The Haagerup property, Property (T) and the Baum-Connes conjecture for locally compact Kac-Moody groups

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Abstract

We indicate which symmetrizable locally compact affine or hyperbolic Kac-Moody groups satisfy Kazhdan's Property (T), and those that satisfy its strong negation, the Haagerup property. This reveals a new class of hyperbolic Kac-Moody groups satisfying the Haagerup property, namely symmetrizable locally compact Kac-Moody groups of rank 2 or of rank 3 noncompact hyperbolic type. These groups thus satisfy the strongest form of the Baum-Connes conjecture, namely the conjecture with coefficients in any C^*-algebra.

For symmetrizable locally compact Kac-Moody groups G of rank 3 compact hyperbolic type, or of affine or hyperbolic type and rank ≥ 4, we deduce from the work of Dymara and Januszkiewicz that G has Property (T). Our results give a dichotomy for hyperbolic Kac-Moody groups of noncompact type, with rank 3 Kac-Moody groups of noncompact hyperbolic type, such as $\widehat{A}_1^{(1)}$, satisfying the Haagerup property and hence the Baum-Connes conjecture with coefficients, and hyperbolic Kac-Moody groups of noncompact type and rank $4 \leq r \leq 10$, such as E_{10}, satisfying Property (T).

We show that Property (T) and the Haagerup property for symmetrizable locally compact affine or hyperbolic Kac-Moody groups can be determined from the Dynkin diagram, or equivalently, from the generalized Cartan matrix.

We deduce from the work of Kasparov and Skandalis that for symmetrizable locally compact Kac-Moody groups G of rank 3 compact hyperbolic type, or of affine or hyperbolic type and rank ≥ 4, the Baum-Connes assembly map on equivariant K-homology of G is injective. We show that if G is of hyperbolic type, of rank $4 \leq r \leq 10$ and G contains a cocompact lattice that acts cocompactly on its Kac-Moody Tits building then the Baum-Connes assembly map is also surjective, so G satisfies the Baum-Connes conjecture without coefficients. This leaves open the possibility that higher rank hyperbolic Kac-Moody groups with Property (T) also satisfy the Baum-Connes conjecture.

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1 Introduction

Let G be a second countable, locally compact and Hausdorff topological group. Let $C^*_{red}(G)$ denote the reduced C^*-algebra of G. Then there exists a universal space $\underline{E}G$ for proper G-actions, unique up to G-equivariant homotopy. Using the KK-theory of Kasparov we may form the equivariant K-homology $K_*^G(\underline{E}G)$ ([Ka], [HK1]). There is then an assembly map from K-homology to K-theory

$$\mu : K_*^G(\underline{E}G) \longrightarrow K_*(C^*_{red}(G)),$$

which is conjectured by Baum and Connes to be an isomorphism (see for example [BCH] and [HK1]). This has become known as the celebrated *Baum-Connes conjecture* in non-commutative geometry, relating the analytic and topological properties of a group and proven for large classes of groups.

A deep result of Higson and Kasparov gives a sufficient condition for locally compact groups to satisfy the Baum-Connes conjecture. They showed that if a locally compact group G admits an affine, isometric and metrically proper action on a Euclidean space, that is, G has the *Haagerup property*, then G satisfies the Baum-Connes conjecture with coefficients in any G-C^*-algebra ([HK1], [HK2]).

However, the Haagerup property is not equivalent to the Baum-Connes conjecture, as is implied by results of Lafforgue who exhibited certain discrete groups satisfying both the Baum-Connes conjecture and Property (T) which is a strong negation of the Haagerup property ([L1]-[L3], see also Sections 5 and 8).

In this work, we study locally compact forms of affine and hyperbolic Kac-Moody groups. These are constructed as completions of the *Tits functor* over finite fields (see Section 2) associated to infinite dimensional Kac-Moody algebras. Complete Kac-Moody groups over finite fields are locally compact and totally disconnected. Even though these groups have exponential growth and behave in many ways like infinite dimensional groups, they contain lattice subgroups which may be studied in analogy with lattices in Lie groups (see [CG], [CC], [Re1] and [RR] for example). Locally compact forms of Kac-Moody groups were anticipated by Tits ([Ti1]) but have only appeared in the literature recently ([CG] and [RR]). The main objective of this work is to explore the representation theoretic and K-theoretic properties of these groups.

We show that a certain class of locally compact hyperbolic Kac-Moody groups and their lattice subgroups have the Haagerup property, and hence satisfy the Baum-Connes conjecture with coefficients.

The results here reveal a new class of locally compact groups satisfying the Baum-Connes conjecture without ever computing K-theory of classifying spaces for proper actions, or K-homology of group C^*-algebras associated to our Kac-Moody groups. Since Kac-Moody groups are amalgams of their parabolic subgroups we believe that it should be straightforward to compute K-homology in terms of the K-homology of the parabolic subgroups. We have not yet determined the reduced C^*-algebra $C^*_{red}(G)$ of G. We hope to take these questions up elsewhere.

When G is a symmetrizable locally compact Kac-Moody group of rank 2 or of rank 3 non-compact hyperbolic type, we are able to deduce that G satisfies the Haagerup property using the following ingredients. We use the work of Dymara and Januszkiewicz giving criteria for groups with a BN-pair to have the Haagerup property and Property (T) in terms of cohomology vanishing theorems and the action on the corresponding Tits building ([DJ]), the existence of nonuniform lattice subgroups ([CG] and [Re1]), and certain aspects of the classification of hyperbolic Kac-Moody algebras ([CCCMNNP], [Li], [Sa] and Section 6). Our results follow in a straightforward but nontrivial way from the work of [DJ].

