Journal of Algebra 325 (2011) 431–439

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Fundamental domains for congruence subgroups of SL_2 in positive characteristic ☆

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ARTICLE INFO

Article history:
Received 28 May 2010
Available online 5 November 2010
Communicated by Gerhard Hiss

MSC:
primary 20E08, 05C25, 20-04
secondary 20F32

Keywords:
Groups acting on trees
Bruhat–Tits tree
Fundamental domain
Special linear group

ABSTRACT

In this work, we construct fundamental domains for congruence subgroups of $\mathrm{SL}_2(\mathbb{F}_q[t])$ and $\mathrm{PGL}_2(\mathbb{F}_q[t])$. Our method uses Gekeler's description of the fundamental domains on the Bruhat–Tits tree $X = X_{q+1}$ in terms of cosets of subgroups. We compute the fundamental domains for a number of congruence subgroups explicitly as graphs of groups using the computer algebra system Magma.

1. Introduction

We construct fundamental domains for congruence subgroups of the group $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t])$ which is a nonuniform lattice subgroup of $G = \mathrm{SL}_2(\mathbb{F}_q((t^{-1})))$. These congruence subgroups have the form $\Gamma(g) = \{A \in \mathrm{SL}_2(\mathbb{F}_q[t]) \mid A \equiv I_2 \bmod g\}$ for some $g \in \mathbb{F}_q[t]$. Our method is to explicitly construct the

☆ We are indebted to Gunther Cornelissen for extremely helpful discussions which led us to the completion of this work. We are also grateful to Gunther for notifying us of the work of Gekeler–Nonnengardt and Rust, and for referring us to Max Gebhardt. We thank Max Gebhardt for providing us with the details of his unpublished computations. We are extremely grateful to Ernst-Ulrich Gekeler for his careful reading of our manuscript which led to a number of improvements and for clarifying the details of this earlier work. We thank Dimitri Leemans for helping us with theory and computation for coset graphs.

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[1] The author was supported in part by NSF grant # DMS-0701176.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2010.08.028

fundamental domain for $\Gamma(g)$ as a graph which is a 'ramified covering' of the quotient graph for Γ on the Bruhat–Tits tree $X = X_{q+1}$ of G. This approach is consistent with the theory of branched topological coverings and coincides with a method suggested by Drinfeld in his theory of modular curves over function fields [Dri77]. This method of fibering the graphs $\Gamma(g)\backslash X$ over $\Gamma\backslash X$ and describing the vertices and edges of $\Gamma(g)\backslash X$ as suitable cosets in Γ first appeared in the doctoral thesis of Gekeler [Gek80] (see also [Gek85]). Gekeler and Nonnengardt [GN95] and Rust [Rus98] have also given independent constructions of fundamental domains of lattices for congruence subgroups.

The structural properties of the quotient graphs obtained as ramified coverings are nontrivial to determine. We use the Magma computer algebra system [BC97] to construct explicit examples. This involves a number of advanced features of Magma including finite matrix groups, graph isomorphism [McK81], and finite geometries [JL04]. We drew some of the resulting graphs with the program `dot` which is part of the Graphviz graph visualization system [GN00].

Our initial motivation for this work was to obtain explicit examples of Morgenstern's construction of fundamental domains for congruence subgroups of the lattice $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q[t])$ [Mor95]. Let X_g denote the quotient graph of the Bruhat–Tits tree $X = X_{q+1}$ by $\Gamma(g)$. Morgenstern proved that certain subgraphs of X_g provide the first known examples of linear families of bounded concentrators [Mor95]. In the abstract and Section 3 of his paper [Mor95], he also gives a construction of the graph X_g in terms of cosets, following the method of Gekeler. We have explicitly constructed these coset graphs, and found that they are disconnected in characteristic 2, and so cannot be quotient graphs by the action of congruence subgroups on the Bruhat–Tits tree. Moreover the subgraphs at levels $0 - 1$, which he claims are bounded concentrators, are also not connected in characteristic 2. We believe that his error is confined to the construction of X_g as a coset graph, and does not effect his main results. We clarify the construction of Morgenstern and we prove that his full graphs are connected only in odd characteristic (Sections 4 and 5).

We mention the following related results. After preparation of this manuscript, we learned that independent computations by Max Gebhardt [Geb08] also show that Morgenstern's graphs are not connected. Chris Hall has notified us that he wrote an explicit algorithm for constructing fundamental domains based on earlier work of Gekeler [Hal03]. The Master's Thesis of Ralf Butenuth [But07] contains a construction of arbitrary congruence subgroups of $\mathrm{PGL}_2(\mathbb{F}_q[t])$. He also implemented an algorithm, using sieving methods but no advanced Magma functions, to compute the quotient graphs of Bruhat–Tits trees by congruence subgroups [But07].

