j-1}^{SU}.$$

3. Sums of real roots

Let Δ be an infinite rank 2 root system of type $H(a,b)$ with $a \geq b$ and $ab \geq 4$. In this section we determine all real roots $\alpha, \beta \in \Delta$ for which $\alpha + \beta$ is also a real root.

We can write $\alpha = w\alpha_i$ for $i = 1$ or 2, and some $w \in W$. So replacing β by $w^{-1}\beta$ reduces the question to determining those $\beta \in \Delta^{\text{re}}$ for which $\alpha_i + \beta \in \Delta^{\text{re}}$. Now replacing β by $-\beta$ if $\beta \in \Delta^{\text{re}}_-$ reduces the question to determining those $\beta \in \Delta^{\text{re}}_+$ for which $\alpha_i \pm \beta \in \Delta^{\text{re}}$.

Lemma 3.1. *If* $a \geq b > 1$*, then*

$$0 = b\gamma_0 < \eta_0 < b\gamma_1 < \eta_1 < b\gamma_2 < \cdots,$$
$$0 = a\gamma_0 < \eta_0 < a\gamma_1 < \eta_1 < a\gamma_2 < \cdots.$$

In fact the gaps between sequence elements are nondecreasing, that is, for $j \geq 0$*,*

$$\eta_{j+1} - b\gamma_{j+1} \geq b\gamma_{j+1} - \eta_j \geq \eta_j - b\gamma_j,$$
$$\eta_{j+1} - a\gamma_{j+1} \geq a\gamma_{j+1} - \eta_j \geq \eta_j - a\gamma_j.$$

Proof. To see that the gaps in the sequences are nondecreasing, we apply Lemma 2.1 as follows:

$$\eta_{j+1} - b\gamma_{j+1} = (a-1)b\gamma_{j+1} - \eta_j \geq b\gamma_{j+1} - \eta_m = (b-1)\eta_j - b\gamma_j \geq \eta_j - b\gamma_j.$$

The other result is similar. □

The inequalities in Lemma 3.1 show that the real roots have the "staircase pattern" shown in Figure 7.

Proposition 3.2. *If* $a \geq b > 1$ *and* $\alpha, \beta \in \Delta^{\text{re}}$*, then* $\alpha + \beta \notin \Delta^{\text{re}}$*.*

Proof. Without loss of generality take $\alpha = \pm\alpha_i$ and $\beta \in \Delta^{\text{re}}_+$. From Figure 7, it is clear that $\beta \pm \alpha_i \in \Delta^{\text{re}}$ could only happen if one of the differences $\eta_{j+1} - \eta_j$ or $\gamma_{j+1} - \gamma_j$ was equal to 1. But the last lemma ensures that this never happens. □

The analogue of this result for the case $a > b = 1$ is considerably more tricky. We need to bound the parameters in the formulas of Lemma 2.2.

Lemma 3.3. *If* $ab > 4$*, then* $\psi_+ > 2.61$ *and* $0 < \psi_- < 0.39$*.*

Proof. Define real functions $\Psi_\pm(x) := \frac{1}{2}\left((x-2) \pm \sqrt{x(x-4)}\right)$ for which $\psi_\pm = \Psi_\pm(ab)$. We can now find

$$\Psi'_\pm(x) = \frac{1}{2}\left(1 \mp \frac{x-2}{\sqrt{x(x-4)}}\right)$$

and so Ψ_+ is increasing for $x \geq 5$ and Ψ_- is positive and decreasing for $x \geq 5$. Hence $\psi_+ \geq \Psi_+(5) > 2.61$ and $0 < \psi_- \leq \Psi_-(5) < 0.45$. □

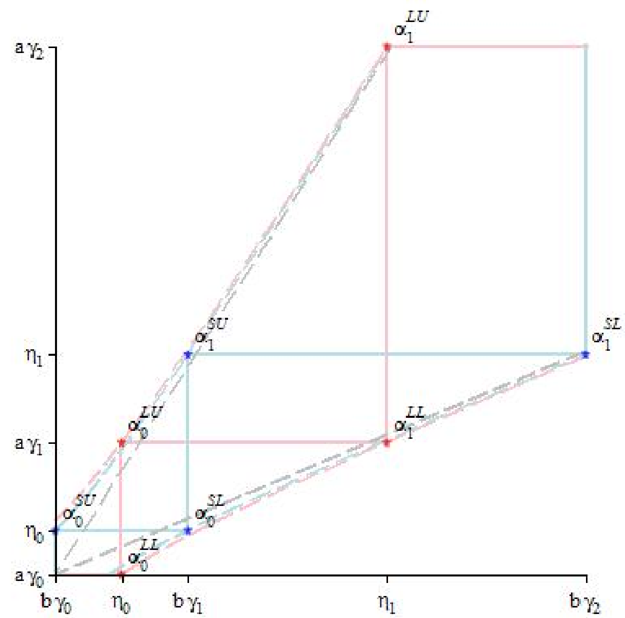

FIGURE 7. The positive real roots for $H(a, b)$ with $a \geq b > 1$

Define

$$\lambda := \frac{\psi_+}{\psi_+ - 1}, \qquad \mu := \frac{1}{\sqrt{ab(ab-4)}}.$$

We can now show that the sequences η_j and γ_j are each within a small constant of being exponential:

Lemma 3.4. *If $ab > 4$, then, for $j \geq 0$,*

$$\lambda\psi_+^j - 1.62 < \eta_j < \lambda\psi_+^j,$$
$$\mu\psi_+^j - 0.45 < \gamma_j < \mu\psi_+^j,$$

where $\psi_+ > 2.61$, $1 < \lambda < 1.62$, and $0 < \mu < 0.45$.

