GROUPS ACTING SIMPLY TRANSITIVELY ON VERTEX SETS OF HYPERBOLIC TRIANGULAR BUILDINGS

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Abstract

We construct and classify all groups, given by triangular presentations associated to the smallest thick generalized quadrangle, that act simply transitively on the vertices of hyperbolic triangular buildings of the smallest non-trivial thickness. Our classification shows 23 non-isomorphic torsion free groups (obtained in an earlier work) and 168 non-isomorphic torsion groups acting on one of two possible buildings with the smallest thick generalized quadrangle as the link of each vertex. In analogy with the $\widetilde{A}_2$ case, we find both torsion and torsion free groups acting on the same building.

1 Introduction

Intensive study of groups acting simply transitively on Euclidean buildings was initiated in [8] and [9]. This work has had a considerable impact in several directions. For example, this led to new examples of fake projective planes ([18]), and, finally, to their full classification ([10]). In the $\widetilde{A}_2$ case, there are two non-isomorphic buildings of minimal non-trivial thickness admitting a simply transitive action, and eight isomorphism classes of groups acting on these two buildings simply transitively and in a type preserving manner. Within this isomorphism class, five groups are torsion free and three have torsion [9].

In this paper we study groups acting simply transitively on vertex sets of hyperbolic buildings with the smallest thick generalized quadrangle as the link of each vertex. The torsion free groups acting simply transitively on such buildings were classified in [16]. Here we classify triangle presentations associated to the smallest thick generalized quadrangle, as well as groups with torsion coming from these presentations. We wish to emphasize that our groups act simply transitively on vertices of hyperbolic buildings, where as other authors have considered groups acting on, for example, chambers (see [17]) or panels (see [12]).

It is known ([19]) that up to isomorphism, there are only two possible triangular hyperbolic buildings, with the smallest generalized quadrangle as the link of each vertex, admitting a simply transitive action. We note that in the formulation of the main theorem in [19] the appropriate polygonal complexes are required to be symmetric, but the proof works also for buildings admitting simply transitive actions.

In [16] the authors constructed, for any n, torsion free groups acting cocompactly on hyperbolic buildings with n-gonal chambers. Our strategy in this paper is to modify the construction in [16] to include the torsion case as well.

Our classification shows 168 non-isomorphic torsion groups acting on vertices of one of two possible buildings with the smallest thick generalized quadrangle as the link of each vertex. In analogy with the $\widetilde{A}_2$ case, we find both torsion and torsion free groups acting on the same building. These groups are listed in the Appendix. The two possible buildings are denoted by (1) and (2) in the Appendix.

The link of order 2 (defined in Section 2) for a Kac-Moody building with the minimal generalized quadrangle as the link of each vertex and equilateral triangular chambers was computed in unpublished paper by the first author and D. Cartwright and T. Steger ([5]), using an invariant for links of order 2 developed by T. Steger. The Kac-Moody building coincides with our building with number (2).

By [19] there are only two possible isomorphism classes of buildings with the smallest thick generalized quadrangle as the link of each vertex and by results of the present paper at least two of them are non-isomorphic. Thus all the groups from the Appendix with building number (2) are cocompact lattices in the automorphism group of the corresponding Kac-Moody building. It remains to determine if it is possible to embed these lattices into the corresponding Kac-Moody group.

The existence of cocompact lattices in certain Kac-Moody groups has already been established. In [7], the authors generalized Lubotzky's construction of Schottky groups of automorphisms in SL_2 over a nonarchimedean local field to give torsion free cocompact lattices in any rank 2 locally compact Kac-Moody group over a finite field $\mathbb{F}_q$. In [4] Capdebosq and Thomas classified cocompact lattices with torsion and with quotient a simplex in rank 2 Kac-Moody groups corresponding to symmetric generalized Cartan matrices. In [6], the first author and Cobbs showed that over the field with 2 elements, rank 3 Kac-Moody groups of noncompact hyperbolic type whose Weyl groups are a free product of $\mathbb{Z}/2\mathbb{Z}$'s contain a cocompact lattice that also acts discretely and cocompactly on a simplicial tree. In [2] and [3], Bourdon constructed a family of cocompact lattices in the automorphism groups of certain hyperbolic Kac-Moody buildings. In [21], Rémy and Ronan showed that Bourdon's cocompact lattices $\Gamma_{r,q+1}$, $r \geq 5$, $q \geq 3$, can be embedded into the closure of right-angled Kac-Moody groups in the automorphism groups of their buildings, $I_{r,q+1}$ for q a prime power.

In all of the above cases, the Kac-Moody buildings are right-angled. The groups we construct here are the first examples of cocompact lattices acting on simply transitively on vertices of hyperbolic triangular Kac-Moody buildings that are not right-angled

By [24] it is known that groups acting cocompactly on hyperbolic buildings, in such a way that the chamber is a polygon with at least six sides, are residually finite. But whether or not groups acting cocompactly on triangular hyperbolic buildings are residually finite remains an open question. Our hyperbolic groups acting simply transitively on triangular hyperbolic buildings are possible candidates of such groups that are not residually finite. The commutator subgroups of many of our examples are perfect groups (that is, they have trivial abelianizations) and an extensive computer search (which was carried out since the paper [16] was completed) did not find any normal subgroups of these commutator subgroups.

To prove our main theorem, we used a program written in Fortran to determine the equivalence classes of triangular presentations. We used Magma to determine isomorphism classes of dual graphs of polyhedra and hence of triangle presentations.

2 Definitions and main results

Recall that a *generalized m-gon* is a connected, bipartite graph of diameter m and girth (the length of shortest circuit) $2m$, in which each vertex lies on at least two edges.

We will call a *polyhedron* a two-dimensional complex which is obtained from several oriented p-gons (Euclidean or hyperbolic) with words on the boundary, by identification of sides with the same labels respecting orientation. We assume that each side of our polygons has length 1.

Consider a sphere of a radius $0 < \epsilon < 1$ at a vertex of the polyhedron. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point. Consider now a sphere of a radius $1+\epsilon$, $0 < \epsilon < 1$ at a vertex of the polyhedron. The intersection of this sphere and the polyhedron will be called a *link of order two.*

We will use the definition of a hyperbolic building given in [15], where an infinite series of examples of hyperbolic buildings, with prescribed local structure, were constructed and studied.

Definition 2.1. *Let $\mathcal{P}(p,m)$ be a tessellation of the hyperbolic plane by regular polygons with p sides, with angles π/m in each vertex where m is an integer. A* hyperbolic building *is a polygonal complex X, which can be expressed as the union of subcomplexes called apartments such that:*

1. *Every apartment is isomorphic to $\mathcal{P}(p,m)$.*
2. *For any two polygons of X, there is an apartment containing both of them.*
3. *For any two apartments $A_1, A_2 \in X$ containing the same polygon, there exists an isomorphism $A_1 \to A_2$ fixing $A_1 \cap A_2$.*

Let C_p be a polyhedron whose faces are p-gons and whose links are generalized m-gons with $mp > 2m + p$. We equip every face of C_p with the hyperbolic metric such that all sides of the polygons are geodesics and all angles are π/m. Then the universal covering of such a polyhedron is a hyperbolic building (see [13]).

Therefore to construct hyperbolic buildings with cocompact group actions, it is sufficient to construct finite polyhedra with appropriate links.

