KAC–MOODY GROUPS AND AUTOMORPHIC FORMS IN LOW DIMENSIONAL SUPERGRAVITY THEORIES

LING BAO AND LISA CARBONE

ABSTRACT. Kac–Moody groups G over $\mathbb{R}$ have been conjectured to occur as symmetry groups of supergravity theories dimensionally reduced to dimensions less than 3, and their integral forms $G(\mathbb{Z})$ conjecturally encode quantized symmetries. In this review paper, we briefly introduce the conjectural symmetries of Kac–Moody groups in supergravity as well as the known evidence for these conjectures. We describe constructions of Kac–Moody groups over $\mathbb{R}$ and $\mathbb{Z}$ using certain choices of fundamental modules that are considered to have physical relevance. Eisenstein series on certain finite dimensional algebraic groups are known to encode quantum corrections in the low energy limit of superstring theories. We describe briefly how the construction of Eisenstein series extends to Kac–Moody groups. The constant terms of Eisenstein series on E_9, E_{10} and E_{11} are predicted to encode perturbative string theory corrections.

1. INTRODUCTION

Kac–Moody groups and algebras are the most natural generalizations to infinite dimensions of finite dimensional simple Lie groups and Lie algebras. Affine Kac–Moody algebras and their generalizations by Borcherds have concrete physical realizations and have wide applications in physical theories. Suitable extensions of the Dynkin diagrams of affine Kac–Moody algebras give rise to hyperbolic and Lorentzian Kac–Moody algebras, such as E_{10} and E_{11}.

As discussed in some of the talks, hyperbolic and Lorentzian Kac–Moody groups and algebras have recently been discovered as symmetries in supergravity theories. To understand the group theoretic questions that arise, it's instructive to first briefly view the symmetries in a physical context. An excellent and detailed survey of the physics we will discuss is given in [FKGP] (see also [D1], [DN], [HPS], [FKP], [KN], [Ni2] and [Ni3]). An expanded version of this article is available at [BC].

We describe a construction of a representation theoretic Kac–Moody group G_V, over $\mathbb{R}$ and $\mathbb{Z}$ using integrable highest weight modules V and a $\mathbb{Z}$–form of the universal enveloping algebra. We describe a choice of fundamental modules V for the Kac–Moody groups E_9, E_{10} and E_{11} that are conjectured to play a role in certain supergravity theories.

Let G be (finite dimensional) semisimple algebraic group and let K be a maximal compact subgroup of G. Eisenstein series on $K \backslash G(\mathbb{R})/G(\mathbb{Z})$ appear naturally in the context of superstring theories, where they encode quantum corrections in the low energy regime described by supergravity theories ([GMRV], [LW]).

We describe briefly how the construction of Eisenstein series extends to Kac–Moody groups. It is also thought that the Eisenstein series on the Kac–Moody groups E_9, E_{10} and E_{11} should encode the quantum correction terms in string theory ([FK]). We outline the construction of Eisenstein series on these groups and we review current developments and open questions in the study of automorphic forms on non–affine Kac–Moody groups.

2. Supergravity theories

The leading candidate in theoretical physics for a single consistent theory of all the fundamental forces is *superstring theory*. However, there is no single theory of superstrings, but rather five theories that describe various aspects of the behaviour of strings. In the 1990's, Witten, building on previous insights by Duff, Townsend, and others, proposed an overarching theory, known as *M–theory*, whose various special limits are the superstring theories ([Wi1], [Wi2]). M–theory has a rich structure, both mathematically and physically, but has not yet been fully formulated and will likely require new mathematics for its full development.

Essential for superstring theory is *supersymmetry*. This symmetry, discovered in 1972, transforms fermions into bosons and vice versa. *Supergravity* is a theory that incorporates both quantum field theory and general relativity using supersymmetry. It was discovered in 1978 as a 4 dimensional theory, but it can occur in dimensions up to 11, where there are 7 additional spacial dimensions.

The possible supergravity theories were classified in the 1970s. Given a physical theory (Lagrangian) in D dimensions, *dimensional reduction* is a procedure that gives rise to a Lagrangian in $d < D$ dimensions, by taking all fields to be independent of location in the extra $D - d$ dimensions. The extra dimensions are contracted to an 'invisible scale' along a choice of vector field, involving the full geometry of spacetime.

The type IIA and type IIB supergravity theories in 10–dimensions are the low energy limit of the type II string theories while the low energy limit of M–theory is known to coincide with 11–dimensional supergravity. These supergravity theories are important, as they encode both perturbative and non–pertubative effects, many of which are not easily calculated from first principles in string theory or M–theory (see for example [GLW]).

The *maximal supergravities* are those with the maximal number of *supercharges*, or fermionic generators of the super–Poincaré algebra. The maximal supergravities in dimensions less than 11 are obtained by dimensional reduction of the classical action of 11–dimensional supergravity on an n–torus. Dimensional reduction may be carried out on other compact manifolds, but on an n–torus all the supercharges are preserved. The resulting $(11 - n)$–dimensional supergravity action in the Einstein frame has the property that the scalar fields of the maximal supergravity theory in $(11-n)$ dimensions take values in the coset $K(E_n(\mathbb{R}))\backslash E_n(\mathbb{R})$, where E_n is a split real form of the simple exceptional Lie group of type E_n and $K(E_n)$ is a subgroup invariant under the Cartan involution.