Proof. The conditions on ψ_+ and λ follow from the previous lemma. Using the similar reasoning we have $0 < \mu < 0.45$. Now

$$\lambda\psi_+^j - \eta_j = -\frac{\psi_-^{j+1}}{\psi_- - 1}.$$

As m increases, this quantity decreases since $0 < \psi_- < 1$. So we have $0 < \lambda\psi_+^j - \eta_j < 1.62$. Also, we have

$$\mu\psi_+^j - \gamma_j = \mu\psi_-^j$$

which implies that $0 < \mu\psi_+^j - \gamma_j < 0.45$. □

Lemma 3.5. *If $ab > 4$ and $b = 1$, then $0 < \mu < 0.45$, $1 < \lambda < 1.62$, $\lambda\psi^+ - a\mu \geq 2$, $\mu\psi_+^2 - \lambda > 3$.*

Proof. We have from above that $0 < \mu < 0.45$ and $1 < \lambda < 1.62$. Also, we have:

$$\lambda\psi^+ - a\mu = \frac{\psi_+^2}{\psi_+ - 1} - \frac{a}{\sqrt{a(a-4)}} \geq 2.62 - 0.45 > 2$$

and

$$\mu\psi_+^2 - \lambda = \frac{\psi_+^2}{\sqrt{a(a-4)}} - \frac{\psi_+}{\psi_+ - 1} > 0.45\psi_+^2 - 0 > 3.$$

□

Lemma 3.6. *If $a > 4$ and $b = 1$, then*

$$0 = \gamma_0 < \eta_0 = \gamma_1 < \gamma_2 < \eta_1 < \gamma_3 < \eta_2 < \cdots,$$
$$0 = a\gamma_0 < \eta_0 < \eta_1 < a\gamma_1 < \eta_2 < a\gamma_2 < \eta_3 < a\gamma_3 < \cdots.$$

Proof. For $j \geq 1$ we have $\gamma_{j+1} = \eta_j - \gamma_j < \eta_j$. Similarly for $j \geq 0$ we have $\eta_{j+1} = a\gamma_{j+1} - \eta_j < a\gamma_j$. Now

$$\begin{aligned} \eta_j - a\gamma_{j-1} &> \lambda\psi_+^j - 1.62 - a\mu\psi_+^{j-1} \\ &= (\lambda\psi^+ - a\mu)\psi^{j-1} - 1.62 \\ &\geq 2\psi^{j-1} - 1.62 > 0 \end{aligned}$$

and so $a\gamma_j < \eta_{j+1}$. And

$$\begin{aligned} \gamma_{j+1} - \eta_{j-1} &> \mu\psi_+^{j+1} - 0.45 - \lambda\psi_+^{j-1} \\ &= \psi_+^{j-1}(\mu\psi_+^2 - \lambda) - 0.45 \\ &\geq 3\psi_+^{j-1} - 0.45 > 0 \end{aligned}$$

and so $\gamma_{j+1} > \eta_{j-1}$. □

This lemma shows that the roots have the "staircase pattern" shown in Figure 8.

Proposition 3.7. *If $a > 4$, $b = 1$, and $\alpha, \beta, \alpha + \beta \in \Delta^{\mathrm{re}}$, then, for some $j \in \mathbb{Z}$,*

(i) $\{\alpha, \beta\} = \{\alpha_j^{LL}, \alpha_{-j}^{SU}\}$ *and* $\alpha + \beta = \alpha_j^{SL}$;
(ii) $\{\alpha, \beta\} = \{\alpha_j^{LU}, \alpha_{-j}^{SL}\}$ *and* $\alpha + \beta = \alpha_j^{SU}$;
(iii) $\{\alpha, \beta\} = \{\alpha_j^{SU}, \alpha_{j+1}^{SU}\}$ *and* $\alpha + \beta = \alpha_j^{LU}$; *or*
(iv) $\{\alpha, \beta\} = \{\alpha_j^{SL}, \alpha_{j+1}^{SL}\}$ *and* $\alpha + \beta = \alpha_j^{LL}$.

Proof. Suppose $\{\alpha, \beta\}$ contains a long root α_j^{LL} (resp. α_j^{LU}). Then this root can be written $w\alpha_1$ for $w = (w_1w_2)^j$ (resp. $w = (w_2w_1)^j w_2$). From Figure 8 and Lemma 3.6, the only real roots that give another real root when α_1 is added are α_2 and $-\alpha_1 - \alpha_2$. The first case can be written $\alpha_0^{LL} + \alpha_0^{SU} = \alpha_0^{SL}$. Multiply this by $w = (w_1w_2)^j$ to get (i); multiply this by $w = (w_2w_1)^j w_2$ to get (ii). The second case can be written $\alpha_0^{LU} + \alpha_{-1}^{SL} = \alpha_{-1}^{SU}$, which gives nothing new when multiplied to w.

Now suppose $\{\alpha, \beta\}$ contains only short roots, one of which is α_j^{SU} (resp. α_j^{SL}). Then this root can be written $w\alpha_2$ for $w = (w_2w_1)^j$ (resp. $w = (w_1w_2)^j w_1$). From Figure 8 and Lemma 3.6, the only real roots that give another short real root when α_2 is added are $\alpha_1 + (a-1)\alpha_2$ and $-\alpha_1 - a\alpha_2$. The first case can be written $\alpha_0^{SU} + \alpha_1^{SU} = \alpha_0^{LU}$. Multiply this by $w = (w_2w_1)^j$ to get (iii); multiply this by $w = (w_1w_2)^j w_1$ to get (iv). Again the second case gives nothing new. □

Note that Proposition 3.2 includes the affine type $H(2,2)$, but Proposition 3.7 does not apply to the affine type $H(4,1)$.

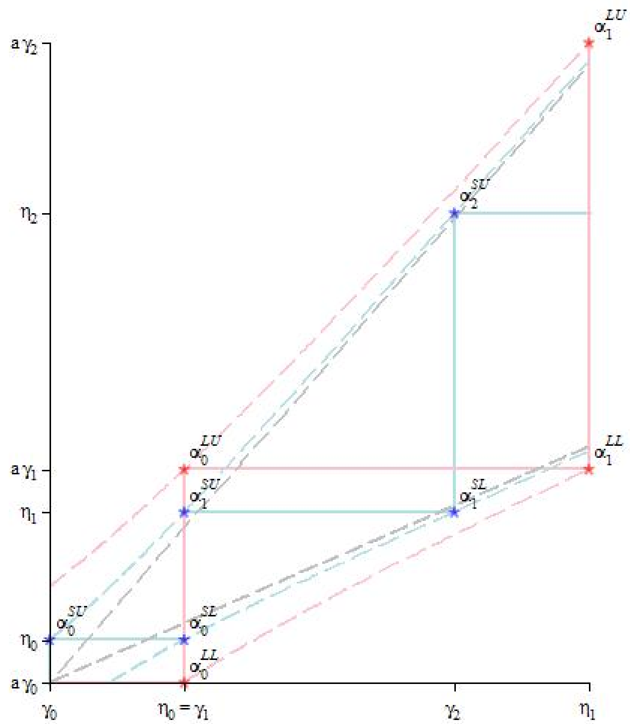

FIGURE 8. The positive real roots for $H(a,1)$ with $a > 4$

Proposition 3.8. *If $a = 4, b = 1$, and $\alpha, \beta, \alpha+\beta \in \Delta^{\mathrm{re}}$, then, for some $j, k \in \mathbb{Z}$,*