2. Fundamental domains for congruence subgroups of $\mathrm{SL}_2(\mathbb{F}_q[t])$ as ramified coverings

In this section we give a construction of quotient graphs for congruence subgroups of $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t])$ acting on the Bruhat–Tits tree $X = X_{q+1}$ of $G = \mathrm{SL}_2(\mathbb{F}_q((t^{-1})))$ as ramified coverings over $\Gamma\backslash X$. This construction is in terms of cosets, following the method of Gekeler [Gek80,Gek85].

2.1. Ramified coverings

Our graphs are connected, oriented and locally finite. A tree is a nonempty graph without closed circuits. Suppose a group Γ acts on a tree X without inversions. Then the quotient graph $\Gamma\backslash X$ is well defined and there is a natural quotient morphism $X \to \Gamma\backslash X$. Given a normal subgroup N in Γ, we can define the quotient graph $N\backslash X$ by

$$V(N\backslash X) = N\backslash V(X) = \{N \cdot v \mid v \in V(X)\}, \qquad E(N\backslash X) = N\backslash E(X) = \{N \cdot e \mid e \in E(X)\}.$$

Then Γ/N acts on $N\backslash X$ by $\gamma N(N \cdot x) = N \cdot \gamma x$, where x denotes either a vertex or an edge of X. Equivalently, we can take the graph $N\backslash X$ to have vertices (respectively edges) given by cosets of $\mathrm{Stab}_\Gamma(x)\Gamma/N$ in Γ/N, $x \in V(\Gamma\backslash X)$ (respectively cosets of $\mathrm{Stab}_\Gamma(e)\Gamma/N$ in Γ/N, $e \in E(\Gamma\backslash X)$). Cosets are adjacent as vertices in the graph $N\backslash X$ if and only if their intersection is nonempty. We call this construction of $N\backslash X$ a *ramified covering* over $\Gamma\backslash X$.

2.2. Fundamental domain for $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t])$

Let $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t]) \leqslant G = \mathrm{SL}_2(\mathbb{F}_q((t^{-1})))$. The Bruhat–Tits building of G is the $(q+1)$-homogeneous tree $X = X_{q+1}$ [Ser03]. Serre [Ser03] gives the fundamental domain for $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t])$ on X as a semi-infinite ray [Ser03, Proposition 3, p. 87]. We construct fundamental domains for congruence subgroups of Γ as ramified coverings over $\Gamma\backslash X$. Since these subgroups are normal, there is also an action by the quotient groups.

Let Γ be a group and X a tree. Suppose Γ acts on X. If N is a normal subgroup of Γ, then Γ/N acts on the connected graph $N\backslash X$. Each $Nx \in N\backslash X$ has stabilizer $\mathrm{Stab}_{\Gamma/N}(Nx) = N\,\mathrm{Stab}_\Gamma(x)/N$.

Therefore, given a normal subgroup N of Γ, we may describe the vertices (respectively edges) of $N\backslash X$ not only as N-orbits with respect to the action of N on X, but as Γ/N-orbits of $\{Nv\colon v \in V(\Gamma\backslash X)\}$ (respectively of $\{Ne\colon e \in E(\Gamma\backslash X)\}$).

2.3. Levelled coset graphs

Let H be a group and let $H_0, H_1, H_2, \ldots$ be a (finite or infinite) sequence of subgroups of H. We define the *levelled coset graph* given by $H_0, H_1, \ldots \leqslant H$ as follows: The vertex set is partitioned into levels $L_0, L_1, \ldots,$ with vertices at level i corresponding to cosets hH_i, for $h \in H$. There is an edge connecting hH_i with kH_{i+1} if, and only if $hH_i \cap kH_{i+1} \neq \emptyset$. There are no edges between vertices in non-adjacent levels.

It is easy to show that the edges between levels i and $i+1$ correspond to the cosets of $H_i \cap H_{i+1}$. The edge connecting hH_i to kH_{i+1} corresponds to $jH_i \cap H_{i+1}$, for some j in the intersection of hH_i and kH_{i+1}.

We consider levelled coset graphs with $H_1 \leqslant H_2 \leqslant \cdots$. The following proposition is a slight modification of a standard result for coset graphs:

Proposition 2.1. *The levelled coset graph given by $H_0, H_1, \ldots, H_{n-1} \leqslant H$ with*

$$H_1 \leqslant H_2 \leqslant \cdots \leqslant H_{n-1}$$

has $|H : \langle H_0, H_{n-1}\rangle|$ connected components.