We have the following.

Theorem 1.1. *Let G be a symmetrizable locally compact affine or hyperbolic Kac-Moody group over a finite field $\mathbb{F}_q$. Assume that q is sufficiently large. If $rank(G) = 2$, or if $rank(G) = 3$ and G has noncompact hyperbolic type, then G has the Haagerup property. Hence G satisfies the Baum-Connes conjecture with coefficients in any $G - C^*$-algebra.*

If $rank(G) = 2$ the assumption of large thickness of the Tits building can be removed ([CG]). We also use [DJ] to deduce Property (T) for certain higher rank Kac-Moody groups.

Theorem 1.2. *Let G be a locally compact affine or hyperbolic Kac-Moody group over a finite field $\mathbb{F}_q$. Suppose that q is sufficiently large. If $r = rank(G) = 3$ and G has compact hyperbolic type, or if $rank(G) \geq 3$ and G has affine type, or if $4 \leq r \leq 10$ and G has hyperbolic type then*

(i) G has Property (T),

(ii) All entries in the Coxeter matrix for G are finite.

We obtain the following (Section 7).

Corollary 1.3. *Property (T) and the Haagerup property for symmetrizable locally compact affine or hyperbolic Kac-Moody groups over sufficiently large finite fields can be determined from the Dynkin diagram, or equivalently from the generalized Cartan matrix.*

Lafforgue's results ([L1]-[L3]) do not rule out the Baum-Connes conjecture for Property (T) groups. The Kac-Moody Tits building X of G is a $CAT(0)$ space. Thus X is a 'bolic space' in the sense of Kasparov and Skandalis ([KS]). For these groups, the Baum-Connes assembly map is injective ([KS]). For Kac-Moody groups of rank 3 compact hyperbolic type or of affine or hyperbolic type and rank ≥ 4 it remains to determine if this map is also surjective. We have the following (see Section 8).

Theorem 1.4. *Let G be a symmetrizable locally compact Kac-Moody group of affine or hyperbolic type. Then the Baum-Connes assembly map is injective. If G is of hyperbolic type, of rank $4 \leq r \leq 10$ and G contains a cocompact lattice that acts cocompactly on its Kac-Moody Tits building, then the Baum-Connes assembly map is also surjective. Thus G satisfies the Baum-Connes conjecture without coefficients.*

The above results can be summarized in the following table, where G is a symmetrizable locally compact affine or hyperbolic Kac-Moody group over a finite field $\mathbb{F}_q$.

Rank r of G	**Assumptions on G**	**Properties of G**
$r = 2$	affine or hyperbolic type	Haagerup property Baum-Connes conjecture with coefficients
$r = 3$	noncompact hyperbolic type q sufficiently large	Haagerup property Baum-Connes conjecture with coefficients
$r = 3$	compact hyperbolic type q sufficiently large	Property (T) Baum-Connes assembly map is injective
$r \geq 3$	affine type q sufficiently large	Property (T) Baum-Connes assembly map is injective
$4 \leq r \leq 10$	hyperbolic type q sufficiently large	Property (T) Baum-Connes assembly map is injective

The Baum-Connes conjecture has not yet been formulated for topological Kac-Moody groups defined over $\mathbb{C}$ since these groups are not locally compact, however for these groups Kitchloo has recently defined the equivariant K-homology in terms of the K-homology of parabolic subgroups ([K2]). A discussion of the Baum-Connes conjecture for such groups G would require a universal space $\underline{E}G$ for proper G-actions. Kitchloo has shown that the space $\underline{E}G$ is the topological Tits building of G, a smooth stack when G is a Kac-Moody group of affine or compact hyperbolic type ([K1]).

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2 Locally compact Kac-Moody groups and Tits buildings

2.1 Introduction to Kac-Moody algebras

The initial construction by Cartan and Killing of finite dimensional simple Lie algebras from the Cartan integers was type dependent. In 1966 Serre showed that relations of Chevalley and Harish-Chandra ([H-C]), with simplifications by Jacobson ([Ja]), give a defining presentation for the Lie algebra ([Se]). One could thus describe a simple Lie algebra in terms of generators and relations using data from the matrix of Cartan integers, which is naturally positive definite.

In his 1967 thesis, R. Moody considered Lie algebras whose Cartan matrix is no longer positive definite ([M1], [M2]). This still gave rise to a Lie algebra, but one which is now infinite dimensional. Simultaneously, $\mathbb{Z}$-graded Lie algebras were being studied in Moscow where I. L. Kantor introduced and studied a general class of Lie algebras including what eventually became known as Kac-Moody algebras ([Kan]). V. Kac was also studying simple or nearly simple

Lie algebras with polynomial growth. A rich mathematical theory of infinite dimensional Lie algebras evolved. An account of the subject, which also includes works of many others, is given in [K] (see also [Sel]).