(i) $\{\alpha, \beta\} = \{\alpha_j^{LL}, \alpha_k^{SU}\}$ *and* $\alpha + \beta = \alpha_{2j-k}^{SL}$*;*
(ii) $\{\alpha, \beta\} = \{\alpha_j^{LU}, \alpha_k^{SL}\}$ *and* $\alpha + \beta = \alpha_{2j-k}^{SU}$*;*
(iii) $\{\alpha, \beta\} = \{\alpha_j^{SU}, \alpha_{j+2k+1}^{SU}\}$ *and* $\alpha + \beta = \alpha_{j+k}^{LU}$*; or*
(iv) $\{\alpha, \beta\} = \{\alpha_j^{SL}, \alpha_{j+2k+1}^{SL}\}$ *and* $\alpha + \beta = \alpha_{j+k}^{LL}$*.*

Proof. The recursion formulas in Lemma 2.1 imply that $\gamma_j = j$ and $\eta_j = 2j+1$. Referring to Figure 2, the result now follows by a similar argument to Proposition 3.7. $\square$

Now considering lengths of sums, we find the following by checking the cases of Propositions 3.2, 3.7, and 3.8:

Theorem 3.9. *Let Δ be an infinite rank 2 root system.*

(i) *If $\alpha, \beta, \alpha+\beta \in \Delta^{\mathrm{re}}$ with α and β short, then $\alpha+\beta$ is long.*
(ii) *If $\alpha, \beta, \alpha+\beta \in \Delta^{\mathrm{re}}$ with α short and β long, then $\alpha+\beta$ is short.*
(iii) *If $\alpha, \beta \in \Delta^{\mathrm{re}}$ with α and β long, then $\alpha+\beta \notin \Delta^{\mathrm{re}}$.*

We note that (i) and (iii) are not true in finite root systems of type A_2 or G_2. However there is a slightly weaker result that holds in any symmetrizable system:

Theorem 3.10. *Let Δ be a symmetrizable root system and suppose $\alpha, \beta, \alpha+\beta \in \Delta^{\mathrm{re}}$.*

(i) *If* $||\alpha||^2 = ||\beta||^2$*, then* $||\alpha+\beta||^2 = a||\alpha||^2$ *for some positive integer* a.
(ii) *If* $||\alpha||^2 \neq ||\beta||^2$*, then* $||\alpha+\beta||^2 = \min(||\alpha||^2, ||\beta||^2)$.

Proof. We only need to consider the rank 2 subsystem $\mathbb{Z}\{\alpha,\beta\}\cap\Delta$. These results are easily shown to be true if the subsystem has finite type A_2, B_2, or G_2, and they follow from the previous proposition if the subsystem is infinite. □

4. Subsystems

Root systems can be used to describe three different structures: Coxeter groups, Kac-Moody algebras and Kac-Moody groups. These three structures lead to two different concepts of subsystem, since the Lie correspondence ensures that Kac-Moody algebras and groups give the same subsystems. In this section, we describe these two types of subsystem, and classify all subsystems of infinite rank 2 root systems. We will see that the two concepts usually coincide, but not always.

Suppose Δ is a symmetrizable root system. For a Coxeter groups $W = W(\Delta)$, only the real roots Δ^{re} need to be considered. The appropriate substructure is a *reflection subgroup*, that is a subgroup generated by a set of reflections. Each reflection corresponds to a real root. For $\Gamma \subseteq \Delta^{\mathrm{re}}$, the reflection subgroup is

$$W_\Gamma = \langle w_\alpha : \alpha \in \Gamma\rangle.$$

Then W_Γ is also a Coxeter group and its real root system is

$$\Phi(\Gamma) = W_\Gamma\Gamma,$$

that is, the closure of Γ under the action of W_Γ. We call this a Φ*-subsystem*. Note that a Φ-subsystem consists entirely of real roots.

Let $\mathfrak{g} = \mathfrak{g}(\Delta)$ be the Kac-Moody algebra of Δ, and recall that the root elements x_α and $x_{-\alpha}$ generate a subalgebra isomorphic to $\mathfrak{sl}_2$ for all real roots α. We call this subalgebra $\mathfrak{sl}_2(\alpha)$. The correct substructure is a *fundamental Kac-Moody subalgebra*, which is a Kac-Moody subalgebra generated by the Cartan subalgebra $\mathfrak{h}$ and the $\mathfrak{sl}_2(\alpha)$ subalgebras for some collection of real roots α. For $\Gamma \subseteq \Delta^{\mathrm{re}}$, the fundamental Kac-Moody subalgebra is

$$\mathfrak{g}_\Gamma = \langle \mathfrak{h}, \mathfrak{sl}_2(\alpha) : \alpha \in \Gamma\rangle.$$

Then $\mathfrak{g}_\Gamma$ is a Kac-Moody algebra and its root system is

$$\Delta(\Gamma) = \mathbb{Z}\Gamma \cap \Delta,$$

that is the set of all roots in Δ that can be written as an integer linear combination of elements of Γ. We call this a $\Delta(\Gamma)$*-subsystem*. The Kac–Moody subalgebra of [FN], Theorem 3.1 is of this type. We also define $\Delta^{\mathrm{re}}(\Gamma) = \mathbb{Z}\Gamma \cap \Delta^{\mathrm{re}}$.