2.4. The levels of X_g

Fix $g \in \mathbb{F}_q[t]$ of degree n. Since $\Gamma(g) = \{A \in \mathrm{PGL}_2(\mathbb{F}_q[t]) \mid A \equiv I_2 \bmod g\}$ is normal in Γ, the quotient graph $X_g = \Gamma(g)\backslash X$ may be viewed as a ramified covering of the quotient graph $\Gamma\backslash X$, which is a semi-infinite ray. We may partition $V(X)$ into Γ-orbits of the Λ_i. Since $\Gamma_i = \mathrm{Stab}_\Gamma(\Lambda_i)$ by [Ser03, Proposition 3, p. 87] the orbit $\Gamma \cdot \Lambda_i$ is in one-to-one correspondence with Γ/Γ_i. Similarly the edges between $\Gamma \cdot \Lambda_i$ and $\Gamma \cdot \Lambda_{i+1}$ correspond to $\Gamma/(\Gamma_i \cap \Gamma_{i+1})$. So we make the identifications

$$V(X) = \bigsqcup_{i \geqslant 0} \Gamma/\Gamma_i, \qquad E(X) = \bigsqcup_{i \geqslant 0} \Gamma/(\Gamma_i \cap \Gamma_{i+1}).$$

Defined in this way, X is the levelled coset graph for $\Gamma_0, \Gamma_1, \Gamma_2, \ldots \leqslant \Gamma$. We can now describe the vertices and edges of $X_g = \Gamma(g)\backslash X$ as follows:

$$V(X_g) = \bigsqcup_{i \geqslant 0} \Gamma(g)\backslash(\Gamma/\Gamma_i), \qquad E(X_g) = \bigsqcup_{i \geqslant 0} \Gamma(g)\backslash\big(\Gamma/(\Gamma_i \cap \Gamma_{i+1})\big).$$

Define groups $H = \Gamma/\Gamma(g)$ and $H_i = \Gamma_i\Gamma(g)/\Gamma(g)$, and coset spaces $L_i = H/H_i$. We have $\mathrm{Stab}_\Gamma(\Lambda_i) = \Gamma_i$ and so $\mathrm{Stab}_H(\Gamma(g) \cdot \Lambda_i) = (\Gamma_i\Gamma(g))/\Gamma(g) = H_i$. Thus we can identify $\Gamma(g)\backslash(\Gamma/\Gamma_i)$ with L_i. Similarly $\Gamma(g)\backslash(\Gamma/(\Gamma_i \cap \Gamma_{i+1}))$ can be identified with $H/H_i \cap H_{i+1}$. So X_g can be viewed as the levelled coset graph of $H_0, H_1, H_2, \ldots \leqslant H$.

We can now establish that $H = \Gamma/\Gamma(g) \cong \mathrm{SL}_2(R_g)$ where $R_g = \mathbb{F}_q[t]/(g)$. The argument is the same as in [Shi94] for $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 2.2. *The map* $\mathrm{SL}_2(\mathbb{F}_q[t]) \to \mathrm{SL}_2(R_g)$ *given by* $A \mapsto A \bmod (g)$ *is surjective.*

2.5. The structure of X_g

Note that we could use Proposition 2.1 to prove that X_g is connected, but this already follows from the fact that X_g is a quotient of a connected graph.

Write $g = \prod_{i=1}^{s} g_i^{n_i}$ where the g_i are distinct irreducible polynomials with $\deg(g_i) = d_i$ and $\sum_i n_i d_i = n$. Then

$$R_g \cong \bigoplus_{i=1}^{s} R_i \quad \text{where } R_i := R_{g_i^{n_i}} \cong \mathbb{F}_{q^{d_i}}[t_i]/\big(t_i^{n_i}\big).$$

By Corollary 2.4 of [Han06],

$$R_g^\times \cong \prod_i R_i^\times \quad \text{and} \quad \mathrm{GL}_2(R_g) \cong \prod_i \mathrm{GL}_2\big(R_i^\times\big).$$

Using Theorem 2.7(3) of [Han06] we get $|H| = |\mathrm{SL}_2(R_g)| = \frac{|\mathrm{GL}_2(R_g)|}{|R^\times|} = q^{3n}\Pi(q)$, where $\Pi(q) := \prod_i (1 - \frac{1}{q^{2d_i}})$. Now $H_{n-1} = H_n = H_{n+1} = \cdots$, and so X_g is a bipartite graph which may be described as a collection of disjoint infinite rays beginning at each vertex of level L_{n-1}. It suffices to describe the graph induced by levels 0 through $n-1$. For $i \leqslant n-1$, we have $\Gamma_i \cap \Gamma(g) = \{1\}$, so $H_i \cong \Gamma_i$. Now $H_0 = \mathrm{SL}_2(q)$, and H_i is a semidirect product of $(\mathbb{F}_q^+)^{\min(n,i+1)}$ by $\mathbb{F}_q^\times$. So we have formulas for the number of vertices in each level:

$$|L_i| = \begin{cases} q^{3n-3}\Pi(q)(1 - \frac{1}{q^2})^{-1} & \text{for } i = 0, \\ q^{3n-2-i}\Pi(q)(1 - \frac{1}{q})^{-1} & \text{for } 0 < i < n, \\ q^{2n-2}\Pi(q)(1 - \frac{1}{q})^{-1} & \text{for } i \geqslant n. \end{cases}$$

Remark 2.3.