We recall also the definition of a polygonal presentation introduced in [23]:

Definition 2.2. *Suppose we have n disjoint connected bipartite graphs $G_1, G_2, \ldots, G_n$. Let P_i and L_i be the sets of black and white vertices respectively in G_i, $i = 1, \ldots, n$; let $P = \bigcup P_i$, $L = \bigcup L_i$, $P_i \cap P_j = \emptyset$, $L_i \cap L_j = \emptyset$ for $i \neq j$ and let λ be a bijection $\lambda : P \to L$.*

A set $\mathcal{K}$ of k-tuples $(x_1, x_2, \ldots, x_k)$, $x_i \in P$, will be called a polygonal presentation *over P compatible with λ if*

(1) $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ implies that $(x_2, x_3, \ldots, x_k, x_1) \in \mathcal{K}$;

(2) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for some $x_3, \ldots, x_k$ if and only if x_2 and $\lambda(x_1)$ are incident in some G_i;

(3) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for at most one $x_3 \in P$.

If there exists such $\mathcal{K}$, we will call λ a basic bijection.

Remark 1. The polygonal presentations with $k = 3$, $n = 1$, and G_1 is the smallest generalized 3-gon have been listed in [8] and [11].

We use the following definition of equivalence, which is similar to the one in [9].

Definition 2.3. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two polygonal presentations with $k = 3$, $n = 1$, and for which the graph G_1 is a generalized 4-gon. Then $\mathcal{K}_1$ and $\mathcal{K}_2$ are equivalent if there exists an automorphism of the generalized 4-gon which transforms the 4-gon of $\mathcal{K}_1$ to the 4-gon of $\mathcal{K}_2$.*

Here we classify all polygonal presentations for $k = 3$, $n = 1$ and G_1 is the smallest thick generalized quadrangle (4-gon). Figure 1 shows the graph G_1.

In [16] the authors classified all polygonal presentations for the case $k = 3$, $n = 1$ and G_1 is the smallest thick generalized quadrangle, when at least two labels in each triangle are different. This corresponds to the case of torsion free groups acting simply transitively on the building.

Theorem 2.4. *([16]) There are 45 non-equivalent torsion free triangle presentations associated to the smallest thick generalized quadrangle. These give rise to 23 non-isomorphic torsion free groups, acting simply transitively on vertices of triangular hyperbolic buildings of smallest non-trivial thickness.*

It turns out that if we allow torsion in the groups acting simply transitively on hyperbolic triangular buildings, the number of non-equivalent presentations and the number of non-isomorphic groups is much larger:

Theorem 2.5. *There are 7159 non-equivalent triangle presentations corresponding to groups with torsion associated to the smallest generalized quadrangle. These give rise to 168 non-isomorphic groups, acting on vertices of one of two possible triangular hyperbolic buildings with the smallest thick generalized quadrangle as the link of each vertex (listed in the Appendix).*

We can associate a polyhedron X on n vertices with each polygonal presentation $\mathcal{K}$ as follows: for every cyclic k-tuple $(x_1, x_2, x_3, \ldots, x_k)$ we take an oriented

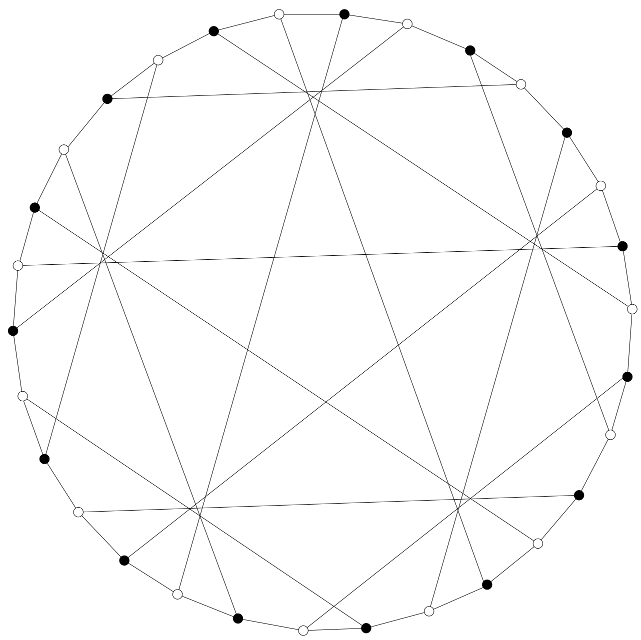

Figure 1: The graph G_1

k-gon on the boundary of which the word $x_1x_2x_3 \ldots x_k$ is written. To obtain the polyhedron we identify the corresponding sides of the polygons, respecting orientation. We say that the polyhedron X corresponds to the polygonal presentation $\mathcal{K}$.

The following lemma was proved in [23]:

Lemma 2.6. *A polyhedron X which corresponds to a polygonal presentation $\mathcal{K}$ has graphs $G_1, G_2, \ldots, G_n$ as vertex-links.*

Polyhedra corresponding to polygonal presentations from Theorem 2.4 have generalized 4-gons as vertex-links and regular hyperbolic triangles with angles $\pi/4$ as faces. The universal covering of such a polyhedron is a hyperbolic building (see [13]). Moreover, with the metric introduced in [1, p. 165] this building is a complete metric space of non-positive curvature in the sense of Alexandrov and Busemann [14]. Examples of hyperbolic buildings with right-angled triangles were constructed by M. Bourdon in [2] and in [13].

Remark. If we have a group with torsion, we may use the method of [13], Cor 2.6 p.176, which applies to torsion free groups by making the following modification. We pass to an index 3 subgroup (obtained in a canonical way by changing alphabets), to form a polyhedron with three vertices which is torsion free, then we pass to the universal cover carrying the labels, and then remove the indices of labels.

3 Proof of Theorem 2.5

We construct all polygonal presentations with $k = 3$ and $n = 1$ and for which the graph G_1 is a generalized 4-gon. The 23 torsion free groups were listed in [16]. Here we give the groups with torsion. Our strategy is to go through all possible incidence tableaus for G_1 and determine if they can be interpreted as triangle presentations.

Let P be the set of black vertices and L be the set of white vertices in G_1. We denote the elements of P by x_i and the elements of L by y_i, $i = 1, 2, \ldots, 15$. In all cases we define the basic bijection $\lambda : P \to L$ by $\lambda(x_i) = y_i$.

By [22], the smallest thick generalized 4-gon can be presented in the following way: its "points" are pairs (i, j), where $i, j = 1, ..., 6$, $i \neq j$ and "lines" are triples $(i_1, j_1), (i_2, j_2), (i_3, j_3)$ of those pairs, where i_1, i_2, i_3, j_1, j_2 and j_3 are all different. Therefore, we build a tableau as follows: For each row we take three pairs $(i_1, j_1), (i_2, j_2)$, and (i_3, j_3), where i_1, i_2, i_3, j_1, j_2 and j_3 are all different and in $1, 2, \ldots, 6$. These are our points: $x_1 = (1, 2)$, $x_2 = (1, 3)$,..., $x_{15} = (5, 6)$.

Table 1: Table of points for incidence tableau

x_1	x_{10}	x_{15}
x_1	x_{11}	x_{14}
x_1	x_{12}	x_{13}
x_2	x_7	x_{15}
x_2	x_8	x_{14}
x_2	x_9	x_{13}
x_3	x_6	x_{15}
x_3	x_8	x_{12}
x_3	x_9	x_{11}
x_4	x_6	x_{14}
x_4	x_7	x_{12}
x_4	x_9	x_{10}
x_5	x_6	x_{13}
x_5	x_7	x_{11}
x_5	x_8	x_{10}

Next we label the rows in Table 1 by $y_1, \ldots, y_{15}$ in such a way that the result is an incidence tableau that gives a triangle presentation with the basic bijection λ. To obtain groups with torsion, we demand that at least one of the triangles is of the form (x_i, x_i, x_i). For example, labeling the rows from top to bottom by y_1, y_2, y_6, y_5, y_{14}, y_{10}, y_7, y_8, y_{12}, y_3, y_4, y_9, y_{15}, y_{13}, and y_{11} gives rise to the presentation T_{24} with the following 17 triangles: (x_1, x_1, x_1), (x_{10}, x_2, x_1), (x_{15}, x_6, x_1), (x_{11}, x_5, x_2), (x_{14}, x_{14}, x_2), (x_4, x_7, x_3) (x_6, x_{12}, x_3), (x_{14}, x_8, x_3), (x_4, x_4, x_4), (x_{12}, x_9, x_4), (x_7, x_{15}, x_5), (x_{15}, x_{13}, x_5), (x_{13}, x_7, x_6), (x_8, x_8, x_8), (x_{12}, x_{11}, x_8), (x_9, x_{10}, x_9), and (x_{13}, x_{11}, x_{10}).