We now classify the Φ-subsystems in an infinite rank 2 root system Δ of type $H(a,b)$ for $a \geq b$ and $ab \geq 4$. Let $\Gamma \subseteq \Delta^{\mathrm{re}}$ be nonempty. First we note that $\Phi(\Gamma)$ is closed under negation, since $w_\alpha\alpha = -\alpha$. So, using the formulas of Lemma 2.3,

$$\Phi(\Gamma) = \{\alpha_j^{LL}, \alpha_{-j-1}^{LU}, \alpha_k^{SU}, \alpha_{-k-1}^{SL} \mid j \in I^L,\, k \in I^S\},$$

for some index sets $I^L, I^S \subseteq \mathbb{Z}$. Every real root has the form $\alpha = w\alpha_i$ for $i = 1,2$ and $w \in W$, so the reflection in α is $w_\alpha = ww_iw^{-1}$. We obtain the following formulas for the reflections corresponding to each real root:

$$w_j^{LL} = w_{-j-1}^{LU} = (w_1w_2)^{2j}w_1, \qquad w_j^{SU} = w_{-j-1}^{SL} = (w_2w_1)^{2j}w_2.$$

We can use this to easily prove formulas for the action of a reflection on a real root:

Lemma 4.1. *For all $j, k \in \mathbb{Z}$,*

$$w_k^{LL}\alpha_j^{LL} = -\alpha_{2k-j}^{LL}, \qquad w_k^{SU}\alpha_j^{SU} = -\alpha_{2k-j}^{SU},$$
$$w_k^{LL}\alpha_j^{SU} = -\alpha_{-2k-j-1}^{SU}, \qquad w_k^{SU}\alpha_j^{LL} = -\alpha_{-2k-j-1}^{LL}.$$

Lemma 4.2. *Given integers j and k:*

(i) *If $j, k \in I^L$, then $j + (k-i)\mathbb{Z} \subseteq I^L$.*
(ii) *If $j, k \in I^S$, then $j + (k-j)\mathbb{Z} \subseteq I^S$.*
(iii) *If $j \in I^L$, $k \in I^S$, then $j + (2j+2k+1)\mathbb{Z} \subseteq I^L$ and $k + (2j+2k+1)\mathbb{Z} \subseteq I^S$.*

Proof. Suppose I^S contains $\ell := j + (n-1)(k-j)$ and $m := j + n(k-j)$. Then Lemma 4.1 shows that $j+(n+1)(k-j) = 2\ell - m \in I^S$ and $j+(n-2)(k-j) = 2m-\ell \in I^S$. Part (i) now follows by bidirectional induction. Part (ii) is similar.

Let $d := 2j+2k+1$. Now suppose $j, j+nd \in I^L$ and $k, k+nd \in I^S$. Then $j-(n+1)d = -2k-(j+nd)-1 \in I^L$ and so $j+(n+1)d = 2j - (j-(n+1)d) \in I^S$. Similar arguments show that $j+(n-1)d \in I^S$ and $k + (n \pm 1)d \in I^L$. Part (iii) now follows by bidirectional induction. □

We can now classify the Φ-subsystems in terms of their index sets:

Proposition 4.3.

(i) *If I^S is empty, then $I^L = r + d\mathbb{Z}$ for some $r, d \in \mathbb{Z}$ with $d \geq 0$ and $0 \leq r < d$.*
(ii) *If I^L is empty, then $I^S = r + d\mathbb{Z}$ for some $r, d \in \mathbb{Z}$ with $d \geq 0$ and $0 \leq r < d$.*
(iii) *Otherwise, $I^L = r + (2d+1)\mathbb{Z}$ and $I^S = d - r + (2d+1)\mathbb{Z}$ for some $d \geq 0$ and $d \leq r \leq d$.*

Proof. (i) Suppose I^S empty and let $J = \{j \in \mathbb{Z} \mid \alpha_j^{LL} \in \Gamma \text{ or } \alpha_{-j-1}^{LU} \in \Gamma\}$, so $\Phi(\Gamma) = \Phi(\{\alpha_j^{LL} \mid j \in J\})$. If J contains a single element, then take r to be that element and $d = 0$. Otherwise, let d be the greatest common divisor of all the integers $j-k$ for $j, k \in J$ with $j \neq k$. Let r be the remainder of $j \in J$ divided by d, which is the same for all $j \in J$. Then standard properties of integer lattices together with Lemma 4.2(i) show that

$$J \subseteq r + d\mathbb{Z} \subseteq I^L.$$

It now suffices to show that $\{\alpha_j^{LL}, \alpha_{-j-1}^{LU} \mid j \in r + d\mathbb{Z}\}$ is a Φ-subsystem, but this follows immediately from Lemmas 2.3 and 4.1. The proof of (ii) is similar to (i).

(iii) The orbits of W on Δ^{re} are $W\alpha_1$ and $W\alpha_2$, so $\Phi(\Gamma) \cap W\alpha_1 = \{\alpha_j^{LL}, \alpha_{-j-1}^{LU} \mid i \in I^L\}$ and $\Phi(\Gamma) \cap W\alpha_2 = \{\alpha_j^{SU}, \alpha_{-j-1}^{SL} \mid j \in I^S\}$ are both Φ-subsystems in their own rights. By (i) and (ii), $I^L = r_1 + d_1\mathbb{Z}$ and $I^S = r_2 + d_2\mathbb{Z}$ for some $d_i \geq 0$, $0 \leq r_i < d_i$, for $i = 1, 2$. For every $m \in \mathbb{Z}$, we have $r_1 \in I^L$ and $r_2 + md_2 \in I^S$, so Lemma 4.2(iii) implies that $r_1 + (2r_1 + 2r_2 + 2md_2 + 1)m\mathbb{Z} \subseteq r_1 + d_1\mathbb{Z}$. Hence

$$d_1 \mid (2r_1 + 2r_2 + 1) + 2md_2, \qquad \text{for all } m \in \mathbb{Z}.$$

So $d_1 \mid 2r_1 + 2r_2 + 1$ and hence d_1 is odd, say $d_1 = 2d+1$. Also $d_1 \mid 2d_2$ and hence $d_1 \mid d_2$. Reversing the roles of I^L and I^S we also get $d_2 \mid d_1$, so $d_1 = d_2 = 2d+1$. We can choose r such that $r \equiv r_1 \pmod{2d+1}$ and $-d \leq r \leq d$, so that $I^L = r + (2d+1)\mathbb{Z}$. Finally $2r + 2r_2 + 1 \equiv 0 \pmod{2d+1}$, so

$$r_2 \equiv r_2 + 2dr + 2dr_2 + d \equiv r_2 - r - r_2 + d \equiv d - r \pmod{2d+1},$$

and hence $I^S = d - r + (2d+1)\mathbb{Z}$. □

Theorem 4.4. *Let Δ be an infinite rank 2 root system of type $H(a,b)$ with $a \geq b$ and $ab \geq 4$. Every nonempty Φ-subsystem of Δ has simple roots, Cartan matrix, and inner product matrix given by one of the rows in Table 3 where $\delta_d := \eta_d - \eta_{d-1}$ and $\epsilon_d := \gamma_{d+1} - \gamma_d$. In particular all Φ-subsystems of Δ have rank at most 2.*