(1) The edges run between consecutive levels, with the edges between L_i and L_{i+1} in the orbit of the edge $\Lambda_i \to \Lambda_{i+1}$ in X.
(2) The subgraph induced by L_0 and L_1 is a $(q+1, q)$-regular bipartite graph.
(3) For $i = 1, \ldots, n-1$, each vertex in L_i has q edges to vertices in L_{i-1} and only 1 edge to a vertex in L_{i+1}. For $i \geqslant n$, each vertex in L_i has one edge to L_{i-1} and one edge to L_{i+1}. So there is a semi-infinite ray, also called a cusp, attached to each vertex in L_{n-1}.

We have

$$\mathrm{Stab}_{\Gamma(g)}(\Lambda_i) = \Gamma_i \cap \Gamma(g) = \begin{cases} \{1\} & \text{if } i < n, \\ U_i = \{\left(\begin{smallmatrix} 1 & gf \\ 0 & 1 \end{smallmatrix}\right) \mid f \in \mathbb{F}_q[t],\ \deg(f) \leqslant i - n\} & \text{if } i \geqslant n. \end{cases}$$

The stabilizer of any vertex in L_i is then conjugate to $\Gamma_i \cap \Gamma(g)$. Thus the 'core' vertices in the graph of groups are labeled with the trivial group, and the 'cusp' vertex groups along each ray are of the form $s_j U_i s_j^{-1}$, where $\{s_j \mid j = 1, \ldots, k = (q+1)q^{2(n-1)}\}$ is a set of conjugacy class representatives.

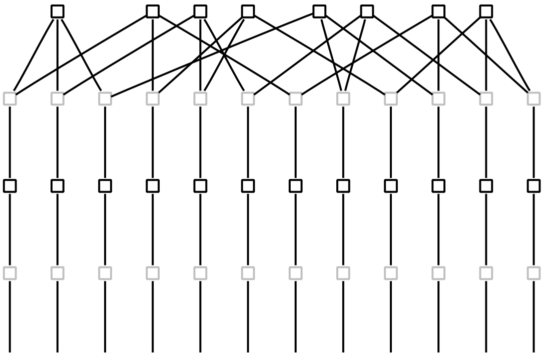

Fig. 1. X_g for $g(t) = t^2$, $q = 2$.

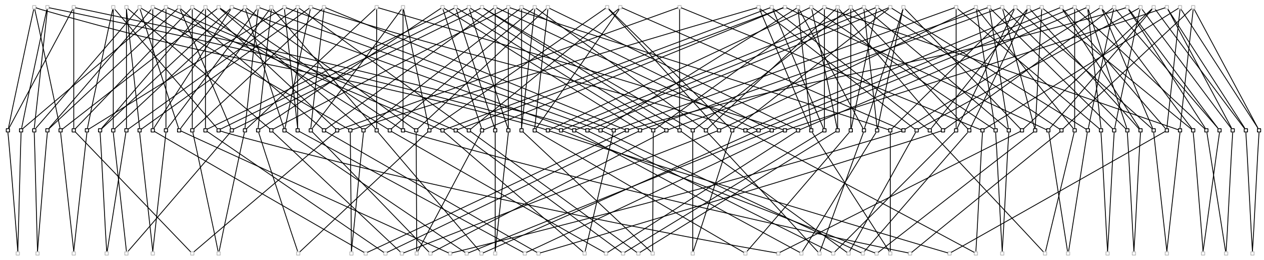

Fig. 2. Core of X_g for $g(t) = t^3$, $q = 2$.

2.6. Detailed examples of fundamental domains for congruence subgroups

In this subsection we construct certain specific examples of the graph X_g for the congruence subgroups of SL_2. When g is linear, we have $|L_0| = 1$ and $|L_i| = q + 1$ for $i \geqslant 1$. Thus X_g consists of a single core vertex plus $q + 1$ cusps which are semi-infinite rays.