The data for constructing a Kac-Moody algebra includes a *generalized Cartan matrix*. This is a square matrix $A = (a_{ij})$, $i, j \in \{1, 2, \dots, \ell\}$ whose entries are integers such that:

(1) $a_{ii} = 2$,

(2) $a_{ij} \leq 0$, $i \neq j$,

(3) $a_{ij} = 0$ implies $a_{ji} = 0$.

A generalized Cartan matrix A is called *indecomposable* if there is no rearrangement of the indices so that A can be written in block diagonal form. A generalized Cartan matrix A is called *symmetrizable* if there exist nonzero rational numbers $d_1, \dots, d_\ell$, such that the matrix DA is symmetric, where $D = diag(d_1, \dots, d_\ell)$. We call DA a *symmetrization* of A. Such a symmetrization always exists and is unique up to a scalar multiple. Symmetrizability is necessary for the existence of a well-defined symmetric invariant bilinear form $(\cdot \mid \cdot)$ on the Kac-Moody algebra which plays the role of 'squared length' of a root.

The generalized Cartan matrix A is *affine* if A is positive semi-definite but not positive definite. If A is neither positive definite nor positive semi-definite, but every proper indecomposable submatrix is either positive definite or positive semi-definite, we say that A has *hyperbolic type*. If every proper indecomposable submatrix of A is positive definite, we say that A has *compact hyperbolic type*. Thus if A has a proper indecomposable affine submatrix, we say that G has *noncompact hyperbolic type*.

Given:

∘ A generalized Cartan matrix $A = (a_{ij})_{i,j \in \ell}$, and

∘ A finite dimensional vector space $\mathfrak{h}$ (Cartan subalgebra) with $dim(\mathfrak{h}) = 2\ell - rank(A)$, and

∘ A choice of *simple roots* $\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subseteq \mathfrak{h}^*$ and *simple coroots* $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subseteq \mathfrak{h}$ such that Π and $\Pi^\vee$ are linearly independent and $\alpha_j(\alpha_i^\vee) = a_{ij}$, $i = 1, \dots \ell$,

we may associate a Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ over K, a field, generated by $\mathfrak{h}$ and elements $(e_i)_{i \in I}$, $(f_i)_{i \in I}$ subject to relations ([K], Theorem 9.11):

$$[\mathfrak{h}, \mathfrak{h}] = 0,$$
$$[h, e_i] = \langle \alpha_i, h \rangle e_i, \ h \in \mathfrak{h},$$
$$[h, f_i] = -\langle \alpha_i, h \rangle f_i, \ h \in \mathfrak{h},$$
$$[e_i, f_i] = \alpha_i^\vee,$$
$$[e_i, f_j] = 0, \ i \neq j,$$
$$(ad \ e_i)^{-a_{ij}+1}(e_j) = 0, \ i \neq j,$$
$$(ad \ f_i)^{-a_{ij}+1}(f_j) = 0, \ i \neq j,$$

where $(ad(x))(y) = [x, y]$ and $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \longrightarrow \mathfrak{h}$ denotes the natural nondegenerate bilinear pairing between $\mathfrak{h}$ and its dual such that $\langle \alpha_j, \alpha_i^\vee \rangle = \alpha_j(\alpha_i^\vee) = a_{ij}$.

2.2 Introduction to Kac-Moody groups and their properties

Subsequent to the discovery of infinite dimensional generalizations of Lie algebras by dropping the assumption that the matrix of Cartan integers is positive definite, the problem of associating groups to Kac-Moody algebras arose, the difficulty being that there is no obvious definition of a general 'Kac-Moody group'. Several appropriate definitions of a Kac-Moody group have been discovered, many of them using a variety of techniques as well as additional external data, such as a $\mathbb{Z}$-form for the universal enveloping algebra ([CG], [Ga], [GW], [KP], [M], [Ma], [MT], [Sl], [Ti2]). Many constructions use some version of the *Tits functor* ([Ti2]).

Though there is no obvious infinite dimensional generalization of finite dimensional Lie groups, Tits associated a group functor G_A on the category of commutative rings, such that for any symmetrizable generalized Cartan matrix A and any ring R there exists a group $G_A(R)$ ([Ti1], [Ti2]). Tits showed that if R is a field, then $G_A(R)$ is characterized uniquely up to isomorphism, apart from some degeneracy in the case of small fields. Tits defined not one group, but rather *minimal* and *maximal* groups. The value of the Tits functor G_A over a field k is called a *minimal Kac-Moody group.* The *maximal* or *complete Kac-Moody group* is defined relative to a completion of the Kac-Moody algebra and contains $G_A(k)$ as a dense subgroup.

Let A be an $\ell \times \ell$ symmetrizable generalized Cartan matrix. The existence of a completion of the Tits functor associated to A and the finite field $\mathbb{F}_q$ was noted by Tits ([Ti1]). Explicit completions have been constructed using distinct methods by Carbone and Garland ([CG]) and by Rémy and Ronan ([RR]). A complete Kac-Moody group G over a finite field is locally compact and totally disconnected.

Locally compact Kac-Moody groups and their lattices have appeared in the literature only recently ([CG], 2003 and [RR], 2006). We mention here some of the known results concerning Property (T) for locally compact Kac-Moody groups.

In [CG], the authors showed that for locally compact Kac-Moody groups G of rank greater than or equal to 3 over sufficiently large finite fields, if G contains a cocompact lattice, then G has Property (T).