For example, on dimensional reduction to $D = 4$ spacetime dimensions, 11–dimensional supergravity has the maximal number $N = 8$ of supercharges and the scalar fields take values in the coset $K(E_7(\mathbb{R}))\backslash E_7(\mathbb{R})$. We discuss this example in more detail in the next subsection.

Dimension	$K(G)$	$G(\mathbb{R})$	$G(\mathbb{Z})$
10, IIA	$\mathbf{1}$	$\mathbb{R}^+$	$\mathbf{1}$
10, IIB	$SO(2)$	$SL_2(\mathbb{R})$	$SL_2(\mathbb{Z})$
9	$SO(2)$	$SL_2(\mathbb{R}) \times \mathbb{R}^+$	$SL_2(\mathbb{Z})$
8	$SO(3) \times SO(2)$	$SL_3(\mathbb{R}) \times SL_2(\mathbb{R})$	$SL_3(\mathbb{Z}) \times SL_2(\mathbb{Z})$
7	$SO(5)$	$SL_5(\mathbb{R})$	$SL_5(\mathbb{Z})$
6	$(Spin(5) \times Spin(5))/\mathbb{Z}_2$	$Spin(5,5;\mathbb{R})$	$Spin(5,5;\mathbb{Z})$
5	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{R})$	$E_6(\mathbb{Z})$
4	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{R})$	$E_7(\mathbb{Z})$
3	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{R})$	$E_8(\mathbb{Z})$
2	$K(E_9)$	$E_9(\mathbb{R})$	$E_9(\mathbb{Z})$
1	$K(E_{10})$	$E_{10}(\mathbb{R})$	$E_{10}(\mathbb{Z})$
0	$K(E_{11})$	$E_{11}(\mathbb{R})$	$E_{11}(\mathbb{Z})$

TABLE 1. Coset symmetries in maximal supergravities.

Symmetries of the scalar cosets were first established for $1 \leq n \leq 8$ in [CJ]. For $n = 9$, it was discovered that the scalar fields take values in the coset $K(E_9)\backslash E_9(\mathbb{R})$ arising from the affine Kac–Moody group E_9 ([Ju1] and [N]). Moreover, the supergravity equations of motion are invariant under the Virasoro algebra.

A correspondence between the scalar fields of 11–dimensional supergravity theory dimensionally reduced to 1 dimension and the hyperbolic E_{10} coset $K(E_{10})\backslash E_{10}(\mathbb{R})$ was established in [DHN1] after certain truncations were made on both sides of the correspondence, in particular, only at 'low levels' of the roots of E_{10}. The subgroup $E_{10}(\mathbb{Z})$ of $E_{10}(\mathbb{R})$ has been conjectured to be a discrete symmetry group of Type II superstring theory ([Ju1], [HT]).

West ([W1]) showed that truncated versions of the bosonic sectors of 11–dimensional supergravity and type IIA supergravity can be derived in terms of a certain truncation of the Lorentzian E_{11} coset $K(E_{11})\backslash E_{11}$.

The full list of symmetry groups in maximal supergravity theories is summarized in Table **??**.

2.1. **Example: maximal supergravity in $D = 4$ spacetime dimensions.** It is known that the 11–dimensional maximal supergravity, dimensionally reduced to $D = 4$ spacetime dimensions, has global $E_7(\mathbb{R})$ symmetry ([CJ]). That is, the equations of motion of the supergravity theory, dimensionally reduced to $D = 4$ are invariant under $E_7(\mathbb{R})$. However, the Lagrangian is not invariant under $E_7(\mathbb{R})$, but under a smaller group which acts on the vector fields. The 70 scalar fields of this theory take values in the coset $[SU(8), \mathbb{R})/\{\pm Id\}]\backslash E_7(\mathbb{R})$ ([CJ]).

In $D = 4$ spacetime dimensions, the electric and magnetic charges of the theory lie in a 56–dimensional vector space V, with 28 electric and 28 magnetic fundamental charges. These charges are subject to the 'Dirac–Schwinger–Zwanziger quantization condition', which constrains the charges to lie on a lattice $V_{\mathbb{Z}}$ in V called the *charge lattice* ([HT]).

The space V is the 56–dimensional fundamental representation of $\mathfrak{e}_7$. This representation gives rise to a faithful representation of $E_7(\mathbb{R})$ in $Sp_{56}(\mathbb{R})$. The charge lattice admits an action of the discrete group $E_7(\mathbb{Z})$ preserving the set of electric and magnetic charges ([HT]).