Let $g(t) = t^2$. Then $|L_0| = q^3$ and $|L_i| = (q + 1)q^2$ for $i \geqslant 1$. The first two levels form a $(q + 1, q)$-regular bipartite graph, and semi-infinite rays are attached to each vertex in level L_1. The graph X_g for $q = 2$ is given in Fig. 1. The odd and even levels of vertices give the bipartition of Remark 2.3(1).

Let $g(t) = t^3$. Here, $|L_0| = q^6$, $|L_1| = (q + 1)q^5$ and $|L_i| = (q + 1)q^4$ for $i \geqslant 2$. The bipartite graph between the first two levels is $(q + 1, q)$-regular, and then the graph collapses once by a factor of q before extending onward as infinite rays. The core graph for $q = 2$ is given in Fig. 2, with the rows of vertices top to bottom corresponding to L_0, L_1 and L_2, respectively.

We used Magma to construct these graphs. The groups H and H_i are constructed as matrix groups of degree $2n$ over $\mathbb{F}_q$, and then the coset graphs are constructed using code due to Leemans [JL04]. We used `dot` to draw Figs. 1 and 2 [GN00].

3. Fundamental domains for congruence subgroups of $\mathrm{PGL}_2(\mathbb{F}_q[t])$

In [Mor95], Morgenstern's motivation was to provide the first known examples of linear families of bounded concentrators. We prove however that, in characteristic 2, Morgenstern's constructions yield graphs that are not connected. The main source of Morgenstern's error was his incorrect assumption that $\Gamma/\Gamma(g) \cong \mathrm{PGL}_2(R_g)$ where $R_g = \mathbb{F}_q[t]/(g)$. The correct formula for $\Gamma/\Gamma(g)$ is somewhat more complicated and is given in this section. We denote the corrected graphs for PGL_2 by $\overline{X}_g$, and Morgenstern's incorrect coset construction by $\widetilde{X}_g$ (in [Mor95], both are denoted X_g).

Let $\overline{\Gamma} = \mathrm{PGL}_2(\mathbb{F}_q[t])$ and let $\overline{\Gamma}(g) = \{A \in \overline{\Gamma} \mid A \equiv I_2 \bmod (g)\}$. Let $\overline{X}_g$ be the graph defined for PGL in the analogous manner to the graph X_g from the previous section.

First we describe the structure of $\overline{H} := \overline{\Gamma}/\overline{\Gamma}(g)$. The proof is straightforward.

Proposition 3.1. *$\overline{H} \cong (\mathrm{SL}_2(R_g) \rtimes F)/Z$ where $F = \{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_q^\times\}$ and $Z = \mathbb{F}_q^\times I_2$.*

Theorem 3.2. *The* PGL_2 *graph* $\overline{X}_g$ *is isomorphic to the* SL_2 *graph* X_g.

Proof. We define a map ϕ from the vertices of X_g to the vertices of $\overline{X}_g$ by $H_i x \mapsto H_i F x/Z$. Note that $H_i F = F H_i$ for all i. Recall that the edge between $H_i x$ and $H_{i+1} x$ corresponds to the coset $(H_i \cap H_{i+1})x$. Similarly the edge between $H_i F x/Z$ and $H_{i+1} F x/Z$ corresponds to the coset $(H_i F \cap H_{i+1} F)x/Z$. So to prove that ϕ takes every edge to an edge it suffices to show that $H_i F \cap H_{i+1} F = (H_i \cap H_{i+1})$. Clearly $(H_i \cap H_{i+1}) \subseteq H_i F \cap H_{i+1} F$. Conversely suppose $hf = kg$ for $h \in H_i$, $k \in H_{i+1}$, $f, g \in F$. Then $f = \mathrm{diag}(a, 1) = g$ where $a = \det(hf) = \det(kg)$, and so $h = k \in H_i \cap H_{i+1}$. Finally we can conclude that ϕ is an isomorphism since the number of edges at level i is the same for the two graphs. □

In particular, $\overline{X}_g$ is always connected, unlike the graph $\widetilde{X}_g$ constructed in [Mor95].

4. Morgenstern's graphs

4.1. Morgenstern's PGL graph

Let $\widetilde{H} = \mathrm{PGL}_2(R_g) = \mathrm{GL}_2(R_g)/\widetilde{Z}$, where $\widetilde{Z} = R_g^\times I_2$. Let $\widetilde{H}_i$ be the subgroup $H_i F \widetilde{Z}/\widetilde{Z}$, and define levels $\widetilde{L}_i = \mathrm{PGL}_2(R_g)/\widetilde{H}_i$. Morgenstern's graph $\widetilde{X}_g$ is now defined as the levelled coset graph for $\widetilde{H}_0, \widetilde{H}_1, \ldots$ in $\widetilde{H}$. This is analogous to the constructions of X_g in Section 3.1 and $\overline{X}_g$ in Section 4. Furthermore