The labeling of rows in Table 1 defines the triangles uniquely: since the last row x_5, x_8, x_{10} has label y_{11} we know that there are triangles (x_{11}, x_5, x_a), (x_{11}, x_8, x_b) and (x_{11}, x_{10}, x_c) for some points x_a, x_b and x_c. For the first of these triangles the missing point is $x_a = x_2$, since the line y_5 has points x_2, x_7

and x_{15} and from the lines with those respective numbers only y_2 has the point x_{11}. That is, line y_{11} has point x_5, line y_5 has point x_2 and line y_2 has point x_{11}, and this gives the triangle (x_{11}, x_5, x_2). Similarly, we must have $x_b = x_{12}$ and $x_c = x_{13}$. Going through all the rows we get the triangles for this presentation. The number of the triangles in each presentation is either 17 or 19, depending of whether there is 3 or 6 triangles of the from (x_i, x_i, x_i).

The presentations are searched by a computer program. The program is written in Fortran in order to keep it fast and simple. It goes through all 15! ways to label the rows of the given tableau, and decides, which of these give an incidence tableau of a triangle presentation with torsion. The program outputs one representative of each equivalence class of triangle presentations. We obtain in this way 7159 different equivalence classes of presentations.

For a polygonal presentation T, take N ($N = 17$ or 19) oriented regular hyperbolic triangles with angles $\pi/4$, write words from the presentation on their boundaries and glue together sides with the same letters, respecting orientation. The result is a hyperbolic polyhedron with one vertex and N triangular faces, and its universal covering is a triangular hyperbolic building. We can draw the link, which is a generalized 4-gon, for any of these buildings: for every triple (x_i, x_j, x_k) the points y_i and x_j, as well as y_j and x_k and y_k and x_i are incident in it. The fundamental group Γ of the polyhedron acts simply transitively on vertices of the building. The group Γ_i has 15 generators and N relations, which come naturally from the polygonal presentation T.

To distinguish groups Γ_i, $i = 1, \ldots, 7159$ it is sufficient to distinguish the isometry classes of polyhedra, according to the Mostow-type rigidity for hyperbolic buildings which was shown, for example, in [25].

Therefore, we consider dual graphs of index 3 subgroups in order to see which of these presentations give rise to isometric polyhedra. First we calculate the index 3 subgroups: we substitute each triple of the form (x_i, x_i, x_i) in the presentation by (x_i^1, x_i^2, x_i^3), each (x_i, x_j, x_k) by three triplets (x_i^1, x_j^2, x_k^3), (x_i^2, x_j^3, x_k^1) and (x_i^3, x_j^1, x_k^2), and each (x_i, x_j, x_j) similarly by three triplets (x_i^1, x_j^2, x_j^3), (x_i^2, x_j^3, x_j^1) and (x_i^3, x_j^1, x_j^2). We then have 45 triangles, which represent the generators of the index 3 subgroup of Γ.

We next construct the dual graph for each of these as follows: we take 90 vertices such that first 45 of them (numbered 1-45) correspond to the edges of the triangles and the second 45 edges (numbered 46-90) correspond to the faces of the triangles. We add an edge between vertices i (from 1-45) and j (from 46-90), if edge i was on the boundary of the face j in a triangle. Thus we obtain trivalent bipartite graphs with 90 vertices.

With the help of the Computational Algebra System Magma we compare the dual graphs of the index 3 subgroups and we find that most of them are isomorphic with some other graph: there are only 168 non-isomorphic dual graphs. Thus we have 168 triangle presentations which give rise to non-isometric polyhedra. We then compute links of order two in buildings defined by our 168 torsion groups and 23 torsion free groups from [16]. There are only two non-isomorphic links of order two and in this case, they are complete invariants of buildings.

The 168 triangle presentations are listed in the Appendix together with the number (1) or (2) indicating the type of building.

This completes the proof of Theorem 2.5.

4 Construction of polyhedra with m-gonal faces using torsion groups

In [16] the authors described how to construct buildings with m-gonal faces, for arbitrary m, starting from torsion free groups acting on triangular buildings with the smallest possible link. We modify this construction to allow torsion groups and an arbitrary generalized polygon as the link of each vertex.

Given generalized polygon G we shall denote by G' the graph arising by calling black vertices of G white vertices of G' and white vertices of G black vertices of G'.

Consider a bipartite graph G with black vertices $P = \{x_1, \ldots, , x_k\}$ and white vertices $L = \{y_1, \ldots, y_k\}$ and a subset $K \subset P \times P \times P$ that defines the triangles. Starting from this triangular presentation K, we construct a polyhedron, whose faces are m-gons and whose m-vertices have links G or G'.

Let $w = z_1 \ldots z_m$ be a word of length m in three letters a, b and c. Assume that $z_1 = a$, $z_2 = b$ and $z_3 = c$ and that w does not contain proper powers of the letters a, b and c, that is, $z_m \neq a$ and $z_t \neq z_{t+1}$ for all $t = 1, \ldots, m-1$.

For each of the triples (x_i, x_j, x_k) in K we take three triples (x_i^1, x_j^2, x_k^3), (x_i^2, x_j^3, x_k^1) , (x_i^3, x_j^1, x_k^2) if at least two of x_i, x_j, x_k are different, and just one (x_i^1, x_i^2, x_i^3), if $i = j = k$. The triples are cyclic, so we can write them as (x_i^1, x_j^2, x_k^3), (x_k^1, x_i^2, x_j^3) and (x_j^1, x_k^2, x_i^3). By glueing together triangles with these words on the boundary, we obtain a polyhedron with triangle faces and 3 vertices, each of them with the graph G as the link of each vertex.

We construct m-tuples, one corresponding to each of these new triples: for triple $(x_\alpha^1, x_\beta^2, x_\gamma^3)$ we define an m-tuple, which corresponds a word w with $a = x_\alpha^1$, $b = x_\beta^2$ and $c = x_\gamma^3$. We have m-tuples whose coordinates start with one of the triples, and then continue with $m-3$ letters in some order defined by the word w in the letters a, b, and c.

If we glue the m-gons with these words on the boundary together by their sides labelled with same letters, respecting orientation, we obtain a polyhedron with m-gonal faces and m vertices, which all have the link G or G'. The type of the link can be seen from the letters of the edges meeting at that vertex. Set

$$Sign(ab) = Sign(bc) = Sign(ca) = 1$$

and

$$Sign(ba) = Sign(cb) = Sign(ac) = -1.$$

Then for vertex $t = 1, \ldots, m-1$ the group G_t of the link is G if $Sign(z_t, z_{t+1}) = 1$ and G' if $Sign(z_t, z_{t+1}) = -1$. For the last vertex we have $G_m = G$ if $Sign(z_m, a) = 1$ and G' if $Sign(z_m, a) = -1$.