Although the symmetry groups of the classical maximal supergravities involve Lie groups constructed over the real numbers $\mathbb{R}$, adding quantum corrections is conjectured to break these continuous groups to discrete subgroups defined over $\mathbb{Z}$ ([HT]). For 11–dimensional maximal supergravity, dimensionally reduced to $D = 4$, the discrete group is $E_7(\mathbb{Z}) = E_7(\mathbb{R}) \cap Sp_{56}(\mathbb{Z})$. Soulé gave a rigorous mathematical proof that the $E_7(\mathbb{Z})$ of [HT] coincides with the Chevalley $\mathbb{Z}$–form $G(\mathbb{Z})$ of $G = E_7$ ([S]). In the appendix of [MS], the authors gave a physics inspired construction of the group $E_8(\mathbb{Z})$.

3. Construction of Kac–Moody groups over $\mathbb{R}$ and $\mathbb{Z}$

Now let $\mathfrak{g}$ be a Kac–Moody algebra over $\mathbb{C}$ with simple roots α_i, $i \in I$. Here we summarize the construction of a Kac–Moody group G associated to $\mathfrak{g}$ in analogy with Chevalley's construction of finite dimensional semisimple algebraic groups. The generalization of this method to Kac–Moody groups over fields is provided by [CG] and over $\mathbb{Z}$ by [Ca]. Some external data is needed: namely an integrable highest weight module V for $\mathfrak{g}$ and a $\mathbb{Z}$-form $V_{\mathbb{Z}}$ constructed from a $\mathbb{Z}$-form of the universal enveloping algebra.

Let $V = V^\lambda$ be the unique irreducible highest weight module for $\mathfrak{g}$ corresponding to dominant integral weight λ. Let $\Lambda \subseteq \mathfrak{h}^*$ be the linear span of the simple roots α_i, for $i \in I$, and let $\Lambda^\vee \subseteq \mathfrak{h}$ be the linear span of the simple coroots $\alpha_i^\vee$, for $i \in I$.

Let e_i and f_i be the Chevalley generators of $\mathfrak{g}$. Let $\mathcal{U}_{\mathbb{C}}$ be the universal enveloping algebra of $\mathfrak{g}$. Let $\mathcal{U}_{\mathbb{Z}} \subseteq \mathcal{U}_{\mathbb{C}}$ be the $\mathbb{Z}$–subalgebra generated by

$$\frac{e_i^m}{m!}, \ \frac{f_i^m}{m!}, \ \binom{h}{m}$$

for $i \in I$, $h \in \Lambda^\vee$ and $m \geq 0$.

Let $v_\lambda \in V$ be a highest weight vector. We set

$$V_{\mathbb{Z}}^\lambda \ = \ \mathcal{U}_{\mathbb{Z}} \cdot v_\lambda.$$

Then $V_{\mathbb{Z}}^\lambda$ is a lattice in $V_R^\lambda = R \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^\lambda$ and a $\mathcal{U}_{\mathbb{Z}}$–module.

For $s, t \in R$ and $i \in I$, set

$$\chi_{\alpha_i}(t) \ = \ exp(\rho(se_i)),$$
$$\chi_{-\alpha_i}(t) \ = \ exp(\rho(tf_i)),$$

where ρ is the defining representation for V. Then these are elements of $Aut(V_R^\lambda)$, thanks to the local nilpotence of e_i, f_i.

We let $G^\lambda(R) \leq Aut(V_R^\lambda)$ be the group:

$$G^\lambda(R) = \langle \chi_{\alpha_i}(s), \ \chi_{-\alpha_i}(t) \mid i \in I, \ s, t \in R \rangle.$$

We summarize the construction as follows.

Theorem 3.1. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra over a commutative ring R with 1. Let α_i, $i \in I$, be the simple roots and e_i, f_i the Chevalley generators of $\mathfrak{g}$. Let V_R^λ be an R–form of an integrable highest weight module V^λ for $\mathfrak{g}$, corresponding to dominant integral weight λ and defining representation $\rho : \mathfrak{g} \longrightarrow End(V_R^\lambda)$. Then*

$$G^\lambda(R) = \langle \chi_{\alpha_i}(s) = exp(\rho(se_i)),\ \chi_{-\alpha_i}(t) = exp(\rho(tf_i)) \mid s, t \in R \rangle \leq Aut(V_R^\lambda)$$

is a representation–theoretic Kac–Moody group associated to $\mathfrak{g}$.

Now we define Kac–Moody groups over $\mathbb{R}$ and $\mathbb{Z}$ as follows:

$$G^\lambda(\mathbb{R}) = \langle \chi_{\alpha_i}(s), \chi_{-\alpha_i}(t) \mid s, t \in \mathbb{R},\ i \in I \rangle.$$
$$G^\lambda(\mathbb{Z}) = \langle \chi_{\alpha_i}(s), \chi_{-\alpha_i}(t) \mid s, t \in \mathbb{Z},\ i \in I \rangle.$$

The following was proven in [Ca].

Theorem 3.2. *([Ca]) The group $G^\lambda(\mathbb{Z})$ has the following generating sets:*

(1) $\chi_{\alpha_i}(1)$ and $\chi_{-\alpha_i}(1)$,

and

(2) $\chi_{\alpha_i}(1)$ and $\widetilde{w}_{\alpha_i}(1) = \chi_{\alpha_i}(1)\chi_{-\alpha_i}(-1)\chi_{\alpha_i}(1)$.