This result was generalized considerably by Dymara and Januszkiewicz ([DJ]) who obtained vanishing theorems for various cohomologies on the Bruhat-Tits building X associated with a BN-pair of our Kac-Moody group G (see subsection 2.4), and on discrete subgroups $\Gamma \leq G$. They also gave criteria for lattices in G to have Property (T) and the Haagerup property. Much of our work in Section 7 involves showing that these (nontrivial) criteria of [DJ] are satisfied.

In [EJ-Z] the authors establish a new spectral criterion for Kazhdan's Property (T) which gives new examples of Property (T) groups. They also apply it to discrete subgroups of Kac-Moody groups over finite fields giving precise Kazhdan constants and generalizing the results of [DJ].

2.3 Tits' presentation of minimal Kac-Moody groups

In this subsection we define minimal Kac-Moody groups over arbitrary fields by generators and relations, following Tits ([Ti2]).

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra with Cartan subalgebra $\mathfrak{h}$. For each simple root α_i, $i \in I = \{1, \dots \ell\}$, we define the simple root reflection

$$w_i(\alpha_j) := \alpha_j - \alpha_j(\alpha_i^\vee)\alpha_i.$$

The w_i generate a subgroup $W = W(A) \subseteq Aut(\mathfrak{h}^*)$, called the *Weyl group* of A. We introduce an auxiliary group $W^* \subseteq Aut(\mathfrak{g})$, generated by elements $\{w_i^*\}_{i\in I}$, where

$$w_i^* \; = \; exp(ad(e_i))exp(-ad(f_i))exp(ad(e_i)) \; = \; exp(-ad(f_i))exp(ad(e_i))exp(-ad(f_i)).$$

There is a surjective homomorphism $\epsilon : W^* \to W$ which sends w_i^* to w_i for all i. We define certain elements of $\mathfrak{g}$, denoted $\{e_\alpha\}_{\alpha\in\Phi}$, where Φ denotes the set of real roots of $\mathfrak{g}$. Given $\alpha \in \Phi$, write α in the form $w\alpha_j$ for some $j \in I$ and $w \in W$, choose $w^* \in W^*$ which maps onto w, and set $e_\alpha = w^* e_{\alpha_j}$. It is clear from [Ti2, (3.3.2)] that e_α belongs to the root space g^α, e_α is uniquely determined up to sign, and for all $i \in I$, $w_i^* e_\alpha = \eta_{\alpha,i} e_{w_i\alpha}$ for some constants $\eta_{\alpha,i} \in \{\pm 1\}$.

Let A be a symmetrizable generalized Cartan matrix. Let k denote an arbitrary field. The group $G = G_A(k)$ defined below is called the *incomplete simply-connected Kac-Moody group* corresponding to A.

By definition, $G_A(k)$ is generated by the set of symbols $\{\chi_\alpha(u) \mid \alpha \in \Phi, u \in k\}$ satisfying relations (R1)-(R7) below. In all the relations i, j are elements of I, u, v are elements of k and α and β are real roots.

(R1) $\chi_\alpha(u+v) = \chi_\alpha(u)\chi_\alpha(v)$;

(R2) Let (α, β) be a *prenilpotent pair,* that is, there exist $w, \ w' \in W$ such that

$$w\alpha, \ w\beta \in \Phi^+ \text{ and } w'\alpha, \ w'\beta \in \Phi^-.$$

Then

$$[\chi_\alpha(u), \chi_\beta(v)] = \prod_{m,n\geq 1} \chi_{m\alpha+n\beta}(C_{mn\alpha\beta}u^m v^n)$$

where the product on the right-hand side is taken over all real roots of the form $m\alpha + n\beta$, $m, n \geq 1$, in some fixed order, and $C_{mn\alpha\beta}$ are integers independent of k (but depending on the order). This product appearing on the right-hand side is finite.

For each $i \in I$ set

$\chi_{\pm i}(u) = \chi_{\pm\alpha_i}(u)$, $u \in k$

$\widetilde{w}_i(u) = \chi_i(u)\chi_{-i}(-u^{-1})\chi_i(u)$, $u \in k$

$\widetilde{w}_i = \widetilde{w}_i(1)$ and $h_i(u) = \widetilde{w}_i(u)\widetilde{w}_i^{-1}$, $u \in k^*$.

The remaining relations are

(R3) $\widetilde{w}_i \chi_\alpha(u) \widetilde{w}_i^{-1} = \chi_{w_i\alpha}(\eta_{\alpha,i} u)$,

(R4) $h_i(u)\chi_\alpha(v)h_i(u)^{-1} = \chi_\alpha(vu^{\langle\alpha,\alpha_i^\vee\rangle})$ for $u \in k^*$,

(R5) $\widetilde{w}_i h_j(u) \widetilde{w}_i^{-1} = h_j(u)h_i(u^{-a_{ji}})$,

(R6) $h_i(uv) = h_i(u)h_i(v)$ for $u, v \in k^*$, and

(R7) $[h_i(u), h_j(v)] = 1$ for $u, v \in k^*$.

An immediate consequence of relations (R3) is that $G_A(k)$ is generated by $\{\chi_{\pm i}(u)\}$. The elements $\widetilde{w}_i$ generate a group $\widetilde{W}$ which is isomorphic to the group W^* above.