We denote the set of m-tuples by T_m. Thus we have the following

Theorem 4.1. *The above constructed subset $T_m \subset P \times \cdots \times P$ is a polygonal presentation. It defines a polyhedron X whose faces are m-gons and whose m vertices have links G or G'.*

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Appendix

The torsion free cases $T_1, \ldots, T_{23}$ have been listed in [16], and thus for those we only denote here in Table 2 whether the link of order 2 is isomorphic to that of T_1 (case 1) or T_2 (case 2). Then in Table 3 we list the labelings of the rows of Table 1 which give rise to the triangle presentations with torsion, denoted by $T_{24}, \ldots, T_{191}$. After the name of the presentation in Table 3 there is (1) resp. (2) if the resulting building is isomorphic with that of T_1 resp. T_2.

Table 2: List of torsion free cases by isomorphism type of the 2-link

(1)	T_1, T_4, T_5, T_8, T_9, T_{11}, T_{13}, T_{15}, T_{17}, T_{18}, T_{19}, T_{23}
(2)	T_2, T_3, T_6, T_7, T_{10}, T_{12}, T_{14}, T_{16}, T_{20}, T_{21}, T_{22}

Table 3: List of labelings giving triangle presentations with torsion

T_{24}	(1)	$y_1, y_2, y_6, y_5, y_{14}, y_{10}, y_7, y_8, y_{12}, y_3, y_4, y_9, y_{15}, y_{13}, y_{11}$
T_{25}	(2)	$y_1, y_2, y_9, y_{10}, y_3, y_5, y_{14}, y_8, y_{15}, y_{13}, y_7, y_4, y_6, y_{12}, y_{11}$
T_{26}	(2)	$y_1, y_2, y_6, y_3, y_{15}, y_7, y_8, y_{14}, y_9, y_{10}, y_{11}, y_{12}, y_4, y_5, y_{13}$
T_{27}	(1)	$y_1, y_2, y_{14}, y_{13}, y_{10}, y_4, y_3, y_6, y_8, y_{15}, y_{11}, y_9, y_5, y_7, y_{12}$
T_{28}	(2)	$y_1, y_2, y_3, y_5, y_{14}, y_{15}, y_8, y_9, y_{10}, y_6, y_7, y_{12}, y_{11}, y_4, y_{13}$
T_{29}	(1)	$y_1, y_4, y_{12}, y_3, y_9, y_6, y_{11}, y_5, y_7, y_{14}, y_{15}, y_{10}, y_{13}, y_2, y_8$
T_{30}	(2)	$y_1, y_2, y_4, y_9, y_6, y_{10}, y_{11}, y_{13}, y_{14}, y_3, y_{15}, y_7, y_5, y_{12}, y_8$
T_{31}	(2)	$y_2, y_6, y_{13}, y_{14}, y_3, y_8, y_7, y_1, y_9, y_{15}, y_4, y_5, y_{12}, y_{11}, y_{10}$
T_{32}	(2)	$y_1, y_2, y_6, y_8, y_{10}, y_4, y_{15}, y_{14}, y_{13}, y_{11}, y_9, y_7, y_5, y_{12}, y_3$
T_{33}	(1)	$y_1, y_2, y_7, y_3, y_{14}, y_{15}, y_9, y_{11}, y_{13}, y_5, y_6, y_4, y_{12}, y_{10}, y_8$
T_{34}	(2)	$y_2, y_7, y_8, y_4, y_6, y_5, y_1, y_3, y_9, y_{14}, y_{15}, y_{12}, y_{13}, y_{11}, y_{10}$
T_{35}	(2)	$y_4, y_9, y_{12}, y_{11}, y_8, y_{15}, y_1, y_3, y_2, y_6, y_{13}, y_{10}, y_5, y_7, y_{14}$
T_{36}	(1)	$y_1, y_{10}, y_{11}, y_2, y_8, y_5, y_{14}, y_7, y_6, y_9, y_{12}, y_3, y_{13}, y_4, y_{15}$
T_{37}	(2)	$y_1, y_2, y_3, y_{15}, y_9, y_4, y_7, y_{12}, y_{10}, y_{11}, y_5, y_{14}, y_6, y_{13}, y_8$
T_{38}	(1)	$y_1, y_2, y_4, y_{15}, y_{13}, y_6, y_{11}, y_5, y_{12}, y_8, y_3, y_{10}, y_{14}, y_7, y_9$
T_{39}	(1)	$y_1, y_2, y_8, y_4, y_6, y_{10}, y_{13}, y_7, y_3, y_{14}, y_{12}, y_5, y_{11}, y_9, y_{15}$
T_{40}	(2)	$y_1, y_2, y_4, y_{15}, y_{14}, y_3, y_5, y_{11}, y_9, y_{10}, y_{12}, y_8, y_6, y_7, y_{13}$
T_{41}	(2)	$y_1, y_2, y_6, y_4, y_{14}, y_{10}, y_3, y_{12}, y_{13}, y_{11}, y_7, y_5, y_{15}, y_8, y_9$
T_{42}	(1)	$y_1, y_2, y_6, y_{10}, y_{13}, y_3, y_5, y_8, y_{11}, y_{15}, y_7, y_4, y_{14}, y_9, y_{12}$
T_{43}	(1)	$y_1, y_2, y_8, y_{13}, y_{15}, y_3, y_6, y_{11}, y_7, y_9, y_4, y_5, y_{14}, y_{12}, y_{10}$
T_{44}	(1)	$y_1, y_2, y_4, y_5, y_3, y_{15}, y_6, y_{11}, y_7, y_8, y_{12}, y_{10}, y_9, y_{14}, y_{13}$
T_{45}	(2)	$y_1, y_2, y_4, y_8, y_{11}, y_{10}, y_{13}, y_9, y_7, y_6, y_{12}, y_{15}, y_5, y_{14}, y_3$
T_{46}	(1)	$y_1, y_2, y_6, y_3, y_8, y_{10}, y_{15}, y_5, y_{11}, y_9, y_{13}, y_{12}, y_7, y_4, y_{14}$
T_{47}	(2)	$y_1, y_2, y_8, y_{15}, y_4, y_6, y_7, y_5, y_3, y_{14}, y_{11}, y_{12}, y_{13}, y_9, y_{10}$
T_{48}	(2)	$y_1, y_2, y_8, y_{15}, y_6, y_7, y_{13}, y_4, y_9, y_{12}, y_{14}, y_5, y_{11}, y_3, y_{10}$
T_{49}	(2)	$y_1, y_2, y_3, y_5, y_4, y_{15}, y_6, y_{10}, y_8, y_9, y_7, y_{11}, y_{12}, y_{14}, y_{13}$
T_{50}	(2)	$y_1, y_2, y_3, y_6, y_{12}, y_{10}, y_8, y_{11}, y_{15}, y_{14}, y_4, y_5, y_9, y_{13}, y_7$
T_{51}	(2)	$y_1, y_2, y_4, y_{12}, y_3, y_{15}, y_5, y_9, y_{14}, y_7, y_{11}, y_{10}, y_6, y_{13}, y_8$
T_{52}	(2)	$y_1, y_2, y_6, y_3, y_{14}, y_{15}, y_{10}, y_{13}, y_{11}, y_{12}, y_8, y_7, y_4, y_5, y_9$
T_{53}	(1)	$y_1, y_2, y_6, y_9, y_{12}, y_{10}, y_{15}, y_4, y_3, y_5, y_{14}, y_{11}, y_8, y_{13}, y_7$
T_{54}	(2)	$y_1, y_2, y_{12}, y_{15}, y_9, y_{11}, y_7, y_8, y_5, y_6, y_{10}, y_3, y_4, y_{14}, y_{13}$
T_{55}	(2)	$y_1, y_2, y_{13}, y_{12}, y_{15}, y_7, y_9, y_{11}, y_4, y_3, y_{14}, y_8, y_{10}, y_6, y_5$
T_{56}	(1)	$y_2, y_9, y_{15}, y_{12}, y_8, y_5, y_6, y_1, y_3, y_{11}, y_7, y_4, y_{13}, y_{10}, y_{14}$
T_{57}	(2)	$y_1, y_2, y_4, y_5, y_3, y_{15}, y_{11}, y_7, y_9, y_6, y_{10}, y_8, y_{12}, y_{14}, y_{13}$
T_{58}	(2)	$y_1, y_2, y_6, y_5, y_{14}, y_{15}, y_{10}, y_3, y_7, y_4, y_8, y_{12}, y_{11}, y_{13}, y_9$
T_{59}	(1)	$y_1, y_2, y_4, y_8, y_{11}, y_{15}, y_6, y_9, y_7, y_{10}, y_{12}, y_5, y_{13}, y_{14}, y_3$
T_{60}	(1)	$y_1, y_2, y_{12}, y_{15}, y_9, y_4, y_7, y_{10}, y_{13}, y_3, y_{11}, y_{14}, y_8, y_6, y_5$
T_{61}	(2)	$y_1, y_2, y_{12}, y_5, y_{14}, y_{15}, y_8, y_{13}, y_3, y_6, y_{10}, y_7, y_{11}, y_4, y_9$
T_{62}	(1)	$y_1, y_2, y_5, y_6, y_4, y_{15}, y_9, y_8, y_{13}, y_{14}, y_{12}, y_{11}, y_{10}, y_3, y_7$
T_{63}	(1)	$y_1, y_3, y_4, y_8, y_{11}, y_9, y_{15}, y_6, y_2, y_{10}, y_7, y_5, y_{14}, y_{13}, y_{12}$
T_{64}	(2)	$y_1, y_2, y_6, y_7, y_{15}, y_3, y_{11}, y_9, y_{12}, y_{10}, y_8, y_{13}, y_4, y_{14}, y_5$
T_{65}	(2)	$y_1, y_2, y_3, y_8, y_6, y_{10}, y_{15}, y_9, y_7, y_{14}, y_5, y_{12}, y_4, y_{13}, y_{11}$