The generating set (2) is the analog of the S, T-generating set for $SL_2(\mathbb{Z})$ and coincides with the generating set obtained in [AC] where the authors also gave a finite presentation for the group $E_{10}(\mathbb{Z})$.

3.1. **Explicit construction of the groups E_9, E_{10} and E_{11}.** An important question in string theory is to determine which representations of the underlying Kac–Moody algebras (or Lie algebras) enter into the construction of the automorphic forms. Here we describe possible choices of modules V for E_9, E_{10} and E_{11} that have physical relevance.

We summarize the constructions of the groups $E_n(\mathbb{R})$ and $E_n(\mathbb{Z})$ for $n = 9, 10, 11$. In each of these constructions, we use the data in Table **??**:

With this data, we have the groups:

$$\boxed{E_n(\mathbb{R}) = \langle \chi_{\alpha_i}(s) = exp(se_i),\ \chi_{-\alpha_i}(t) = exp(tf_i) \mid s, t \in \mathbb{R}\ i = 1, \ldots, n \rangle}$$

$$\boxed{E_n(\mathbb{Z}) = E_n(\mathbb{R}) \cap Aut(V_\mathbb{Z}) = \langle \chi_{\alpha_i}(s) = exp(se_i),\ \chi_{-\alpha_i}(t) = exp(tf_i) \mid s, t \in \mathbb{Z},\ i = 1, \ldots, n \rangle}$$

TABLE 2

Algebra	Generators	Simple roots	Fund. weights	Highest weight module
$\mathfrak{e}_9(\mathbb{C})$	$e_1, \ldots, e_9$ $f_1, \ldots, f_9$	$\alpha_1, \ldots, \alpha_9$	$\omega_1, \ldots, \omega_9$	$V = V^{\omega_1}$ V integrable with high. wt. vec. v^{ω_1} corresp. to fund. weight ω_1 labeled as in Figure **??**
$\mathfrak{e}_{10}(\mathbb{C})$	$e_1, \ldots, e_{10}$ $f_1, \ldots, f_{10}$	$\alpha_1, \ldots, \alpha_{10}$	$\omega_1, \ldots, \omega_{10}$	$V = V^{\omega_1 + \cdots + \omega_{10}}$ V irred. and integrable with high. wt. vec. $v^{\omega_1 + \cdots + \omega_{10}}$
$\mathfrak{e}_{11}(\mathbb{C})$	$e_1, \ldots, e_{11}$ $f_1, \ldots, f_{11}$	$\alpha_1, \ldots, \alpha_{11}$ labeled as in Figure **??**	$\omega_1, \ldots, \omega_{11}$	$V = \omega_{11}$–rep. of $\mathfrak{e}_{11}(\mathbb{C})$. V integrable with high. wt. vec. $v^{\omega_{11}}$ corresp. to fund. weight ω_{11}

We now outline the choice of representation for the above group constructions.

○ $E_9(\mathbb{Z})$ *and* $E_9(\mathbb{R})$

For E_9, we choose V to be the fundamental representation V^{ω_1} which is the highest weight module with highest weight ω_1, where ω_1 is the fundamental weight dual to α_1 as labeled in Figure **??**, viewing the Dynkin diagram for E_9 as a subdiagram of E_{11}. Our motivation for this is the following. Certain coefficients of quantum correction terms (the $\mathcal{R}^4$ and $\partial^4 \mathcal{R}^4$ correction terms for instance) in type IIB supergravity, dimensionally reduced to $D \geq 3$, can be expressed in terms of Eisenstein series on various finite dimensional Lie groups. In particular, the coefficient of the $\mathcal{R}^4$ correction term is an Eisenstein series whose complex parameter is a scalar multiple of ω_1 ([GMRV]). It is an open question to determine if this symmetry structure extends to 2 dimensions with corresponding Eisenstein series on E_9.

○ $E_{10}(\mathbb{Z})$ *and* $E_{10}(\mathbb{R})$

When the root lattice Q equals the weight lattice P, as is the case for $\mathfrak{e}_{10}$, that is, $|det(A)| = 1$, where A is the generalized Cartan matrix, then we may choose $V = V^{\omega_1 + \cdots + \omega_{10}}$, where ω_i are the fundamental weights. Then V is an irreducible integrable highest weight module with lattice of weights equal to P and with highest weight $\omega_1 + \cdots + \omega_{10}$. We have

$$wts(V) \subseteq \{\omega_1 + \cdots + \omega_{10} - \sum_{j=1}^{10} k_j \alpha_j \mid k_j \in \mathbb{Z}_{\geq 0}\},$$

where α_i are the simple roots, and

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{10} \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{10} \end{pmatrix}.$$