Type	Integer conditions	Simple roots	Cartan Matrix	Inner product matrix
I_L	r arbitrary	α_r^{LL}	A_1	$\frac{a}{b}A_1$
I_S	r arbitrary	α_r^{SU}	A_1	A_1
II_L	$d > 0, 0 \leq r < d$	$\alpha_r^{LL}, \alpha_{d-r-1}^{LU}$	$H(\delta_d, \delta_d)$	$\frac{a}{b}H(\delta_d, \delta_d)$
II_S	$d > 0, 0 \leq r < d$	$\alpha_r^{SU}, \alpha_{d-r-1}^{SL}$	$H(\delta_d, \delta_d)$	$H(\delta_d, \delta_d)$
II_{LS}	$d \geq 0, -d \leq r \leq d$	$\alpha_r^{LL}, \alpha_{d-r}^{SU}$	$H(a\epsilon_d, b\epsilon_d)$	$B(a\epsilon_d, b\epsilon_d)$

TABLE 3. Φ-subsystems of rank 2 root systems

d	$\delta_d = \eta_d - \eta_{d-1}$	$\epsilon_d = \gamma_{d+1} - \gamma_d$
0		1
1	$ab-2$	$ab-3$
2	$a^2b^2 - 4ab + 2$	$a^2b^2 - 5ab + 5$
3	$a^3b^3 - 6a^2b^2 + 9ab - 2$	$a^3b^3 - 7a^2b^2 + 14ab - 7$
4	$a^4b^4 - 8a^3b^3 + 20a^2b^2 - 16ab + 2$	$a^4b^4 - 9a^3b^3 + 27a^2b^2 - 30ab + 9$
5	$a^5b^5 - 10a^4b^4 + 35a^3b^3 - 50a^2b^2 + 25ab - 2$	$a^5b^5 - 11a^4b^4 + 44a^3b^3 - 77a^2b^2 + 55ab - 11$
6	$a^6b^6 - 12a^5b^5 + 54a^4b^4 - 112a^3b^3 + 105a^2b^2 - 36ab + 2$	$a^6b^6 - 13a^5b^5 + 65a^4b^4 - 156a^3b^3 + 182a^2b^2 - 91ab + 13$

TABLE 4. Values of δ_d and ϵ_d for small d

Proof. Let Φ' be a Φ-subsystem of Δ. First suppose that $\Phi' \subseteq W\alpha_1$. Then Proposition 4.3(i) implies that $\Phi' = \{\alpha_j^{LL}, \alpha_{-j-1}^{LU} \mid j \in r + d\mathbb{Z}\}$, for some $d \geq 0$ and $0 \leq r < d$. If $d = 0$, this gives us type I_L. Otherwise it is easily shown that every positive root in Φ' is a positive linear combination of α_r^{LL} and α_{d-r-1}^{LU}, so this forms a base. The Cartan matrix and inner product matrix can be computed directly from the base. For example, if the Cartan matrix is (c_{ij}) then

$$\begin{aligned} c_{12} &= \frac{2(\alpha_r^{LL}, \alpha_{d-r-1}^{LU})}{(\alpha_r^{LL}, \alpha_r^{LL})} = \frac{b}{a}(\alpha_r^{LL}, \alpha_{d-r-1}^{LU}) = \frac{b}{a}\left((w_1w_2)^r\alpha_1, (w_1w_2)^r\alpha_{d-1}^{LU}\right) = \frac{b}{a}(\alpha_1, \alpha_{d-1}^{LU}) \\ &= \frac{b}{a}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}2a/b & -a \\ -a & 2\end{pmatrix}\begin{pmatrix}\eta_{d-1} \\ a\gamma_d\end{pmatrix} = \frac{b}{a}\left(2\frac{a}{b}\eta_{d-1} - a^2\gamma_d\right) = 2\eta_{d-1} - ab\gamma_d = \eta_{d-1} - \eta_d = -\delta_d, \end{aligned}$$

where the second last equality follows from Lemma 2.1(ii). This gives type II_L.

Similarly we get types I_S and II_S from Proposition 4.3(ii), and type II_{LS} from Proposition 4.3(iii). $\square$

Values of δ_d and ϵ_d for small d are given in Table 4.

We can also classify of Φ-subsystems as finite, affine or hyperbolic systems:

Theorem 4.5. *Let Δ be a rank 2 root system and let Γ be a nonempty set of real roots in Δ.*

(i) *If Δ is finite, then $\Phi(\Gamma)$ is finite.*
(ii) *If Δ is affine of type $\widetilde{A}_1$, then $\Phi(\Gamma)$ has finite type A_1 or affine type $\widetilde{A}_1$.*
(iii) *If Δ is affine of type $\widetilde{A}_2^{(2)}$, then $\Phi(\Gamma)$ has finite type A_1, or affine type $\widetilde{A}_1$ or $\widetilde{A}_2^{(2)}$.*
(iv) *If Δ is hyperbolic, then $\Phi(\Gamma)$ has finite type A_1 or hyperbolic type.*

Proof. Part (i) is clear. The finite type A_1 occurs exactly when $\Gamma \subseteq \{\pm\alpha\}$, so we will assume from now on that this is not the case.

If Δ is affine, then $ab = 4$ and it is easy to show from the recursion formulas in Lemma 2.1 that $\delta_d = \eta_d - \eta_{d-1} = 2$ and $\epsilon_d = \gamma_{d+1} - \gamma_d = 1$. Parts (ii) and (iii) now follow.