T_{66}	(2)	$y_1, y_2, y_3, y_4, y_{14}, y_{10}, y_{15}, y_5, y_7, y_6, y_9, y_{11}, y_8, y_{13}, y_{12}$
T_{67}	(1)	$y_1, y_2, y_7, y_{10}, y_{14}, y_8, y_3, y_{11}, y_4, y_{13}, y_6, y_{12}, y_9, y_{15}, y_5$
T_{68}	(1)	$y_1, y_2, y_7, y_{15}, y_5, y_4, y_{13}, y_{12}, y_8, y_{11}, y_3, y_6, y_9, y_{10}, y_{14}$
T_{69}	(2)	$y_1, y_2, y_3, y_{10}, y_{14}, y_8, y_5, y_4, y_{15}, y_{13}, y_6, y_9, y_{12}, y_7, y_{11}$
T_{70}	(2)	$y_1, y_2, y_4, y_6, y_9, y_{15}, y_{14}, y_8, y_{12}, y_5, y_{10}, y_{11}, y_{13}, y_7, y_3$
T_{71}	(2)	$y_1, y_2, y_{11}, y_{15}, y_{12}, y_6, y_7, y_4, y_8, y_{14}, y_9, y_3, y_5, y_{10}, y_{13}$
T_{72}	(1)	$y_1, y_2, y_{12}, y_5, y_9, y_{10}, y_7, y_{15}, y_{11}, y_{13}, y_4, y_6, y_8, y_3, y_{14}$
T_{73}	(2)	$y_2, y_6, y_{13}, y_{15}, y_3, y_8, y_4, y_1, y_9, y_{14}, y_{11}, y_7, y_{12}, y_5, y_{10}$
T_{74}	(2)	$y_1, y_2, y_4, y_9, y_8, y_{10}, y_7, y_3, y_{14}, y_5, y_{12}, y_{15}, y_6, y_{13}, y_{11}$
T_{75}	(1)	$y_1, y_{10}, y_{11}, y_5, y_{14}, y_{13}, y_6, y_7, y_2, y_3, y_{12}, y_{15}, y_9, y_4, y_8$
T_{76}	(1)	$y_1, y_2, y_{12}, y_{10}, y_{14}, y_5, y_9, y_{13}, y_7, y_6, y_{15}, y_4, y_8, y_{11}, y_3$
T_{77}	(2)	$y_1, y_2, y_4, y_7, y_{10}, y_9, y_5, y_3, y_{13}, y_{15}, y_6, y_{11}, y_{12}, y_{14}, y_8$
T_{78}	(1)	$y_1, y_2, y_5, y_8, y_{12}, y_{10}, y_4, y_7, y_3, y_{13}, y_{11}, y_9, y_6, y_{15}, y_{14}$
T_{79}	(1)	$y_1, y_2, y_4, y_3, y_{14}, y_{15}, y_5, y_{11}, y_9, y_{10}, y_{12}, y_8, y_6, y_7, y_{13}$
T_{80}	(2)	$y_1, y_2, y_5, y_{12}, y_{14}, y_{10}, y_{13}, y_{11}, y_6, y_4, y_8, y_9, y_3, y_7, y_{15}$
T_{81}	(2)	$y_1, y_2, y_8, y_4, y_{14}, y_{15}, y_{12}, y_6, y_7, y_{13}, y_9, y_{11}, y_3, y_5, y_{10}$
T_{82}	(1)	$y_1, y_2, y_4, y_9, y_{15}, y_{11}, y_7, y_{12}, y_{14}, y_{10}, y_8, y_3, y_6, y_5, y_{13}$
T_{83}	(1)	$y_1, y_2, y_4, y_{13}, y_{15}, y_8, y_6, y_5, y_{12}, y_3, y_7, y_{10}, y_{14}, y_9, y_{11}$
T_{84}	(2)	$y_1, y_2, y_8, y_6, y_4, y_{15}, y_9, y_3, y_{13}, y_{11}, y_5, y_{12}, y_{14}, y_7, y_{10}$
T_{85}	(1)	$y_1, y_2, y_{11}, y_{15}, y_{14}, y_{13}, y_7, y_3, y_{10}, y_5, y_4, y_6, y_9, y_8, y_{12}$
T_{86}	(2)	$y_1, y_6, y_8, y_{11}, y_9, y_{13}, y_{15}, y_{12}, y_4, y_{14}, y_7, y_5, y_2, y_3, y_{10}$
T_{87}	(1)	$y_1, y_2, y_4, y_3, y_6, y_{15}, y_5, y_{11}, y_9, y_{10}, y_{12}, y_8, y_{14}, y_7, y_{13}$
T_{88}	(2)	$y_1, y_2, y_{14}, y_{15}, y_5, y_{13}, y_3, y_4, y_7, y_{10}, y_6, y_8, y_9, y_{11}, y_{12}$
T_{89}	(1)	$y_1, y_2, y_3, y_{10}, y_{13}, y_5, y_7, y_8, y_{15}, y_4, y_9, y_6, y_{14}, y_{12}, y_{11}$
T_{90}	(2)	$y_1, y_2, y_4, y_5, y_{14}, y_{15}, y_8, y_{12}, y_7, y_3, y_6, y_{10}, y_{13}, y_{11}, y_9$
T_{91}	(2)	$y_1, y_2, y_4, y_6, y_{15}, y_{12}, y_8, y_{14}, y_{13}, y_{11}, y_9, y_{10}, y_5, y_3, y_7$
T_{92}	(1)	$y_1, y_2, y_6, y_{10}, y_9, y_4, y_{15}, y_{12}, y_{14}, y_8, y_5, y_{11}, y_{13}, y_7, y_3$
T_{93}	(1)	$y_1, y_2, y_9, y_{14}, y_{10}, y_7, y_{12}, y_4, y_3, y_{11}, y_5, y_{15}, y_6, y_{13}, y_8$
T_{94}	(2)	$y_1, y_2, y_{11}, y_{10}, y_3, y_{12}, y_{14}, y_8, y_{15}, y_7, y_5, y_4, y_{13}, y_6, y_9$
T_{95}	(2)	$y_2, y_{14}, y_{13}, y_6, y_3, y_4, y_{15}, y_8, y_{11}, y_5, y_{12}, y_{10}, y_1, y_9, y_7$
T_{96}	(2)	$y_1, y_3, y_{14}, y_9, y_{15}, y_2, y_{10}, y_{12}, y_{11}, y_4, y_5, y_{13}, y_6, y_7, y_8$
T_{97}	(2)	$y_2, y_3, y_{13}, y_{15}, y_4, y_8, y_6, y_{12}, y_5, y_{14}, y_1, y_9, y_7, y_{11}, y_{10}$
T_{98}	(1)	$y_2, y_{10}, y_{15}, y_{13}, y_9, y_1, y_6, y_8, y_{11}, y_4, y_7, y_3, y_{12}, y_{14}, y_5$
T_{99}	(2)	$y_2, y_{11}, y_{15}, y_8, y_1, y_9, y_6, y_{12}, y_{14}, y_{13}, y_7, y_{10}, y_5, y_3, y_4$
T_{100}	(1)	$y_2, y_{14}, y_{15}, y_{12}, y_7, y_5, y_6, y_8, y_9, y_4, y_1, y_3, y_{13}, y_{11}, y_{10}$
T_{101}	(1)	$y_2, y_4, y_{15}, y_8, y_{13}, y_1, y_3, y_{12}, y_5, y_{14}, y_7, y_9, y_6, y_{11}, y_{10}$
T_{102}	(2)	$y_1, y_{10}, y_{11}, y_7, y_5, y_9, y_{13}, y_2, y_3, y_{14}, y_{12}, y_{15}, y_8, y_4, y_6$
T_{103}	(2)	$y_2, y_4, y_{13}, y_7, y_6, y_9, y_8, y_{14}, y_{12}, y_3, y_5, y_{10}, y_1, y_{15}, y_{11}$
T_{104}	(1)	$y_2, y_{10}, y_{12}, y_{15}, y_1, y_{14}, y_6, y_4, y_{11}, y_7, y_8, y_{13}, y_5, y_9, y_3$
T_{105}	(1)	$y_2, y_6, y_{11}, y_8, y_{13}, y_7, y_9, y_{12}, y_{14}, y_5, y_{15}, y_4, y_1, y_3, y_{10}$
T_{106}	(1)	$y_3, y_6, y_{13}, y_9, y_2, y_4, y_7, y_{10}, y_8, y_{14}, y_1, y_{15}, y_{12}, y_{11}, y_5$
T_{107}	(2)	$y_2, y_3, y_{13}, y_8, y_{15}, y_7, y_{10}, y_9, y_{11}, y_6, y_{12}, y_4, y_5, y_1, y_{14}$