Since the root lattice $Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{10}$ and the weight lattice P coincide, the weights of V are contained in the $\mathbb{Z}$-span of the simple roots. Hence $wts(V)$ contains all the fundamental weights ω_i.
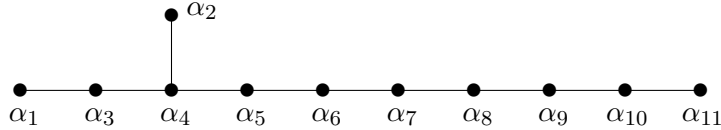
FIGURE 1. The Dynkin diagram of E_{11}

○ $E_{11}(\mathbb{Z})$ *and* $E_{11}(\mathbb{R})$

Symmetries of the discrete group $E_{11}(\mathbb{Z})$ were discussed in [GW1], where the authors conjecture that the group $E_{11}(\mathbb{Z})$ preserves the brane charge lattice of Type II superstring theory. This charge lattice belongs to the fundamental representation V corresponding to the vertex at the end of the long tail of the Dynkin diagram for E_{11}. We call this representation the ω_{11}–representation. In [GW1], due to a different labeling of the vertices of the Dynkin diagram, this is called the ℓ_1–representation.

We note that the above constructions depend on choices of modules for which there is a physical motivation. These choices give rise to group constructions which are useful in the physical contexts described above (for example our generating set for E_{11} was used in [GW1]).

In general, the dependence of the representation theoretic construction of Kac–Moody groups on the choice of module is not completely understood. Garland gave a representation theoretic construction of affine Kac–Moody groups as central extensions of loop groups, where each central extension corresponds to a unique cohomology class represented by a cocycle, known as the *Steinberg cocycle* restricted to the torus H. Garland characterized the dependence on the choice of highest weight module V for affine groups in terms of the Steinberg cocycle. In [CW], the authors proved that for different choices of highest weight module V, the discrepancy between the groups $E_{10}^V(\mathbb{Z})$ is contained in a finite abelian group of order at most $(\mathbb{Z}/2\mathbb{Z})^{10}$.

4. EISENSTEIN SERIES IN SUPERGRAVITY

The first quantum corrections to the classical supergravity action were found by computing the scattering amplitudes of four closed interacting strings. In particular, these scattering amplitudes give rise to quantum corrections of the form $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$. The coefficient of the $\mathcal{R}^4$ correction term is an Eisenstein series whose complex parameter is a scalar multiple of ω_1 ([GMRV]).

Schematically, this scattering amplitude is separable into analytic and non–analytic parts

$$A = A^{\text{analytic}} + A^{\text{non-analytic}}.$$

The analytic part of the scattering amplitude in $d = (11 - n)$ dimensions is an expansion in automorphic forms on the double coset $K(E_n(\mathbb{R}))\backslash E_n(\mathbb{R})/E_n(\mathbb{Z})$ ([GMRV]). Each automorphic form has a Fourier expansion, which naturally separates the perturbative and non–perturbative quantum corrections.

Being the perturbative limit, the scattering amplitudes only capture the constant terms (zero–th order terms in the Fourier expansion) of the automorphic forms ([GRV]).

The remaining parts of the Fourier expansion correspond to 'non–perturbative instanton corrections' ([GG], [GV], [GMV]). The direct determination of instanton effects in string theory

is usually very difficult. The benefit here is that the action of the integral group $G(\mathbb{Z})$ mixes perturbative and non–perturbative effects, which allows indirect computation of the instanton corrections. It is conjectured that different types of instanton effects correspond to Fourier expansions of Eisenstein series with respect to different parabolic subgroups ([GMV], [FK]).

By dimensionally reducing the higher derivative corrections of ten–dimensional IIB theory on a torus, the authors of [GLW] and [GW1] describe some natural constraints on the E_{n+1} automorphic forms that occur in $d = 10 - n$ dimensions. They also argue that these automorphic forms involve the representation of E_{n+1} with highest weight ω_1.

With regard to Eisenstein series on Kac–Moody groups, in analogy with $SL_2(\mathbb{Z})$, the group $E_{10}(\mathbb{Z})$ is conjectured to be a 'modular group' for certain automorphic forms that are expected to arise in the context of 11 dimensional supergravity ([DN], [Ga]). Automorphic forms on E_{10} and E_{11} are conjectured to encode higher derivative corrections of string theory and M–theory ([DN2], [DHHKN], [W1]). The constant terms of Eisenstein series on E_9, E_{10} and E_{11} have been studied in [Fl] in relation to string scattering amplitudes. We discuss this further in the next section.

5. Eisenstein series on non–affine Kac–Moody groups

In his seminal work [**?, ?, ?, ?, ?, ?, ?, ?**], Garland extended the classical theory of Eisenstein series to arithmetic quotients $K(G(\mathbb{R}))\backslash G(\mathbb{R})/G(\mathbb{Z})$ of affine Kac–Moody groups G. He proved absolute convergence in a half space and meromorphic continuation to the full complex plane. Garland's theory of Eisenstein series has been extended to affine groups over fields such as number fields ([Li]) and function fields ([BK], [LL], [P]) and to hyperbolic and more general Kac–Moody groups ([CLL], [CGLLM]).