If Δ hyperbolic, then $ab > 4$, and so for $d > 1$

$$\delta_d = \eta_d - \eta_{d-1} = (ab-2)\eta_{d-1} - \eta_{d-2} - \eta_{d-1} > 2\eta_{d-1} - \eta_{d-2} - \eta_{d-1} = \eta_{d-1} - \eta_{d-2} = \delta_{d-1}.$$

By induction we get $\delta_d \geq \delta_1 = (ab-1)-1 = ab-2 > 2$ for all $d > 0$. It now follows that $H(\delta_d, , \delta_d)$ is hyperbolic since ${\delta_d}^2 > 4$.

A similar argument shows that $\epsilon_d > \epsilon_{d-1}$ for $d > 0$, and so $\epsilon_d \geq \epsilon_0 = 1$ for $d \geq 0$. and so $H(a\epsilon_d, b\epsilon_d)$ is hyperbolic. □

As part of the last proof we showed that the sequences δ_d and ϵ_d are strictly increasing when Δ is hyperbolic, so Theorem 1.2 is now proved for Φ-subsystems.

We now consider the classification of Δ-subsystems of Δ. Let $\Gamma \subseteq \Delta^{\mathrm{re}}$ nonempty and recall that $\Delta(\Gamma) = \mathbb{Z}\Gamma \cap \Delta$, $\Delta^{\mathrm{re}}(\Gamma) = \mathbb{Z}\Gamma \cap \Delta^{\mathrm{re}}$. Since the imaginary roots of an affine or hyperbolic root system are just the linear combinations of real roots with nonpositive norm, it will suffice to describe $\Delta^{\mathrm{re}}(\Gamma)$. From the definition of a reflection, we can see that $w_\alpha \Delta^{\mathrm{re}}(\Gamma) \subseteq \Delta^{\mathrm{re}}(\Gamma)$ for all $\alpha \in \Gamma$, and so

$$\Phi(\Gamma) \subseteq \Delta^{\mathrm{re}}(\Gamma).$$

We also have $\Phi(\Delta^{\mathrm{re}}(\Gamma)) = \Delta^{\mathrm{re}}(\Gamma)$, so the real roots of a Δ-subsystem always form a Φ-subsystem, but possibly for a different set of generators. The classification of Δ subsystems reduces to divisibility properties for the sequences η_j and γ_j.

Lemma 4.6. *Let $a \geq b \geq 1$ with $ab \geq 4$, and let $d \geq 0$, $i \in \mathbb{Z}$. Then*

$$\gamma_d \delta_{j-d} = \gamma_j - \gamma_{j-2d}, \tag{1}$$
$$\eta_d \epsilon_{j-d-1} = \gamma_j - \gamma_{j-2d-1}, \tag{2}$$
$$\eta_d \delta_{j-d} = \eta_j - \eta_{j-2d-1}, \tag{3}$$
$$ab\gamma_d \epsilon_{j-d} = \eta_j - \eta_{j-2d}. \tag{4}$$

Proof. The equations are easy to prove for $d = 0$. Note that

$$\delta_{j-1} = \eta_{j-1} - \eta_{j-2} = (ab\gamma_j - \eta_j) - (ab\gamma_{j-1} - \eta_{j-1}) = ab\epsilon_{j-1} - \delta_j, \quad \text{and}$$
$$\epsilon_{j-1} = \gamma_j - \gamma_{j-1} = (\eta_j - \gamma_{j+1}) - (\eta_{j-1} - \gamma_j) = \delta_j - \epsilon_j.$$

Assume all of the equations hold for $d \leq e$. First we prove (1) and (4) for $d = e+1$:

$$\begin{aligned}
\gamma_{e+1}\delta_{j-e-1} &= (\eta_e - \gamma_e)(ab\epsilon_{j-e-1} - \delta_{j-e}) \\
&= ab\eta_e\epsilon_{j-e-1} - ab\gamma_e\epsilon_{j-e-1} - \eta_e\delta_{j-e} + \gamma_e\delta_{j-e} \\
&= ab(\gamma_j - \gamma_{j-2e-1}) - (\eta_{j-1} - \eta_{j-2e}) - (\eta_j - \eta_{j-2e-1}) + (\gamma_j - \gamma_{j-2e}) \\
&= \gamma_j + (ab\gamma_j - \eta_{j-1} - \eta_j) + (\eta_{j-2e-1} - ab\gamma_{j-2e-1}) + (\eta_{j-2e} - \gamma_{j-2e}) \\
&= \gamma_j + 0 - \eta_{j-2e-2} - \gamma_{j-2e-1} = \gamma_j - \gamma_{j-2e-2}, \\
ab\gamma_{e+1}\epsilon_{j-e-1} &= ab(\eta_e - \gamma_e)(\delta_{j-e} - \epsilon_{j-e}) \\
&= ab\eta_e\delta_{j-e} - ab\gamma_e\delta_{j-e} - ab\eta_e\epsilon_{j-e} + ab\gamma_e\epsilon_{j-e} \\
&= ab(\eta_j - \eta_{j-2e-1}) - ab(\gamma_j - \gamma_{j-2e}) - ab(\gamma_{j+1} - \gamma_{j-2e}) + (\eta_j - \eta_{j-2e}) \\
&= \eta_j + ab(\eta_j - \gamma_j - \gamma_{j+1}) - ab(\eta_{j-2e-1} - \gamma_{j-2e}) + (ab\gamma_{j-2e} - \eta_{j-2e}) \\
&= \eta_j + 0 - ab\gamma_{j-2e-1} + \eta_{j-2e-1} = \eta_j - \eta_{j-2e-2}.
\end{aligned}$$