T_{108}	(1)	$y_2, y_3, y_{10}, y_{13}, y_8, y_{12}, y_{11}, y_{15}, y_4, y_6, y_1, y_9, y_5, y_{14}, y_7$
T_{109}	(1)	$y_2, y_7, y_{13}, y_6, y_3, y_9, y_{15}, y_5, y_{12}, y_1, y_8, y_{10}, y_{14}, y_4, y_{11}$
T_{110}	(1)	$y_1, y_2, y_4, y_{15}, y_{13}, y_{12}, y_3, y_8, y_7, y_9, y_{14}, y_{10}, y_{11}, y_5, y_6$
T_{111}	(1)	$y_1, y_2, y_3, y_8, y_9, y_{10}, y_{15}, y_6, y_{11}, y_{12}, y_5, y_7, y_4, y_{13}, y_{14}$
T_{112}	(2)	$y_1, y_2, y_3, y_{10}, y_{14}, y_8, y_6, y_5, y_{15}, y_{13}, y_7, y_4, y_9, y_{12}, y_{11}$
T_{113}	(2)	$y_1, y_2, y_4, y_{15}, y_6, y_3, y_5, y_{14}, y_{12}, y_{11}, y_9, y_{10}, y_8, y_{13}, y_7$
T_{114}	(2)	$y_1, y_2, y_4, y_{15}, y_{13}, y_5, y_9, y_3, y_7, y_{10}, y_{12}, y_8, y_{14}, y_{11}, y_6$
T_{115}	(1)	$y_1, y_2, y_4, y_{15}, y_{13}, y_{11}, y_{12}, y_7, y_{14}, y_8, y_6, y_{10}, y_3, y_5, y_9$
T_{116}	(1)	$y_1, y_2, y_4, y_{15}, y_{14}, y_6, y_8, y_{13}, y_9, y_3, y_{10}, y_5, y_{11}, y_{12}, y_7$
T_{117}	(1)	$y_1, y_2, y_6, y_8, y_{13}, y_{15}, y_{10}, y_4, y_3, y_7, y_5, y_9, y_{14}, y_{12}, y_{11}$
T_{118}	(1)	$y_1, y_6, y_{11}, y_3, y_9, y_{12}, y_{15}, y_5, y_{10}, y_4, y_{14}, y_7, y_{13}, y_2, y_8$
T_{119}	(2)	$y_1, y_2, y_4, y_{10}, y_9, y_{11}, y_{13}, y_8, y_3, y_6, y_{15}, y_5, y_7, y_{14}, y_{12}$
T_{120}	(1)	$y_1, y_2, y_6, y_{10}, y_3, y_5, y_9, y_8, y_{14}, y_7, y_{12}, y_{13}, y_{15}, y_{11}, y_4$
T_{121}	(2)	$y_1, y_2, y_8, y_4, y_7, y_{10}, y_9, y_3, y_{12}, y_{11}, y_{14}, y_5, y_6, y_{13}, y_{15}$
T_{122}	(2)	$y_1, y_2, y_3, y_9, y_{15}, y_8, y_{10}, y_5, y_6, y_7, y_4, y_{14}, y_{13}, y_{12}, y_{11}$
T_{123}	(1)	$y_1, y_2, y_6, y_3, y_{10}, y_8, y_5, y_{14}, y_{11}, y_{15}, y_9, y_{13}, y_4, y_7, y_{12}$
T_{124}	(2)	$y_1, y_2, y_8, y_{13}, y_7, y_{10}, y_3, y_{12}, y_5, y_9, y_6, y_4, y_{14}, y_{11}, y_{15}$
T_{125}	(2)	$y_1, y_2, y_4, y_{10}, y_8, y_9, y_{12}, y_5, y_7, y_{15}, y_6, y_{11}, y_3, y_{13}, y_{14}$
T_{126}	(2)	$y_1, y_2, y_6, y_3, y_{10}, y_7, y_{15}, y_{13}, y_{14}, y_5, y_8, y_4, y_{12}, y_{11}, y_9$
T_{127}	(2)	$y_{10}, y_{11}, y_{13}, y_2, y_5, y_{12}, y_{14}, y_7, y_1, y_8, y_4, y_9, y_3, y_6, y_{15}$
T_{128}	(1)	$y_1, y_4, y_7, y_{15}, y_{11}, y_8, y_6, y_3, y_{13}, y_{12}, y_{14}, y_{10}, y_2, y_9, y_5$
T_{129}	(1)	$y_1, y_2, y_6, y_{15}, y_5, y_9, y_7, y_8, y_{13}, y_3, y_4, y_{11}, y_{10}, y_{12}, y_{14}$
T_{130}	(1)	$y_8, y_{14}, y_{15}, y_7, y_{11}, y_6, y_3, y_{12}, y_9, y_4, y_1, y_2, y_5, y_{10}, y_{13}$
T_{131}	(2)	$y_1, y_2, y_{13}, y_{15}, y_9, y_3, y_{12}, y_{14}, y_8, y_6, y_7, y_{11}, y_{10}, y_5, y_4$
T_{132}	(1)	$y_1, y_2, y_3, y_4, y_{10}, y_{13}, y_7, y_{15}, y_9, y_{14}, y_{12}, y_6, y_{11}, y_8, y_5$
T_{133}	(2)	$y_1, y_2, y_4, y_{15}, y_3, y_6, y_{11}, y_{12}, y_{13}, y_{10}, y_8, y_9, y_{14}, y_7, y_5$
T_{134}	(1)	$y_1, y_2, y_8, y_{15}, y_6, y_9, y_4, y_{12}, y_{11}, y_{14}, y_7, y_5, y_{13}, y_3, y_{10}$
T_{135}	(2)	$y_1, y_2, y_{12}, y_{15}, y_8, y_3, y_7, y_{10}, y_{11}, y_6, y_4, y_{13}, y_5, y_9, y_{14}$
T_{136}	(1)	$y_1, y_2, y_{13}, y_3, y_{14}, y_{15}, y_{12}, y_{11}, y_9, y_4, y_7, y_8, y_{10}, y_6, y_5$
T_{137}	(1)	$y_1, y_4, y_{12}, y_2, y_{13}, y_7, y_{14}, y_8, y_9, y_5, y_{15}, y_{10}, y_6, y_3, y_{11}$
T_{138}	(1)	$y_1, y_2, y_{12}, y_{10}, y_4, y_9, y_7, y_{15}, y_{11}, y_{14}, y_5, y_6, y_{13}, y_3, y_8$
T_{139}	(1)	$y_1, y_2, y_3, y_4, y_{10}, y_{13}, y_{15}, y_8, y_9, y_{11}, y_7, y_6, y_{14}, y_5, y_{12}$
T_{140}	(2)	$y_1, y_2, y_4, y_3, y_{14}, y_{10}, y_{13}, y_9, y_{11}, y_5, y_6, y_{15}, y_7, y_{12}, y_8$
T_{141}	(2)	$y_1, y_2, y_4, y_8, y_{14}, y_{10}, y_{13}, y_9, y_7, y_{11}, y_6, y_{15}, y_5, y_{12}, y_3$
T_{142}	(1)	$y_1, y_2, y_{13}, y_5, y_{10}, y_{15}, y_{14}, y_8, y_7, y_4, y_6, y_{12}, y_9, y_{11}, y_3$
T_{143}	(2)	$y_1, y_2, y_{12}, y_6, y_{14}, y_{15}, y_{11}, y_{10}, y_8, y_9, y_7, y_5, y_4, y_{13}, y_3$
T_{144}	(2)	$y_1, y_2, y_4, y_{15}, y_{13}, y_6, y_{11}, y_3, y_{12}, y_{10}, y_8, y_9, y_{14}, y_7, y_5$
T_{145}	(1)	$y_1, y_2, y_6, y_{10}, y_5, y_9, y_7, y_8, y_{12}, y_{15}, y_{13}, y_{14}, y_{11}, y_4, y_3$
T_{146}	(2)	$y_1, y_2, y_9, y_{15}, y_{14}, y_8, y_5, y_{12}, y_6, y_{13}, y_3, y_{10}, y_7, y_4, y_{11}$
T_{147}	(2)	$y_1, y_2, y_4, y_5, y_{12}, y_{10}, y_3, y_{14}, y_{13}, y_{15}, y_8, y_9, y_6, y_7, y_{11}$
T_{148}	(1)	$y_1, y_2, y_5, y_{15}, y_{12}, y_4, y_6, y_{11}, y_{13}, y_7, y_9, y_{14}, y_8, y_{10}, y_3$
T_{149}	(2)	$y_1, y_{10}, y_{11}, y_3, y_{14}, y_7, y_{13}, y_8, y_4, y_{12}, y_5, y_{15}, y_9, y_6, y_2$