We outline the construction of general Eisenstein series on Kac–Moody groups. Let $G = G(\mathbb{R})$ be a non–affine Kac–Moody group associated to a Kac–Moody algebra $\mathfrak{g}$. Let $G(\mathbb{R}) = KA^+N$ be the Iwasawa decomposition of G, where K is the fixed point subgroup of the involution on $G(\mathbb{R})$ induced from the Cartan involution on $\mathfrak{g}$, $A^+ \cong (\mathbb{R}^+)^{rank(G)}$ is an abelian subgroup and N is a completion of the subgroup generated by all positive real root groups ([DGH], [CLL]).

Let $g = k_g a_g n_g \in G(\mathbb{R})$, written in Iwasawa form. Let $\nu : A^+ \longrightarrow \mathbb{C}^\times$ be a quasi–character and define

$$\Phi_\nu : G(\mathbb{R}) \longrightarrow \mathbb{C}^\times$$

to be the function

$$\Phi_\nu(g) = \nu(a_g).$$

Then ν is well defined since the Iwasawa decomposition is unique. For convenience, we write $\Gamma = G(\mathbb{Z})$.

Let B denote the minimal parabolic subgroup of $G(\mathbb{R})$. Define the Eisenstein series on $G(\mathbb{R})$ to be the infinite formal sum ([CLL], [CGLLM])

$$E_\nu(g) \quad := \sum_{\gamma \in \Gamma/\Gamma \cap B} \Phi_\nu(g\gamma).$$

When G is finite dimensional, we define the *constant term* ([CLL], [CGLLM])

$$\int_{N/N \cap \Gamma} E_\nu(gu) du$$

where du is a measure on the compact space $N/N \cap \Gamma$. For example, when $G = SL_2$, $\Gamma = SL_2(\mathbb{Z})$, $N/N \cap \Gamma = S^1$.

The constant term corresponds to the zero–th Fourier coefficient in the Fourier expansion of the Eisenstein series.

Regarding the complex question of convergence of Eisenstein series, in [CLL], the authors proved convergence of the constant term and hence almost everywhere convergence of the full Eisenstein series on rank 2 hyperbolic Kac–Moody groups over $\mathbb{R}$. The techniques in [CLL] have been extended to Kac–Moody groups in general ([CGLLM]).

5.1. **The Weyl group and the constant term.** Let G be a semi simple algebraic group or a Kac–Moody group. Then G has Bruhat decomposition $G = \sqcup_w G_w$ where

$$G_w = BwB,$$

$w \in W$, the Weyl group and B is the 'upper triangular', or Borel subgroup. Each Bruhat cell

$$\Gamma/\Gamma \cap B = \sqcup_{w \in W}(\Gamma \cap G_w)/(\Gamma \cap B)$$

then contributes one term to the constant term. The number of terms in the constant term thus equals the cardinality of the Weyl group W.

When G is a Kac–Moody group, W is infinite. In [FK], the authors showed that remarkably, for certain choices of the complex parameter ν, there are only finitely many terms in the constant terms for E_9, E_{10} and E_{11}. This uses a reduction method of [GMRV] for eliminating terms of the constant term. For E_9, this also uses the structure of the affine Weyl group and the Weyl group orbits. This is consistent with the (conjectural) notion that the constant terms of Eisenstein series on Kac–Moody groups should encode a finite number of perturbative string theory corrections. A detailed analysis of this question is given in [Fl].

Eisenstein series on non–affine Kac–Moody groups $G(\mathbb{R})$ are invariant under translations by $G(\mathbb{Z})$ and hence have a Fourier expansion. In [CLL], the authors defined and calculated the degenerate Fourier coefficients for Eisenstein series on rank 2 hyperbolic Kac–Moody groups over $\mathbb{R}$. In [Fl], the author gave the Fourier integrals needed to obtain the constant term and the higher order Fourier modes for Eisenstein series in E_9, E_{10} and E_{11} and showed how the 'collapse mechanism' of [FK] extends to the higher order Fourier modes. He also gave explicit expressions for the constant terms and Fourier modes of some Kac–Moody Eisenstein series.

6. Some open questions

There are certain obstacles to overcome in order to determine the precise role of Kac–Moody Eisenstein series in supergravity. For example, in the study of $E_9(\mathbb{Z})$ in 2 dimensions, the supergravity action cannot be cast in the Einstein frame, that is, the coordinate frame where the coset symmetries are manifest. The objective is to interpret Eisenstein series as coefficients of higher derivative corrections in the string scattering amplitudes. However, without the supergravity

action in the Einstein frame, identifying symmetries in the string scattering amplitudes may be considerably more difficult. A formulation of the quantum corrections on the level of equations of motion may be required.

The Kac–Moody Eisenstein series studied in [Fl] have been seen to appear in string scattering amplitudes ([Fl], Sec 2.2.1). Characterizing the underlying physical mechanism at work here will require a more detailed knowledge of string scattering in 2 and 1 dimensions.

The Eisenstein series should be an eigenfunction of the Casimir operators including the Laplacian on $K(G(\mathbb{R}))\backslash G(\mathbb{R})/G(\mathbb{Z})$. It would be interesting to find the explicit forms of these eigenvalue equations using the higher order Casimir operators introduced in [K1].