Now we can prove (2) and (3) for $d = e + 1$:

$$\begin{aligned}
\eta_{e+1}\epsilon_{j-e-2} &= (ab\gamma_{e+1} - \eta_e)(\delta_{j-e-1} - \epsilon_{j-e-1}) \\
&= ab\gamma_{e+1}\delta_{j-e-1} - \eta_e\delta_{j-e-1} - ab\gamma_{e+1}\epsilon_{j-e-1} + \eta_e\epsilon_{j-e-1} \\
&= ab(\gamma_j - \gamma_{j-2e-2}) - (\eta_{j-1} - \eta_{j-2e-2}) - (\eta_j - \eta_{j-2e-2}) + (\gamma_j - \gamma_{j-2e-1}) \\
&= \gamma_j + (ab\gamma_j - \eta_{j-1} - \eta_j) - (ab\gamma_{j-2e-2} - \eta_{j-2e-2}) + (\eta_{j-2e-2} - \gamma_{j-2e-1}) \\
&= \gamma_j + 0 - \eta_{j-2e-3} + \gamma_{j-2e-2} = \gamma_j - \gamma_{i-2e-3}, \\
\eta_{e+1}\delta_{j-e-1} &= (ab\gamma_{e+1} - \eta_e)(ab\epsilon_{j-e-1} - \delta_{j-e}) \\
&= (ab)^2\gamma_{e+1}\epsilon_{j-e-1} - ab\eta_e\epsilon_{j-e-1} - ab\gamma_{e+1}\delta_{j-e} + \eta_e\delta_{j-e} \\
&= ab(\eta_j - \eta_{j-2e-2}) - ab(\gamma_j - \gamma_{j-2e-1}) - ab(\gamma_{j+1} - \gamma_{j-2e-1}) + (\eta_i - \eta_{j-2e-1}) \\
&= \eta_j + ab(\eta_j - \gamma_j - \gamma_{j+1}) - ab(\eta_{j-2e-2} - \gamma_{j-2e-1}) + (ab\gamma_{j-2e-1} + \eta_{j-2e-1}) \\
&= \eta_j + 0 - ab\gamma_{j-2e-2} + \eta_{j-2e-2} = \eta_j - \eta_{j-2e-3}.
\end{aligned}$$

By induction, the equations are now proved for $d \geq 0$. □

Lemma 4.7. *Let $a \geq b \geq 1$ with $ab \geq 4$, and let $d \geq 0$, $j \in \mathbb{Z}$.*

(i) $\gcd(a, \eta_j) = \gcd(b, \eta_j) = 1$.
(ii) $\gamma_d \mid \gamma_j$ *iff* $j \in d\mathbb{Z}$.
(iii) $\eta_d \mid \gamma_j$ *iff* $j \in (2d+1)\mathbb{Z}$.
(iv) $\eta_d \mid \eta_j$ *iff* $j \in d + (2d+1)\mathbb{Z}$.
(v) $\gamma_d \mid \eta_j$ *iff* $d = 1$, *when* $ab > 4$.
(vi) $\gamma_d \mid \eta_j$ *iff* $d = 2e+1$ *is odd and* $j \in e + (2e+1)\mathbb{Z}$, *when* $ab = 4$.

Proof. (i) This follows from the fact that $\eta_j \equiv (-1)^j \pmod{ab}$, which is easily proved by induction.

(ii) Let $j = r + 2md$ for some $m, r \in \mathbb{Z}$ with $-d < r \leq d$. Then repeated application of (1) gives $\gamma_j \equiv \gamma_r \pmod{\gamma_d}$. If $j \in d\mathbb{Z}$, then $r = 0$ or d, and so $\gamma_j \equiv 0 \pmod{\gamma_d}$. Otherwise we have $0 < r < d$, so that $0 < \gamma_r < \gamma_d$; or $-d < r < 0$, so that $-\gamma_d < \gamma_r < 0$ since $\gamma_r = -\gamma_{-r}$. In either case $\gamma_j \not\equiv 0 \pmod{\gamma_d}$.

(iii) Let $j = r + m(2d+1)$ for some $m, r \in \mathbb{Z}$ with $-d \leq r \leq d$. Then repeated application of (2) gives $\gamma_j \equiv \gamma_r \pmod{\eta_d}$. If $j \in (2d+1)\mathbb{Z}$, then $r = 0$ and so $\gamma_j \equiv 0 \pmod{\gamma_d}$. Otherwise we have $0 < r \leq d$, so that $0 < \gamma_r < \eta_d$; or $-d < r < 0$, so that $-\eta_d < \gamma_r < 0$. In either case $\gamma_j \not\equiv 0 \pmod{\gamma_d}$.

(iv) Let $j = r + m(2d+1)$ for some $m, r \in \mathbb{Z}$ with $-d \leq r \leq d$. Then repeated application of (3) gives $\eta_j \equiv \eta_r \pmod{\eta_d}$. If $j \in d + (2d+1)\mathbb{Z}$, then $r = d$ and so $\eta_j \equiv \eta_d \equiv 0 \pmod{\eta_d}$. Otherwise we have $0 < r < d$, so that $0 < \eta_r < \eta_d$; or $-d \leq r < 0$, so that $-\eta_d < \eta_r < 0$ since $\eta_r = -\eta_{-r+1}$. In either case $\eta_j \not\equiv 0 \pmod{\eta_d}$.

(v) Suppose $ab > 4$. Let $j = r + 2md$ for some $m, r \in \mathbb{Z}$ with $-d < r \leq d$. If $d = 1$, then $\gamma_d = 1$ and so $\gamma_d \mid \eta_d$. Otherwise repeated application of (4) gives $\eta_j \equiv \eta_r \pmod{\gamma_d}$. Now $\eta_r = \gamma_r + \gamma_{r+1}$ by Lemma 2.1(iii). Since $ab > 4$, we have $\gamma_{j+1} \geq 2\gamma_j$ for all $j \geq 0$. If $0 < r < d-1$, then $0 < \gamma_r + \gamma_{r+1} \leq \frac{3}{4}\gamma_d$, so $\gamma_r + \gamma_{r+1} \not\equiv 0 \pmod{\gamma_d}$. If $r = d$, then $\gamma_r + \gamma_{r+1} \equiv \gamma_{d+1} \not\equiv 0 \pmod{\gamma_d}$ by (ii). If $r = d-1$, then $\gamma_r + \gamma_{r+1} \equiv \gamma_{d-1} \not\equiv 0 \pmod{\gamma_d}$. If $-d < r < 0$, then we can use the fact that $\gamma_{-j} = -\gamma_j$.