T_{150}	(1)	$y_5,\ y_{14},\ y_{15},\ y_3,\ y_8,\ y_9,\ y_{11},\ y_{12},\ y_7,\ y_6,\ y_2,\ y_4,\ y_{10},\ y_{13},\ y_1$
T_{151}	(1)	$y_6,\ y_8,\ y_{10},\ y_3,\ y_{14},\ y_{15},\ y_4,\ y_2,\ y_{12},\ y_1,\ y_7,\ y_9,\ y_{13},\ y_{11},\ y_5$
T_{152}	(1)	$y_4,\ y_{11},\ y_{12},\ y_{15},\ y_6,\ y_{13},\ y_7,\ y_8,\ y_9,\ y_{10},\ y_2,\ y_{14},\ y_3,\ y_5,\ y_1$
T_{153}	(2)	$y_6,\ y_{11},\ y_{14},\ y_{10},\ y_8,\ y_2,\ y_{15},\ y_9,\ y_5,\ y_7,\ y_{12},\ y_4,\ y_3,\ y_1,\ y_{13}$
T_{154}	(2)	$y_2,\ y_5,\ y_{11},\ y_{13},\ y_8,\ y_9,\ y_{12},\ y_4,\ y_{15},\ y_{14},\ y_3,\ y_7,\ y_{10},\ y_6,\ y_1$
T_{155}	(1)	$y_1,\ y_8,\ y_{12},\ y_{11},\ y_{14},\ y_{13},\ y_6,\ y_2,\ y_5,\ y_3,\ y_{15},\ y_9,\ y_4,\ y_7,\ y_{10}$
T_{156}	(2)	$y_1,\ y_6,\ y_{12},\ y_2,\ y_3,\ y_{13},\ y_{10},\ y_{15},\ y_5,\ y_4,\ y_9,\ y_{14},\ y_7,\ y_{11},\ y_8$
T_{157}	(2)	$y_8,\ y_1,\ y_5,\ y_3,\ y_{14},\ y_{10},\ y_{15},\ y_9,\ y_7,\ y_{12},\ y_2,\ y_4,\ y_{13},\ y_{11},\ y_6$
T_{158}	(1)	$y_6,\ y_{12},\ y_{14},\ y_{15},\ y_7,\ y_3,\ y_8,\ y_{13},\ y_9,\ y_{11},\ y_5,\ y_4,\ y_2,\ y_1,\ y_{10}$
T_{159}	(2)	$y_6,\ y_9,\ y_{12},\ y_{15},\ y_7,\ y_{10},\ y_2,\ y_{14},\ y_{11},\ y_{13},\ y_8,\ y_3,\ y_5,\ y_4,\ y_1$
T_{160}	(1)	$y_2,\ y_{11},\ y_{12},\ y_3,\ y_5,\ y_7,\ y_{15},\ y_6,\ y_9,\ y_{13},\ y_1,\ y_{14},\ y_{10},\ y_4,\ y_8$
T_{161}	(2)	$y_2,\ y_9,\ y_{13},\ y_{15},\ y_{12},\ y_8,\ y_6,\ y_{10},\ y_5,\ y_3,\ y_1,\ y_4,\ y_7,\ y_{11},\ y_{14}$
T_{162}	(1)	$y_2,\ y_3,\ y_{13},\ y_{11},\ y_8,\ y_6,\ y_{15},\ y_{10},\ y_5,\ y_{12},\ y_9,\ y_{14},\ y_7,\ y_1,\ y_4$
T_{163}	(1)	$y_1,\ y_2,\ y_7,\ y_{15},\ y_4,\ y_9,\ y_{13},\ y_8,\ y_{11},\ y_{14},\ y_{12},\ y_6,\ y_5,\ y_{10},\ y_3$
T_{164}	(1)	$y_2,\ y_5,\ y_{10},\ y_{15},\ y_{13},\ y_1,\ y_6,\ y_{14},\ y_{11},\ y_7,\ y_{12},\ y_4,\ y_3,\ y_9,\ y_8$
T_{165}	(1)	$y_2,\ y_9,\ y_{10},\ y_{12},\ y_6,\ y_5,\ y_8,\ y_{11},\ y_3,\ y_1,\ y_{13},\ y_4,\ y_{15},\ y_7,\ y_{14}$
T_{166}	(2)	$y_1,\ y_2,\ y_{11},\ y_{15},\ y_{14},\ y_{12},\ y_7,\ y_6,\ y_5,\ y_4,\ y_8,\ y_3,\ y_{13},\ y_{10},\ y_9$
T_{167}	(2)	$y_1,\ y_2,\ y_{12},\ y_3,\ y_5,\ y_{10},\ y_9,\ y_{14},\ y_7,\ y_{13},\ y_{15},\ y_4,\ y_8,\ y_{11},\ y_6$
T_{168}	(1)	$y_2,\ y_3,\ y_{11},\ y_9,\ y_8,\ y_4,\ y_6,\ y_{13},\ y_{10},\ y_7,\ y_{15},\ y_{12},\ y_{14},\ y_5,\ y_1$
T_{169}	(1)	$y_2,\ y_3,\ y_{13},\ y_8,\ y_5,\ y_7,\ y_6,\ y_9,\ y_4,\ y_{11},\ y_{12},\ y_{14},\ y_1,\ y_{15},\ y_{10}$
T_{170}	(1)	$y_2,\ y_8,\ y_{13},\ y_9,\ y_{14},\ y_{12},\ y_{11},\ y_{15},\ y_{10},\ y_1,\ y_7,\ y_3,\ y_6,\ y_5,\ y_4$