In some cases, certain constraints are required on Eisenstein series defined using representation theory in order that these automorphic forms are eigenfunctions of the Laplacian ([OP1]). It is not yet understood how this might generalize to the Kac–Moody case.

References

[AC]   Allcock, D. and Carbone, L. *Presentation of hyperbolic Kac–Moody groups over rings*, J. Algebra 445 (2016), 232–243, arXiv:1409.5918

[BC]   Bao, L. and Carbone, L. *Integer forms of Kac–Moody groups and Eisenstein series in low dimensional supergravity theories*, Preprint, arXiv:1308.6194

[BK]   A. Braverman and D. Kazhdan, *Representations of affine Kac–Moody groups over local and global fields: a suvery of some recent results,* arXiv:1205.0870.

[CGLLM]   Carbone, L., Garland, H. Lee, K-H, Liu D. and Miller, S. D. *Eisenstein series on Kac–Moody groups*, In preparation (2016)

[CLL]   Carbone, L., Lee, K.–H. and Liu, D. *Eisenstein series on rank 2 hyperbolic Kac–Moody groups over* $\mathbb{R}$, To appear in Math. Annalen, (2015), arXiv:1306.3280,

[Ca]   Carbone, L., *Infinite dimensional Chevalley groups and Kac–Moody groups over* $\mathbb{Z}$, In preparation (2016)

[CG]   Carbone, L. and Garland, H. *Existence of lattices in Kac–Moody groups over finite fields*, Communications in Contemporary, Math, Vol 5, No.5, 813–867, (2003).

[CW]   Carbone, L. and Wagner, F. *Uniqueness of representation–theoretic hyperbolic Kac–Moody groups over* $\mathbb{Z}$, Submitted to Contemporary Math, Conference on Lie Algebras, Vertex Operator Algebras, and Related topics, edited by K. Barron, E. Jurisich, H. Li, A. Milas and K. C.Misra, (2016)

[CJ]   E. Cremmer and B. Julia, *The N=8 supergravity theory. 1. The Lagrangian*, Phys. Lett. B **80** (1978) 48.

[D1]   Damour, T., *Poincare, Relativity, Billiards and Symmetry*, arXiv:hep-th/0501168v1

[DGH]   De Medts, T., Gramlich, R., and Horn, M., *Iwasawa decompositions of split Kac–Moody groups*, Journal of Lie Theory 19 (2009), No. 2, 311–337.

[DHN1]   T. Damour, M. Henneaux and H. Nicolai, *Cosmological billiards*, Class. Quant. Grav. **20** (2003) R145, hep-th/0212256.

[DN]   Damour, T. and Nicolai, H. *Symmetries, singularities and the de–emergence of space*, International Journal of Modern Physics D. Vol 17, Nos 3 and 4 (2008) 525–531.

[DN2]   Damour, T. and Nicolai, H. *Higher order M–theory corrections and the Kac–Moody algebra* E_{10}, Class.Quant.Grav. 22 (2005) 2849–2880

[DHHKN]   Damour, T., Hanany, A., Henneaux, M., Kleinschmidt, A. and Nicolai, H. *Curvature corrections and Kac–Moody compatibility conditions*, Gen. Rel. Grav. 38:1507–1528 (2006)

[Fl]   Fleig, P. *Kac–Moody Eisenstein series in string theory*, Dissertation, University of Berlin, (2013)

[FK]   Fleig, P. and Kleinschmidt, A. *Eisenstein series for infinite–dimensional U–duality groups*, arXiv:1204.3043.

[FKP]   Fleig, P., Kleinschmidt, A. and Persson, D. *Fourier expansions of Kac–Moody Eisenstein series and degenerate Whittaker vectors*, Commun. Num. Theor. Phys. 08 (2014) 41–100, arXiv:1312.3643v2 [hep-th]

[FKGP]   Fleig, P., Kleinschmidt, A., Gustafsson, H. and Persson, D. *Eisenstein series and automorphic representations*, arXiv:1511.04265v1 [math.NT]

[Ga]   Ganor, O. J., *Two conjectures on gauge theories, gravity and infinite dimensional Kac–Moody groups*, arXiv:hep-th/9903110

[G99]   Garland, H. *Eisenstein series on arithmetic quotients of loop groups*, Math. Res. Lett. 6 (1999), no. 5-6, 723?733

[G04]   Garland, H. *Certain Eisenstein series on loop groups: convergence and the constant term*, Algebraic Groups and Arithmetic, Tata Inst. Fund. Res., Mumbai, (2004), 275–319.

[G06]   Garland, H. *Absolute convergence of Eisenstein series on loop groups*, Duke Math. J. **135** (2006), no. 2, 203–260.

[G11]   Garland, H. *On extending the Langlands–Shahidi method to arithmetic quotients of loop groups*, Representation theory and mathematical physics, Contemp. Math., vol. 557, Amer. Math. Soc., Providence, RI, 2011, pp. 151–167 arXiv:1009.4507.

