THE EXISTENCE THEOREM FOR TREE LATTICES

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Appendix [BCR] in 'Tree Lattices', by Hyman Bass and Alex Lubotzky, Birkhauser, 2000

This document contains Sections 0. and 1.

Contents

0. Statement of the main results

Let X be a locally finite tree, and $G = Aut(X)$. Then G is naturally a locally compact group with compact open vertex stabilizers G_x, $x \in VX$ ([BL], (3.1)). A subgroup $\Gamma \leq G$ is discrete if and only if Γ_x is a finite group for some (hence for every) $x \in VX$. A discrete subgroup $\Gamma \leq G$ is called an *X-lattice* if

$$(1) \qquad Vol(\Gamma\backslash\backslash X) := \sum_{x \in V(\Gamma\backslash X)} \frac{1}{|\Gamma_x|}$$

is finite, and a *uniform X-lattice* if $\Gamma\backslash X$ is a finite graph.

We wish to determine conditions that will ensure that G contains X-lattices. Let μ be a (left) Haar measure on G. Let $\Gamma \leq G$ be a discrete subgroup with quotient $p : G \longrightarrow \Gamma\backslash G$. Then μ induces a measure, also denoted μ, on $\Gamma\backslash G$. We call Γ a *G-lattice* if $\mu(\Gamma\backslash G) < \infty$, and a *uniform* G-lattice if $\Gamma\backslash G$ is compact.

For $x \in VX$, $0 < \mu(G_x) < \infty$. When G is unimodular, $\mu(G_x)$ is constant on G-orbits, so we can define ([BL], (1.5)):

$$(2) \qquad \mu(G\backslash\backslash X) := \sum_{x \in V(G\backslash X)} \frac{1}{\mu(G_x)}.$$

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(0.1) Theorem ([BL], (1.6)). *For a discrete subgroup $\Gamma \leq G = Aut(X)$, the following conditions are equivalent:*

(a) Γ *is an X-lattice, that is,* $Vol(\Gamma\backslash\backslash X) < \infty$.
(b) Γ *is a G-lattice (hence G is unimodular), and* $\mu(G\backslash\backslash X) < \infty$.

In this case:

$$Vol(\Gamma\backslash\backslash X) = \mu(\Gamma\backslash G) \cdot \mu(G\backslash\backslash X).$$

The main result proved here is the following theorem, originally conjectured in an earlier version of [BL]:

(0.2) Lattice Existence Theorem. *Let X be a locally finite tree, let $G = Aut(X)$, and let μ be a (left) Haar measure on G. The following conditions are equivalent:*

(a) *G contains an X-lattice Γ.*
(b) *(U)* *G is unimodular, and*
(FV) $\mu(G\backslash\backslash X) < \infty$.

(0.3) Remarks.

(1) The implication (a) $\Longrightarrow$ (b) follows from Theorem (0.1).
(2) When (FV) is replaced by the stronger condition:
(F) *$G\backslash X$ is finite,*
then we have the:

(0.4) Uniform Existence Theorem ([BK], (4.10)). *Let X be a locally finite tree and let $G = Aut(X)$. The following conditions are equivalent:*

(a) *G contains a uniform X-lattice Γ, which is also a uniform G-lattice.*
(b) *G contains a uniform X-lattice Φ such that $\Phi\backslash X = G\backslash X$.*
(c) *(U)* *G is unimodular, and*
(F) *$G\backslash X$ is finite.*
(d) *X is the universal cover of a finite connected graph.*

Under these conditions, X is called a 'uniform tree'.

In light of (0.4), to prove the Lattice Existence Theorem (0.2), we are reduced to proving:

(0.5) Theorem. *Let X be a locally finite tree, let $G = Aut(X)$, and let μ be a (left) Haar measure on G. Assume that:*
(U) *G is unimodular,*
(FV) $\mu(G\backslash\backslash X) < \infty$, *and*
(INF) *$G\backslash X$ is infinite.*
Then G contains a (necessarily non-uniform) X-lattice Γ.

This theorem will be deduced from the following result about 'edge-indexed graphs'. Here we follow the notations and terminology of [BL], Ch 2, and we defer explanation until Section 2.