T_{171}	(2)	$y_2,\ y_{11},\ y_{13},\ y_5,\ y_7,\ y_3,\ y_1,\ y_{12},\ y_4,\ y_{14},\ y_{10},\ y_9,\ y_{15},\ y_6,\ y_8$
T_{172}	(2)	$y_2,\ y_7,\ y_9,\ y_3,\ y_6,\ y_8,\ y_{15},\ y_1,\ y_{11},\ y_{14},\ y_4,\ y_5,\ y_{13},\ y_{12},\ y_{10}$
T_{173}	(1)	$y_1,\ y_2,\ y_4,\ y_9,\ y_{15},\ y_{11},\ y_7,\ y_{12},\ y_{14},\ y_{10},\ y_5,\ y_3,\ y_6,\ y_8,\ y_{13}$
T_{174}	(2)	$y_1,\ y_2,\ y_{13},\ y_4,\ y_7,\ y_{15},\ y_{12},\ y_5,\ y_6,\ y_8,\ y_{11},\ y_9,\ y_{10},\ y_{14},\ y_3$
T_{175}	(1)	$y_1,\ y_2,\ y_4,\ y_6,\ y_{10},\ y_{12},\ y_{14},\ y_7,\ y_3,\ y_9,\ y_{11},\ y_{15},\ y_8,\ y_{13},\ y_5$
T_{176}	(2)	$y_1,\ y_6,\ y_7,\ y_2,\ y_5,\ y_{13},\ y_9,\ y_{11},\ y_4,\ y_3,\ y_{12},\ y_{14},\ y_{15},\ y_{10},\ y_8$
T_{177}	(2)	$y_1,\ y_2,\ y_6,\ y_{15},\ y_9,\ y_{11},\ y_{10},\ y_{12},\ y_5,\ y_8,\ y_3,\ y_4,\ y_{13},\ y_{14},\ y_7$
T_{178}	(1)	$y_1,\ y_6,\ y_8,\ y_4,\ y_2,\ y_{13},\ y_9,\ y_{12},\ y_{11},\ y_{14},\ y_7,\ y_5,\ y_{15},\ y_3,\ y_{10}$
T_{179}	(1)	$y_1,\ y_2,\ y_3,\ y_9,\ y_{15},\ y_{11},\ y_{10},\ y_7,\ y_{14},\ y_{12},\ y_{13},\ y_8,\ y_6,\ y_4,\ y_5$
T_{180}	(1)	$y_1,\ y_2,\ y_6,\ y_4,\ y_{10},\ y_{12},\ y_{15},\ y_{14},\ y_9,\ y_5,\ y_{11},\ y_3,\ y_{13},\ y_8,\ y_7$
T_{181}	(1)	$y_2,\ y_{10},\ y_{12},\ y_{15},\ y_1,\ y_{14},\ y_6,\ y_8,\ y_4,\ y_7,\ y_{13},\ y_9,\ y_3,\ y_{11},\ y_5$
T_{182}	(1)	$y_1,\ y_2,\ y_6,\ y_{13},\ y_{10},\ y_{11},\ y_7,\ y_3,\ y_9,\ y_{15},\ y_8,\ y_{12},\ y_4,\ y_{14},\ y_5$
T_{183}	(2)	$y_1,\ y_2,\ y_3,\ y_8,\ y_9,\ y_{15},\ y_5,\ y_{11},\ y_{10},\ y_6,\ y_7,\ y_{14},\ y_4,\ y_{13},\ y_{12}$
T_{184}	(1)	$y_2,\ y_5,\ y_{10},\ y_{15},\ y_1,\ y_{14},\ y_6,\ y_{12},\ y_{11},\ y_7,\ y_{13},\ y_4,\ y_3,\ y_9,\ y_8$
T_{185}	(1)	$y_1,\ y_2,\ y_{12},\ y_8,\ y_{14},\ y_{15},\ y_4,\ y_9,\ y_5,\ y_{11},\ y_{10},\ y_{13},\ y_7,\ y_3,\ y_6$
T_{186}	(1)	$y_1,\ y_2,\ y_4,\ y_{10},\ y_{12},\ y_6,\ y_7,\ y_5,\ y_3,\ y_{14},\ y_{11},\ y_{15},\ y_8,\ y_{13},\ y_9$
T_{187}	(1)	$y_2,\ y_{14},\ y_{13},\ y_{15},\ y_3,\ y_4,\ y_{10},\ y_8,\ y_{11},\ y_1,\ y_{12},\ y_5,\ y_6,\ y_9,\ y_7$
T_{188}	(2)	$y_1,\ y_3,\ y_4,\ y_9,\ y_{14},\ y_6,\ y_{15},\ y_2,\ y_{11},\ y_{10},\ y_5,\ y_8,\ y_{12},\ y_7,\ y_{13}$
T_{189}	(1)	$y_1,\ y_2,\ y_7,\ y_{10},\ y_{12},\ y_4,\ y_6,\ y_{14},\ y_3,\ y_{11},\ y_8,\ y_5,\ y_9,\ y_{15},\ y_{13}$
T_{190}	(2)	$y_1,\ y_2,\ y_4,\ y_{15},\ y_{12},\ y_6,\ y_7,\ y_5,\ y_3,\ y_{14},\ y_{11},\ y_{10},\ y_8,\ y_{13},\ y_9$
T_{191}	(2)	$y_1,\ y_2,\ y_3,\ y_4,\ y_{14},\ y_{15},\ y_{10},\ y_7,\ y_6,\ y_{13},\ y_{12},\ y_{11},\ y_8,\ y_5,\ y_9$

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