$$|H| = |\overline{H}| = |\widetilde{H}|, \qquad |H_i| = |\overline{H}_i| = |\widetilde{H}_i|, \qquad |H_i \cap H_{i+1}| = |\overline{H}_i \cap \overline{H}_{i+1}| = |\widetilde{H}_i \cap \widetilde{H}_{i+1}|$$

for all $i \geqslant 0$. Hence the properties of Remark 2.3 hold for all three graphs. We have already seen that $X_g \cong \overline{X}_g$. Morgenstern claims that the graphs $\overline{X}_g$ and $\widetilde{X}_g$ are isomorphic, but we will see that this is not always the case. This is a consequence of the fact that Morgenstern fails to prove that he has the desired ramified covering. We now consider connectedness properties of $\widetilde{X}_g$.

Proposition 4.1. *Morgenstern's graph* $\widetilde{X}_g$ *has* $|R_g^\times : \mathbb{F}_q^\times R_g^{\times 2}|$ *connected components, where* $R_g^{\times 2} = \{x^2 \mid x \in R_g^\times\}$.

Proof. By the connectedness of X_g and Proposition 2.1, we know $\langle H_0, H_{n-1}\rangle = H$. Hence

$$\langle \widetilde{H}_0, \widetilde{H}_{n-1}\rangle = \langle H_0 F\widetilde{Z}, \widetilde{H}_{n-1} F\widetilde{Z}\rangle/\widetilde{Z} = \langle H_0, H_{n-1}\rangle F\widetilde{Z}/\widetilde{Z} = HF\widetilde{Z}/\widetilde{Z}.$$

Since det maps $\mathrm{GL}_2(R_g)$ onto $R_n^\times$ with kernel H, we have

$$\mathrm{GL}_2(R_g)/HF\widetilde{Z} \cong \mathbb{F}_q^\times R_n^\times / \det(F\widetilde{Z}) = R_n^\times/\mathbb{F}_q^\times R_n^{\times 2}. \qquad \square$$

Lemma 4.2. *Let* $R = \mathbb{E}[u]/(u^n)$ *where* $\mathbb{E} := \mathbb{F}_{q^d}$.

(1) *If q is odd, then* $R^{\times 2} = \mathbb{E}^{\times 2} + \mathbb{E}u + \mathbb{E}u^2 + \cdots$ *and so* $\mathbb{E}^\times R^{\times 2} = R^\times$.
(2) *If q is even, then* $R^{\times 2} = \mathbb{E}^\times R^{\times 2} = \mathbb{E}^\times + \mathbb{E}u^2 + \mathbb{E}u^4 + \cdots$.

Proof. For q even, $(a_0 + a_1 u + a_2 u^2 + \cdots)^2 = a_0^2 + a_1^2 u^2 + a_2^2 u^4 + \cdots$, for all $a_i \in \mathbb{E}$. Using the fact that $\mathbb{E}^{\times 2} = \mathbb{E}^\times$, we get $R^{\times 2} = \mathbb{E}^\times + \mathbb{E}u^2 + \mathbb{E}u^4 + \cdots$.

Now let q be odd. It suffices to show that every element of the form $1 + a_1 u + \cdots$ is in $R^{\times 2}$. Suppose this is not true, and take $a = 1 + a_i u^i + \cdots \notin R^{\times 2}$ with i maximal such that $a_i \neq 0$. But $R^{\times 2}$ is a subgroup of $R^\times$, and so $a(1 - \frac{a_i}{2}u)^2 \notin R^{\times 2}$. Since the coefficients of $u, u^2, \ldots, u^i$ are all zero in this element, we have a contradiction. □

Theorem 4.3. *Morgenstern's graph $\widetilde{X}_g$ is connected if and only if q is odd or g is squarefree.*

Proof. This follows immediately from the previous two results and the decomposition $R_g \cong \bigoplus_{i=1}^{s} \mathbb{F}_{q^{d_i}}[t_i]/(t_i^{n_i})$. $\square$

In particular, $\widetilde{X}_g$ is not isomorphic to $\overline{X}_g$ when q is even and g is not squarefree. By Magma computation using the algorithm of [McK81], we found that X_{t^n} and $\widetilde{X}_{t^n}$ are also nonisomorphic for $q = 3$ and $n = 2, 3, 4$.