2.4 The (B, N)-pair and Tits building of a complete Kac-Moody group

A Kac-Moody group G may be also described by certain group theoretic data, called a *Tits system* or (B, N)*-pair*. This data carries a great deal of information about the group and its subgroups, and in particular determines a simplicial complex, a *Tits building* X on which the group acts.

Let A be an $\ell \times \ell$ symmetrizable generalized Cartan matrix. From now on, we shall only consider complete (maximal) Kac-Moody groups. By a slight abuse of notation, we will also denote these by $G_A(k)$ where k is a field. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field $\mathbb{F}_q$ ([CG] and [RR]). The Tits building X of a complete Kac-Moody group G over a finite field is locally finite. In this section we give a brief description of the Tits system for G and its corresponding Tits building.

A completion G of Tits' functor over the finite field $\mathbb{F}_q$ has subgroups $B^\pm \subseteq G$, $N \subseteq G$, and Weyl group $W = N/H$, where $H = N \cap B^\pm$ is a normal subgroup of N. We have $B^\pm = HU^\pm$ where $U^+ = \langle \chi_i(u) \mid i \in I,\ u \in k\rangle$, $U^- = \langle \chi_{-i}(u) \mid i \in I,\ u \in k\rangle$, B^+ is compact, in fact a profinite neighborhood of the identity in G, and B^- is discrete. The group W coincides with the Weyl group of the previous section, and $H = \langle h_i(u) \mid i \in I,\ u \in k^*\rangle$.

Tits ([Ti1]) and Carbone and Garland ([CG]) showed that (G, B^+, N) and (G, B^-, N) are BN-pairs, and

$$G \quad = \quad B^+NB^- \quad = \quad B^-NB^+.$$

It follows that G has Bruhat decomposition

$$G \quad = \quad \sqcup_{w\in W}\ B^\pm w B^\pm.$$

Let S be the standard generating set for the Weyl group W consisting of simple root reflections. Let $U \subsetneq S$. The *standard parabolic subgroups* are

$$P_U \quad = \quad \sqcup_{w\in\langle U\rangle}\ B^\pm w B^\pm.$$

A *parabolic* subgroup is any subgroup containing a conjugate of $B^\pm$. The Tits building of G is a simplicial complex X of dimension $dim(X) = |S| - 1$. In fact we associate a building $X^\pm$ to each BN-pair (G, B^+, N) and (G, B^-, N). The buildings X^+ and X^- are isomorphic as chamber complexes and have constant thickness $q + 1$ (see [DJ, Appendix KMT]).

The vertices of X are given by the cosets of G by the maximal parabolic subgroups of G. The incidence relation is described as follows. The $r + 1$ vertices $Q_1, \ldots, Q_{r+1}$ span an r-simplex if and only if the intersection $Q_1 \cap \cdots \cap Q_{r+1}$ is parabolic, that is, contains a conjugate of $B\pm$. In our case, the Weyl group W is infinite, so by the Solomon-Tits theorem, X is contractible. The group G acts by left multiplication on cosets.

2.5 Lattices in Kac-Moody groups

Let G be a locally compact group and let μ be a (left) Haar measure on G. Let $\Gamma \leq G$ be a discrete subgroup with quotient $p : G \longrightarrow \Gamma\backslash G$. We call Γ a *lattice* in G if $\mu(\Gamma\backslash G) < \infty$, and a *cocompact* lattice if $\Gamma\backslash G$ is compact.

Symmetrizable locally compact Kac-Moody groups over finite fields $\mathbb{F}_q$ are known to contain lattice subgroups. If q is sufficiently large then G contains nonuniform lattice subgroups ([CG] and [Re1]) for $\ell \geq 3$, and both cocompact and nonuniform lattice subgroups if $\ell = 2$, with no restriction on q ([CG], [CC] and [RR]).

The discreteness of a lattice subgroup Γ in a locally compact Kac-Moody group G is equivalent to the property that Γ acts on the Tits building X with finite vertex stabilizers ([BL]). Thus locally compact Kac-Moody groups G come equipped with lattices that act properly and isometrically on the locally finite Tits building X.

3 The Baum-Connes conjecture

Let G be a second countable, locally compact and Hausdorff topological group. The reduced C^*-algebra $C^*_{red}(G)$ of G is the completion in the operator norm of the convolution algebra $L^1(G)$ viewed as an algebra of operators on $L^2(G)$.

If G is a discrete and torsion free group, it is relatively straightforward to formulate the Baum-Connes conjecture for G. For such G, there is a natural homomorphism, or 'assembly map' on K-homology

$$\mu : K_*(BG) \longrightarrow K_*(C^*_{red}(G)),$$

where BG is the classifying space for G, and $C^*_{red}(G)$ is the reduced C^*-algebra of G. The Baum-Connes conjecture states that μ is an isomorphism of abelian groups ([BCH]).

Now suppose that G is non-discrete or discrete with torsion. There exists a universal space $\underline{E}G$ for proper G-actions, unique up to G-equivariant homotopy. Using the KK-theory of Kasparov ([Ka]), we may form the equivariant K-homology $K^G_*(\underline{E}G)$. There is then an assembly map from K-homology to K-theory

$$\mu : K^G_*(\underline{E}G) \longrightarrow K_*(C^*_{red}(G)),$$

which is conjectured by Baum and Connes to be an isomorphism.