(vi) Since $\gamma_d = d$ and $\eta_j = 2j+1$, $\gamma_d \mid \eta_j$ iff $d = 2e+1$ is odd and $d \mid 2j+1$. In this case, $2j+1 \equiv 0 \pmod{2e+1}$ iff $j \equiv (-2e)j \equiv 2j(-e) \equiv (-1)(-e) \equiv e \pmod{2e+1}$. □

Theorem 4.8. *Let Δ be an infinite rank 2 root system of type $H(a,b)$ with $a \geq b$ and $ab \geq 4$. Let $\Gamma \subseteq \Delta^{\mathrm{re}}$ be nonempty.*

(i) *If $a > 4$, $b = 1$ and $\Phi(\Gamma)$ is the subsystem consisting of all short roots in Δ^{re}, then $\Delta^{\mathrm{re}}(\Gamma) = \Delta^{\mathrm{re}}$.*

(ii) *If $a = 4$, $b = 1$ and $\Phi(\Gamma)$ is a subsystem of type II_S with base $\alpha_r^{SU}, \alpha_{d-r-1}^{SL}$ for some odd $d = 2e+1$ and $0 \le r < d$, then $\Delta^{\rm re}(\Gamma)$ is a subsystem of type II_{LS} with base $\alpha_s^{LL}, \alpha_{e-s}^{SU}$ where $s \equiv e - r \pmod d$ and $-e \le s \le e$.*

(iii) *In all other cases, $\Delta^{\rm re}(\Gamma) = \Phi(\Gamma)$.*

Proof. If $\Phi(\Gamma)$ has type I_L or I_S, then it is clear that $\Phi(\Gamma) = \Delta^{\rm re}(\Gamma)$.

Suppose $\Phi(\Gamma)$ has type II_L. Since $\Phi(\Gamma) = (w_1w_2)^r\Phi(\{\alpha_0^{LL}, \alpha_{d-1}^{LU}\})$, it suffices to consider $r = 0$. Now $\alpha_0^{LL} = \alpha_1$ and $\alpha_{d-1}^{LU} = \eta_{d-1}\alpha_1 + a\gamma_d\alpha_2$, so

$$\Delta^{\rm re}(\Gamma) = \mathbb{Z}\{\alpha_0^{LL}, \alpha_{d-1}^{LU}\} \cap \Delta^{\rm re} = \mathbb{Z}\{\alpha_1, a\gamma_d\alpha_2\} \cap \Delta^{\rm re}.$$

Now $\alpha_j^{LL} = \eta_j\alpha_1 + a\gamma_j\alpha_2$ is in $\Delta^{\rm re}(\Gamma)$ iff $\gamma_d \mid \gamma_j$ iff $j \in d\mathbb{Z}$ by Lemma 4.7(ii). And $\alpha_j^{SU} = b\gamma_j\alpha_1 + \eta_i\alpha_2$ is in $\Delta^{\rm re}(\Gamma)$ iff $a\gamma_d \mid \eta_j$ which is not possible by Lemma 4.7(i) since $a > 1$. Hence $\Delta^{\rm re}(\Gamma) = \Phi(\Gamma)$.

Suppose $\Phi(\Gamma)$ has type II_{LS}. Since $\Phi(\Gamma) = (w_2w_1)^{d-r}\Phi(\{\alpha_d^{LL}, \alpha_0^{SU}\})$, it suffices to consider $r = d$. Now $\alpha_d^{LL} = \eta_d\alpha_1 + a\eta_d\alpha_2$ and $\alpha_0^{SU} = \alpha_2$, so

$$\Delta^{\rm re}(\Gamma) = \mathbb{Z}\{\alpha_d^{LL}, \alpha_0^{SU}\} \cap \Delta^{\rm re} = \mathbb{Z}\{\eta_d\alpha_1, \alpha_2\} \cap \Delta^{\rm re}.$$

Now $\alpha_j^{LL} = \eta_j\alpha_1 + a\gamma_j\alpha_2$ is in $\Delta^{\rm re}(\Gamma)$ iff $\eta_d \mid \eta_j$ iff $j \in d + (2d+1)\mathbb{Z}$ by Lemma 4.7(iv). And $\alpha_j^{SU} = b\gamma_j\alpha_1 + \eta_j\alpha_2$ is in $\Delta^{\rm re}(\Gamma)$ iff $\eta_d \mid b\gamma_i$ iff $j \in (2d+1)\mathbb{Z}$ by Lemma 4.7(iii). Hence $\Delta^{\rm re}(\Gamma) = \Phi(\Gamma)$.

Finally suppose $\Phi(A)$ has type II_S. Since $\Phi(\Gamma) = (w_2w_1)^r\Phi(\{\alpha_0^{SU}, \alpha_{d-1}^{SL}\})$, it suffices to consider $r = 0$. Now $\alpha_0^{SU} = \alpha_2$ and $\alpha_{d-1}^{SL} = b\gamma_d\alpha_1 + \eta_{d-1}\alpha_2$, so

$$\Delta^{\rm re}(\Gamma) = \mathbb{Z}\{\alpha_0^{LL}, \alpha_{d-1}^{LU}\} \cap \Delta^{\rm re} = \mathbb{Z}\{b\gamma_d\alpha_1, \alpha_2\} \cap \Delta^{\rm re}.$$

Now $\alpha_j^{SU} = b\gamma_i\alpha_1 + \eta_j\alpha_2$ is in $\Delta^{\rm re}(\Gamma)$ iff $\gamma_d \mid \gamma_j$ iff $j \in d\mathbb{Z}$. And $\alpha_j^{LL} = \eta_j\alpha_1 + a\gamma_j\alpha_2$ is in $\Delta^{\rm re}(\Gamma)$ iff $b\gamma_d \mid \eta_j$. By Lemma 4.7(i), (v), and (vi), this can only happen if $a > 4$, $b = 1$ and $d = 1$; or $a = 4$, $b = 1$ and d odd. If $a > 4$, $b = 1$, and $d = 1$, then $\Phi(\Gamma)$ is the set of all short real roots, and $b\gamma_d \mid \eta_j$ for all j so $\Delta^{\rm re}(\Gamma) = \Delta^{\rm re}$. If $a = 4$, $b = 1$, and $d = 2e+1$, then $b\gamma_d \mid \eta_i$ iff $j \in e + (2e+1)\mathbb{Z}$, so $I^S = (2e+1)\mathbb{Z}$, $I^L = e + (2e+1)\mathbb{Z}$, and hence $\Delta^{\rm re}(\Gamma)$ has type II_{LS} with the given basis. In all other cases $\Delta^{\rm re}(\Gamma) = \Phi(\Gamma)$. □

Theorem 1.1 and Theorem 1.2 for Δ-subsystems now follow.

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