4.2. The subgraphs of levels 0–1

Morgenstern constructed $\widetilde{X}_g$ as a means of providing examples of linear families of bounded concentrators. These examples were obtained as the subgraph $\widetilde{D}_g(0-1)$ induced by the vertices of $\widetilde{X}_g$ in the first two levels $\widetilde{L}_0$ and $\widetilde{L}_1$. However, a necessary property for a bounded concentrator is connectedness. We will show in characteristic 2 that the subgraphs $\widetilde{D}_g(0-1)$ are not connected. This contradicts the following claim of Morgenstern:

> [Mor95], *Proposition* 4.2: If $q \geqslant 4$, or $q = 3$ and $g(x)$ is irreducible of degree greater than 2, then $\widetilde{D}_g(0-1)$ is connected.

This in turn is based on an incorrect lower bound for $N_0(S)$, the set of vertices in $\widetilde{L}_0$ which are adjacent to a subset $S \subseteq \widetilde{L}_1$ of vertices in $\widetilde{L}_1$:

> [Mor95], *Lemma* 4.1: For every $S \subseteq \widetilde{L}_1$, $\frac{|N_0(S)|}{|S|} \geqslant \frac{q|\widetilde{L}_1|}{(q-3)|S|+4|\widetilde{L}_1|}$.

This bound fails if we take S to be a connected component of one of the disconnected graphs described below. We believe that these two results are correct when applied to the correct fundamental domain $\overline{X}_g$ for PGL_2 described in Section 4. We note that, when $\widetilde{D}_g(0-1)$ is not connected, all the connected components are isomorphic. Furthermore H acts transitively on the set of components. This follows from general properties of coset graphs.

In the remainder of this section we consider connectedness properties of $\widetilde{D}_g(0-1)$ and the corresponding subgraph $D_g(0-1)$ induced on the first two levels of X_g (or equivalently $\overline{X}_g$). By Proposition 2.1, the number of components of $D_g(0-1)$ is

$$C := \bigl|H : \langle H_0, H_1\rangle\bigr|,$$

and the number of components $\widetilde{D}_g(0-1)$ is

$$\widetilde{C} := \bigl|\widetilde{H} : \langle \widetilde{H}_0, \widetilde{H}_1\rangle\bigr| = \bigl|\mathrm{GL}_2(R_g) : \langle H_0, H_1\rangle F\widetilde{Z}\bigr|.$$

This allows us to count components using Magma's matrix group machinery. These results, for even q and $g(t) = t^n$, are summarized in Table 1. For odd q we found both graphs to be connected in every example we computed.

Based on these experimental results, we conjecture formulas:

Conjecture 4.4. *For $g(t) = t^n$ over $\mathbb{F}_q$,*

$$C = \begin{cases} q^{\lfloor (3n-5)/2\rfloor} & \textit{for } q = 2, n > 2, \\ 1 & \textit{for } q > 2, \end{cases}$$

Table 1
Number of components of the first two levels for q even

q	2												
n	2	3	4	5	6	7	8	9	10	11	12	13	14
C	1	2^2	2^3	2^5	2^6	2^8	2^9	2^{11}	2^{12}	2^{14}	2^{15}	2^{17}	2^{18}
$\widetilde{C}$	2^1	2^3	2^4	2^6	2^7	$2^1$0	2^{11}	2^{13}	2^{14}	2^{17}	2^{18}	2^{20}	2^{21}
q	2												
n	15	16	17	18	19	20	21	22	23	24	25	26	
C	2^{20}	2^{21}	2^{23}	2^{24}	2^{26}	2^{27}	2^{29}	2^{30}	2^{32}	2^{33}	2^{35}	2^{36}	
$\widetilde{C}$	2^{24}	2^{25}	2^{27}	2^{28}	2^{31}	2^{32}	2^{34}	2^{35}	2^{38}	2^{39}	2^{41}	2^{42}	
q	4												
n	2	3	4	5	6	7	8	9	10	11	12	13	
C	1	1	1	1	1	1	1	1	1	1	1	1	
$\widetilde{C}$	2^2	2^2	2^4	2^4	2^6	2^6	2^8	2^8	2^{10}	2^{10}	2^{12}	2^{12}	
q	8					16				32		64	
n	2	3	4	5	6	7	2	3	4	2	3	2	
C	1	1	1	1	1	1	1	1	1	1	1	1	
$\widetilde{C}$	2^3	2^3	2^6	2^6	2^9	2^9	2^4	2^4	2^8	2^5	2^5	2^6	