[GMS1]   Garland, H. *Eisenstein series on loop groups: Maass-Selberg relations. I*, Algebraic groups and homogeneous spaces, 275–300, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007.

[GMS2]   Garland, H. *Eisenstein series on loop groups: Maass-Selberg relations. II,* Amer. J. Math. **129** (2007), no. 3, 723–784.

[GMS3]   Garland, H. *Eisenstein series on loop groups: Maass-Selberg relations. III,* Amer. J. Math. 129 (2007), no. 5, 1277–1353.

[GMS4]   Garland, H. *Eisenstein series on loop groups: Maass-Selberg relations. IV*, Lie algebras, vertex operator algebras and their applications, 115–158, Contemp. Math., **442**, Amer. Math. Soc., Providence, RI, 2007.

[GG]   M. B. Green and M. Gutperle, *Effects of D instantons*, Nucl. Phys. B **498** (1997) 195, hep-th/9701093.

[GLW]   Gubay, F., Lambert, N. and West, P. C. *Constraints on Automorphic Forms of Higher Derivative Terms from Compactification*, JHEP, arXiv:1002.1068v3 [hep-th].

[GMRV]   M. B. Green, S. D. Miller, J. G. Russo and P. Vanhove, *Eisenstein series for higher-rank groups and string theory amplitudes*, arXiv:1004.0163.

[GMV]   M. B. Green, S. D. Miller and P. Vanhove, *Small representations, string instantons, and Fourier modes of Eisenstein series (with an appendix by D. Ciubotaru and P. Trapa)*, arXiv:1111.2983.

[GRV]   M. B. Green, J. G. Russo and P. Vanhove, *Automorphic properties of low energy string amplitudes in various dimensions*, Phys. Rev. D **81** (2010) 086008, arXiv:1001.2535.

[GV]   M. B. Green and P. Vanhove, *D instantons, strings and M theory*, Phys. Lett. B **408** (1997) 122, hep-th/9704145.

[GW1]   Gubay, F. and West, P. C. *Higher derivative type II string effective actions, automorphic forms and E_n*, arXiv:1111.0464v1.

[HPS]   Henneaux, M., Persson, D, and Spindel, P. *Spacelike Singularities and Hidden Symmetries of Gravity* Living Reviews in Relativity, vol. 11, no. 1, 2008

[HT]   C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. B **438** (1995) 109, hep-th/9410167.

[Ju1]   B. Julia, *Infinite Lie Algebras In Physics*, In Baltimore (1981), Proceedings, Unified Field Theories and Beyond, 23–41.

[K1]   V. G. Kac, *Laplace operators of infinite-dimensional Lie algebras and theta functions*, Proc. Natl. Acad. Sci. USA 81 (1984), 645–647.

[KN]   Kleinschmidt, A. and Nicolai, H. *Maximal supergravities and the E_{10} coset model*, Internat. J. Modern Phys. D 15 (2006), no. 10, 1619–1642

[LW]   N. Lambert and P. West, *Perturbation theory from automorphic forms*, JHEP **1005** (2010) 098, arXiv:1001.3284.

[LL]   K.-H. Lee and P. Lombardo, *Eisenstein series on affine Kac-Moody groups over function fields*, Trans. Amer. Math. Soc. **366** (2014), no. 4, 2121–2165.

[Li]   D. Liu, *Eisenstein series on loop groups*, Tran. Amer. Math. Soc. **367** (2015), no. 3, 2079–2135.

[MS]   Mizoguchi, S. and Schroeder, G. *On Discrete U-duality in M–theory*, Class.Quant.Grav. 17 (2000) 835–870.

[N]   H. Nicolai, *The integrability of N=16 supergravity*, Phys. Lett. B **194** (1987) 402.

[Ni2]   Nicolai, H., *Gravitational billiards, dualities and hidden symmetries* (English summary) 100 years of relativity, 39–75, World Sci. Publ., Hackensack, NJ, (2005)

[Ni3]   Nicolai, H., *Infinite Dimensional Symmetries*, Lecture notes on Infinite Dimensional Symmetries given by Hermann Nicolai at the 15th Saalburg summer school in Wolfersdorf, Thuringia, in September 2009. Typed by Oliver Schlotterer.

[OP1]   N. A. Obers and B. Pioline, *Eisenstein series and string thresholds*, Commun. Math. Phys. **209** (2000) 275, hep-th/9903113.

[S]   Soulé, *An introduction to arithmetic groups*, Proceedings of 'Frontiers in Number Theory, Physics and Geometry', Les Houches, March 9–21, (2003), arXiv:math/0403390v1.

[W1]   P. C. West, *E(11) and M theory*, Class. Quant. Grav. **18** (2001) 4443, hep-th/0104081.

[Wi1]   Witten, E., *Magic, mystery, and matrix*, Notices of the Amer. Math. Soc.45: 1124–1129, (1998)

[Wi2]   Witten, E., *Five-branes and M theory on an orbifold*, hep-th/9512219 and Proceedings of Strings '95, hep-th/9507121