3.1 The Baum-Connes conjecture with coefficients

A more general version of the Baum-Connes conjecture can be formulated with coefficients. Let A be a C^*-algebra on which G acts as C^*-algebra automorphisms. Let $C^*_{red}(G, A)$ denote the reduced crossed-product C^*-algebra. By [BCH], there is a homomorphism of abelian groups

$$\mu : K_*^G(\underline{E}G, A) \longrightarrow K_*(C^*_{red}(G, A)),$$

where $K_*(C^*_{red}(G, A))$ denotes the K-theory of $C^*_{red}(G, A)$ and $K_*(\underline{E}G, A)$ denotes the G-equivariant K-homology of $\underline{E}G$. As above the Baum-Connes conjecture states that μ is an isomorphism of abelian groups ([BCH]).

4 Metrically proper, isometric, affine actions of discrete groups

Let G be a locally compact group. We say that there is a continuous, isometric action of G on some affine Hilbert space H if there is a a continuous map $G \longrightarrow Isom(H)$. We say that the action of G on H is metrically proper if for any bounded subset B in H the set

$$K(G, B) := \{g \in G \ s.t. \ gB \cap B \neq \emptyset\}$$

has compact closure in G. The locally compact group G satisfies the *Haagerup property*, (or is *a-T-menable*) if it admits a continuous, isometric, proper action on an affine Hilbert space.

Higson and Kasparov have shown that if a locally compact group G has the Haagerup property, then G satisfies the Baum-Connes conjecture with coefficients in any G-C^*-algebra ([HK1], [HK2]).

5 Property (T), the Haagerup property and rapid decay

A locally compact group G has Property (T) if and only if every continuous action of G by isometries on a Hilbert space has a fixed point. Other equivalent definitions in representation theory and ergodic theory all indicate that the Haagerup property is a strong negation of Kazhdan's Property (T). In addition, if Γ is a discrete group that has both Kazhdan's Property (T) and the Haagerup property, then Γ is finite. Furthermore, a theorem of Wang ([W]) shows that Property (T) for a lattice subgroup Γ of a locally compact group G is equivalent to Property (T) for G itself.

It is known that if a lattice subgroup Γ of a locally compact group G has the Haagerup property, then G has the Haagerup property ([CCJJV]). It is also easy to verify that the Haagerup property for a locally compact group G implies the Haagerup property for a lattice subgroup $\Gamma \leq G$.

Lafforgue has shown that there exist groups satisfying both Property (T) and the Baum-Connes conjecture. In [L1]-[L3] he showed that if a discrete group Γ acts properly, isometrically and with the analytical property of 'rapid decay' on a space with non-positive curvature then Γ satisfies the Baum-Connes conjecture without coefficients (see also Section 8). Thus Lafforgue's work reveals groups satisfying both Property (T) and the Baum-Connes conjecture, such as cocompact lattices in $SL_3(\mathbb{R})$ (see [L1] for a proof of rapid decay for lattices in $SL_3(\mathbb{R})$, adapted from earlier work of Ramagge, Robertson and Steger on rapid decay for lattices in $SL_3(\mathbb{Q}_p)$ ([RRS]).)

In [CR], the authors give new examples of groups with the rapid decay property, including finitely generated nonuniform lattices in rank one Lie groups over $\mathbb{R}$ or $\mathbb{C}$. Rapid decay is known to fail for all nonuniform lattices in simple Lie groups of rank > 1, and to hold for only certain cocompact lattices in rank > 1.

If G is a locally compact rank 2 Kac-Moody group, then G contains cocompact lattices ([CG], [CC] and [RR]). These lattices are Gromov hyperbolic groups and hence have rapid decay. It is not currently known if lattices in Kac-Moody groups of rank $r > 2$ have the rapid decay property.

We ask the following:

Question Let G be a locally compact hyperbolic Kac-Moody group of rank ≥ 3. Let Γ be a lattice subgroup of G. Does Γ have the rapid decay property? Assume now that G has rank 2 and Γ is nonuniform. Does Γ have rapid decay?

6 Classification of hyperbolic Dynkin diagrams

It is known that the maximal rank of a hyperbolic Kac-Moody algebra is 10. This is determined by the following restrictive conditions:

(1) The fundamental chamber $\mathcal{C}$ of the Weyl group, viewed as a hyperbolic reflection group, must be a Coxeter polyhedron. The dihedral angles between adjacent walls must be of the form π/k, where $k \geq 2$.

(2) The fundamental chamber $\mathcal{C}$ of the Weyl group must be a simplex, which gives a bound on the number of faces.

Such a 'Coxeter simplex' $\mathcal{C}$ exists in hyperbolic n-space $\mathbb{H}^n$ for $n \leq 9$. The bound on the rank of a hyperbolic Dynkin diagram can also be deduced by purely combinatorial means ([K], [Li], [Sa]).

There are 18 hyperbolic algebras of rank 7-10, classified by Kac ([K]). There are 142 possible symmetric or symmetrizable hyperbolic Dynkin diagrams between the ranks 3 and 10 (136 of these were classified and exhibited by Saçlioğlu, but he omitted 6, as was observed by [dBS], p 4491). Li independently classified the hyperbolic Dynkin diagrams without the assumption of symmetrizability ([Li]). A complete account of the classification of hyperbolic Dynkin diagrams is given in [CCCMNNP].