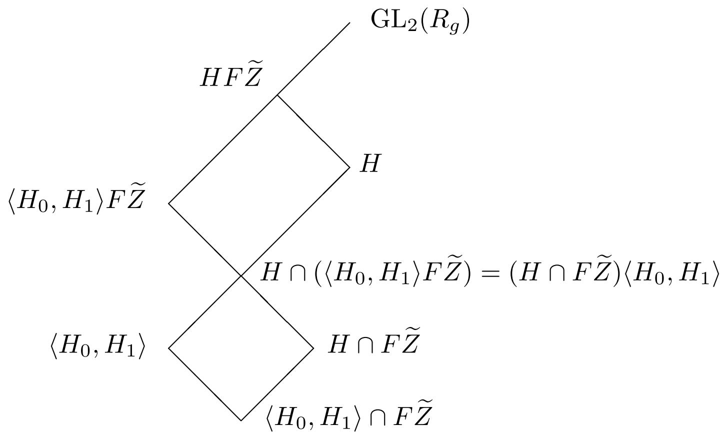

Fig. 3. Subgroup lattice.

$$\widetilde{C} = \begin{cases} q^{\lfloor (3n-5)/2\rfloor + \lfloor (n+1)/4\rfloor} & \text{for } q = 2, n > 2, \\ q^{\lfloor n/2\rfloor} & \text{for } q > 2 \text{ even}, n > 1, \\ 1 & \text{for } q \text{ odd}. \end{cases}$$

We now give some theoretical results on the number of components for arbitrary g.

Proposition 4.5.

$$C \cdot \left|R_g^\times : \mathbb{F}_q^\times R_g^{\times 2}\right| = \widetilde{C} \cdot |S : T|$$

where $S := \{a \in R_g^\times \mid a^2 \in \mathbb{F}_q^\times\}$ *and* $T := \{a \in S \mid \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \in \langle H_0, H_1\rangle\}$.

Proof. From Fig. 3, we can see that

$$C \cdot \left|\mathrm{GL}_2(R_g) : HF\widetilde{Z}\right| = \widetilde{C} \cdot \left|H \cap F\widetilde{Z} : \langle H_0, H_1\rangle \cap F\widetilde{Z}\right|.$$

Since det maps G onto $R_n^\times$ with kernel H, we have $\mathrm{GL}_2(R_g)/HF\widetilde{Z} \cong R_g^\times/\det(F\widetilde{Z}) = R_g^\times/\mathbb{F}_q^\times R_g^{\times 2}$. An element of $F\widetilde{Z}$ has the form $x = \left(\begin{smallmatrix}\lambda a & 0\\ 0 & a\end{smallmatrix}\right)$, for $\lambda \in \mathbb{F}_q^\times$ and $a \in R_g^\times$. And $x \in H$ is equivalent to $a^2 = \lambda^{-1} \in \mathbb{F}_q^\times$, so projection onto the bottom right entry gives an isomorphism from $H \cap F\widetilde{Z}$ to S. Clearly the subgroup $\langle H_0, H_1\rangle \cap F\widetilde{Z}$ corresponds to the T under this isomorphism. □

Proposition 4.6. *If q is odd and $g(t) = t^n$, then $C = \widetilde{C}$.*

Proof. We have $\mathbb{F}_q^\times R_g^{\times 2} = R_g^\times$ by Lemma 4.2. If $a = a_0 + a_i t^i + \cdots \in S$ with a_i the smallest nonzero coefficient other than a_0, then $a^2 = a_0 + 2a_i t^i + \cdots = 1$ and so $i \geqslant n$. Hence $S = \mathbb{F}_q^\times$, and it is now easy to prove that $T = S$. □

Proposition 4.7. *If q is even and g is not squarefree, then $\widetilde{C} > C$.*

Proof. By Lemma 9 and the decomposition $R = \bigoplus_r R_i$, we get $|R_g^\times : \mathbb{F}_q^\times R_g^{\times 2}| = \prod_i q^{d_i \lfloor n_i/2 \rfloor}$. Now suppose $a = a_0 + a_1 t_i + a_2 t_i^2 + \cdots \in R_i$ with $a^2 = 1$. This is equivalent to $a_0 = 1$, and $a_i = 0$ for all $j > 0$ with $2j < n_i$. Hence $|S| = \prod_i q^{d_i \lfloor n_i/2 \rfloor}$.

We now have $\widetilde{C} = |T|C$. But if $2^e < n_i \leqslant 2^{e+1}$, then $a = 1 + t_i^{2^e}$ is a nontrivial element which squares to the identity. And $\langle H_0, H_1\rangle$ contains

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \left[\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]^3,$$

so T is nontrivial. □

So, for q even and g not squarefree, we know that $\widetilde{D}_g(0-1)$ is not connected, and also that it cannot be isomorphic to $D_g(0-1)$. By Magma computation using the algorithm of [McK81], we found that $D_{t^n}(0-1)$ and $\widetilde{D}_{t^n}(0-1)$ are also nonisomorphic for $q = 3$ and $n = 2, 3, 4$. However they are isomorphic for $q = 5, 7$ and $n = 2$.

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