The rank 2 hyperbolic generalized Cartan matrices are:

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}_{ab>4}$$

The only 2×2 affine generalized Cartan matrices are:

$$A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \; A_2^{(2)} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Proposition 6.1. *A symmetrizable hyperbolic generalized Cartan matrix contains an $A_1^{(1)}$ or $A_2^{(2)}$ proper indecomposable submatrix if and only if rank $A = 3$ and A has noncompact type.*

Proof: A symmetrizable hyperbolic generalized Cartan matrix cannot contain an $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable proper submatrix if rank $A = \ell > 3$, since the Dynkin diagram has ℓ vertices, and the 3 vertex connected subdiagram consisting of $A_1^{(1)}$ or $A_2^{(2)}$ plus an additional vertex would then be neither affine nor finite. Thus if A is a symmetrizable hyperbolic generalized Cartan matrix with an $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrix, then the rank of A is 3, and A has noncompact type.

Conversely, every hyperbolic diagram of rank 3 of noncompact type must contain contain an $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable subdiagram, since it must contain a subdiagram of affine type with 2 vertices. □

The following corollary summarizes the properties of rank 3 hyperbolic generalized Cartan matrix of noncompact type.

Corollary 6.2. *Let A be a rank 3 symmetrizable hyperbolic generalized Cartan matrix of noncompact type. Let $W = W(A)$ be the corresponding Weyl group, let $G = G_A(\mathbb{F}_q)$ be a complete Kac-Moody group corresponding to A over the finite field $\mathbb{F}_q$, and let X denote the Tits building of G. Then we have the following equivalent conditions.*

(a) The Dynkin diagram for A has an $A_1^{(1)}$ or $A_2^{(2)}$ proper connected subdiagram.

(b) A has a proper indecomposable affine submatrix $B = (b_{ij})$ such that for some i and j, $i, j \in \{1, 2, 3\}$, $i \neq j$, $b_{ij} b_{ji} = 4$.

(c) Every infinite parabolic subgroup of W is dihedral.

(d) X has a noncompact link which contains a linear subtree $\cong \mathbb{Z}$.

7 The Haagerup property and Property (T) for locally compact Kac-Moody groups

Let A be an $\ell \times \ell$ symmetrizable generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field $\mathbb{F}_q$ ([CG], [Ti1], [Ti2]). Let X be the Tits building of G. If q is sufficiently large we recall that G contains nonuniform lattice subgroups ([CG] and [Re1]) for $\ell \geq 3$, and both cocompact and nonuniform lattice subgroups if $\ell = 2$, with no restriction on q ([CG] and [CC]).

We recall the following conditions from [DJ], in order to define the class of group actions on buildings denoted $\mathcal{B}+$ of [DJ].

$\mathcal{B}1$. 0-dimensional links in X (i.e., links of faces of codimension 1) are finite.

$\mathcal{B}2$. Links of simplices in X are connected in the following sense: for any two chambers in a link, there exists a path of chambers connecting them, such that each pair of consecutive elements of the path share a face of codimension 1.

$\mathcal{B}3$. The link of every simplex in X is either compact or contractible; in particular, this holds for $X = Lk(\varnothing)$.

$\mathcal{B}4$. The group G acts transitively on chambers in X and the quotient map $X \longrightarrow X/G$ restricts to an isomorphism on each chamber.

$\mathcal{B}\delta$. 1-dimensional links are compact and the nonzero eigenvalues of the Laplacian on 1dimensional links are $\geq 1 - \delta$,

The class $\mathcal{B}+$ of [DJ] is then defined to be all pairs (X, G) for groups G acting on buildings X satisfying $\mathcal{B}1 - \mathcal{B}4$ and $\mathcal{B}\delta$ for $\delta = \frac{13}{28^n}$, where n is the dimension of the Tits building X.

Lemma 7.1. *Let A be a symmetrizable affine or hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field $\mathbb{F}_q$. Let X be the Tits building of G. If A has no $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrices, and q is sufficiently large, then (X, G) is in the class $\mathcal{B}+$ of [DJ].*

Proof: The hypothesis that A has no $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrices ensures that all entries of the Coxeter matrix associated to A are finite. We then apply Proposition A of [DJ] to conclude that (X, G) is in the class $\mathcal{B}+$. $\square$

Proposition 7.2. *Let A be a symmetrizable affine or hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field $\mathbb{F}_q$. Let $\Gamma \leq G$ be a lattice subgroup of G.*

(1) If A has an $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrix (not necessarily proper), then Γ does not have Property (T).

(2) If A has no $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrices, and q is sufficiently large, then Γ has Property (T).

Proof: For (1), the hypothesis that A has an $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrix (not necessarily proper) implies that X has a noncompact 1-dimensional link by Corollary 6.2. We then apply part (1) of Corollary G of [DJ] to conclude that Γ does not have Property (T). Part (2) follows from Lemma 7.1 and (2) of Corollary G of [DJ]. $\square$

One of our main tools will be the following theorem of [DJ].

Theorem 7.3. *([DJ], Thm 10.1) Let X be the building of a BN-pair (G, B, N) such that X has sufficiently large finite thickness. The following conditions are equivalent.*

(a) G acts with a proper orbit on a product of trees.

(b) G has the Haagerup property.

(c) Every closed Property (T) subgroup of G is compact.

(d) For every simplex σ of X with noncompact link, there is a codimension 2 simplex containing σ with noncompact link.

(e) Every infinite parabolic subgroup of the Weyl group of G contains an infinite parabolic dihedral subgroup.

We now deduce our main theorems :

Theorem 7.4. *Let A be an $\ell \times \ell$ symmetrizable hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field $\mathbb{F}_q$, with q sufficiently large. Let Γ be a lattice subgroup of G. If the rank of A is 2, or if the rank of A is 3 and A has noncompact hyperbolic type, then Γ has the Haagerup property. Hence G satisfies the Baum-Connes conjecture with coefficients in any $G - C^*$-algebra.*

Proof: If the rank of A is 2, then (e) of [DJ] Theorem 10.1 holds, so Γ has the Haagerup property. If the rank of A is 3 and A is of noncompact hyperbolic type, by Proposition 6.1 and Corollary 6.2, if A is a $3{\times}3$ generalized Cartan matrix of noncompact hyperbolic type then every infinite parabolic subgroup of the Weyl group of G contains an infinite parabolic dihedral subgroup. By [DJ], Theorem 10.1, Γ has the Haagerup property. □

Theorem 7.5. *Let A be an $\ell \times \ell$ symmetrizable affine or hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field $\mathbb{F}_q$, with q sufficiently large. Let Γ be a lattice subgroup of G. If $r = rank(G) = 3$ and G has compact hyperbolic type, or if $rank(G) \geq 3$ and G has affine type, or if $4 \leq r \leq 10$ and G has hyperbolic type then*

(i) Γ has Property (T).

(ii) All entries in the Coxeter matrix for G are finite.

Proof: For (i), if the rank of A is 3 and A is of compact hyperbolic type, then A has no $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrices. By (2) of Proposition 7.2, Γ has Property (T). If the rank of A is ≥ 4, we use Proposition 6.1 to deduce that A has no $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrices. By (2) of Proposition 7.2, Γ has Property (T).

For (ii), since G is affine or hyperbolic, the Coxeter matrix for G contains an ∞ if and only if the generalized Cartan matrix A contains an $A_1^{(1)}$ or $A_2^{(2)}$ proper indecomposable submatrix. By Proposition 6.1 this occurs if and only if rank $A = 3$ and A has noncompact type, which is impossible under the assumptions here.□

Since G has Property (T), if G acts on a tree then G must fix a vertex ([Wa]).

Corollary 7.6. *Property (T) and the Haagerup property for symmetrizable locally compact affine or hyperbolic Kac-Moody groups over sufficiently large finite fields can be determined from the Dynkin diagram, or equivalently from the generalized Cartan matrix.*

8 The Baum-Connes conjecture for higher rank Kac-Moody groups

Let G be a symmetrizable locally compact Kac-Moody group of rank 3 compact hyperbolic type or of affine or hyperbolic type and rank ≥ 4. We deduced from [DJ] that G has Property (T) (Section 7). Let X be the Tits building of G. Then X is a $CAT(0)$ space. Thus X is a 'bolic space' in the sense of Kasparov and Skandalis ([KS]). By [KS], the Baum-Connes assembly map is injective.

Now let $\Gamma \leq G$ be a lattice subgroup. Since G has Property (T), Γ is finitely generated. If further G is of hyperbolic type and Γ is a cocompact lattice that acts cocompactly on the Tits building X, a hyperbolic space, it follows that Γ is a Gromov-hyperbolic group. Since Gromov hyperbolic groups have rapid decay ([delaH]), we conclude that cocompact lattices $\Gamma \leq G$ acting cocompactly on X have rapid decay.

In [L1]-[L3], Lafforgue used 'Banach KK-theory' to define an analog of the Baum-Connes assembly map. However, this does not involve the reduced C^*-algebra, but an unconditional completion of $C_c(G)$, the convolution algebra of compactly supported functions. For groups with rapid decay Lafforgue showed that there is an induced isomorphism of the Baum-Connes assembly map ([L1]-[L3]). Thus we have the following.

Theorem 8.1. *Let G be a symmetrizable locally compact Kac-Moody group of affine or hyperbolic type. Then the Baum-Connes assembly map is injective. If G is of hyperbolic type, of rank $4 \leq r \leq 10$ and G contains a cocompact lattice that acts cocompactly on its Kac-Moody Tits building, then the Baum-Connes assembly map is also surjective. Thus G satisfies the Baum-Connes conjecture without coefficients.*

Unfortunately we do not know if G as in Theorem 8.1 contains cocompact lattices that act cocompactly on the Kac-Moody Tits building. Our work has thus left open the possibility that higher rank hyperbolic Kac-Moody groups with Property (T) also satisfy the Baum-Connes conjecture. Lafforgue's work provides an analogue of the Baum-Connes conjecture for groups acting properly isometrically on 'strongly bolic spaces' ([L3]), a strengthening of the Kasparov-Skandalis notion of bolicity. To get the full Baum-Connes conjecture for our Kac-Moody groups with Property (T) using Lafforgue's Banach KK-theory, we require an unconditional completion such that the inclusion into $C^*_{red}(G)$ induces isomorphisms in K-theory. If we succeed to use Lafforgue's work to give the full Baum-Connes conjecture for Property (T) Kac-Moody groups, this will give the weaker form of Baum-Connes, namely the conjecture without coefficients. We may also try to strengthen this result to give the Baum-Connes conjecture with coefficients. We hope to take this up elsewhere.

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