(0.6) The Bounding Denominators Theorem. *Let (A, i) be an edge-indexed graph, and let $a_0 \in VA$. Assume*

$(U)_{(A,i)}$ $\qquad$ *(A, i) is unimodular.*

Then there is a canonical covering

$$p : (B, j) \longrightarrow (A, i)$$

of edge indexed graphs, and a $b_0 \in p^{-1}(a_0)$, with the following properties:

$(U)_{(B,j)}$ $\qquad$ *(B, j) is unimodular.*

(FF) $\qquad$ *p has finite fibers. Hence B is infinite if and only if A is infinite.*

(V) $\qquad$ *$Vol_{b_0}(B, j) = Vol_{a_0}(A, i)$. Hence $Vol(B, j) < \infty$ if and only if $Vol(A, i) < \infty$.*

$(BD)_{(B,j)}$ $\qquad$ *(B, j) has bounded denominators.*

The utility of this result is indicated by the following.

(0.7) Theorem ([BK], (2.4)). *Let (B, j) be an edge-indexed graph. Then (B, j) admits a finite faithful grouping $\mathbb{B} = (B, \mathcal{B})$, if and only if (B, j) satisfies:*

$(U)_{(B,j)}$ $\qquad$ *(B, j) is unimodular, and*

$(BD)_{(B,j)}$ $\qquad$ *(B, j) has bounded denominators.*

With the notations of (0.6) and (0.7), if $b_0 \in VB$ and $a_0 = p(b_0) \in VA$, then we put

$$(\widetilde{A, i, a_0}) = X = (\widetilde{B, j, b_0})$$

so that

$$\begin{array}{ccc} & X & \\ {\scriptstyle p_B}\swarrow & & \searrow{\scriptstyle p_A} \\ B & \xrightarrow{p} & A \end{array}$$

commutes. Let $G = Aut(X)$ and

$$G_{(B,j)} := \{g \in G \mid p_B \circ g = p_B\} \quad \leq \quad G_{(A,i)} := \{g \in G \mid p_A \circ g = p_A\}.$$

Let $\Gamma = \pi_1(\mathbb{B}, b_0) \leq G_{(B,j)}$.

(0.9) Theorem. *Assuming*

$(U)_{(A,i)}$ $\qquad$ *(A, i) is unimodular, and*

$(FV)_{(A,i)}$ $\qquad$ $Vol(A, i) < \infty$,

then Γ is an X-lattice, $\Gamma \leq G_{(A,i)}$, and Γ is a uniform $G_{(A,i)}$-lattice. In fact, if $x_0 \in p_B^{-1}(b_0)$, then

$$Vol(\Gamma\backslash\backslash X) = \frac{1}{|\Gamma_{x_0}|} Vol_{a_0}(A, i).$$

(0.10) Corollary. *Let X be a locally finite tree, $G = Aut(X)$, $H \leq G$ a subgroup acting without inversions, $p_H : X \longrightarrow A = H\backslash X$, and $(A, i) = I(H\backslash\backslash X)$. Assume that $H = G_{(A,i)}$. Then the following conditions are equivalent:*

(a) *There is an X-lattice $\Gamma \leq H$.*
(b) $(U)_H$ $\quad$ *H is unimodular, and*
$\quad$ $(FV)_H$ $\quad$ $\mu(H\backslash\backslash X) < \infty$.

Under these conditions, we can choose Γ to be a uniform H-lattice.

Proof. By Remark ((0.3), (1)), we need only verify the implication $(b) \implies (a)$. We verify that the hypotheses of Theorem (0.9) are satisfied. We have $(U)_H \iff (U)_{(A,i)}$ by ([BK], (3.7)), and we have $(FV)_H \iff Vol(A, i) < \infty$ by ([BL], (3.6)(2)). Taking $H = G$ gives (0.2) and (0.5). □

1. Related existence results and questions

Under the assumptions that G is unimodular, and $\mu(G\backslash\backslash X) < \infty$, Theorem (0.5) gives existence of an X-lattice Γ that is of course non-uniform when $G\backslash X$ is infinite. However, the Γ we produce in Theorem (0.5) is a *uniform* G-lattice. This naturally raises the following:

(1.1) Question. *Let X be a locally finite tree that admits a non-uniform X-lattice. Does X admit one that is also a non-uniform G-lattice?*

Question (1.1) has a positive answer if X is a uniform tree ([C1], [C2]). Hence it remains to answer Question (1.1) in the case that $G = Aut(X)$ is unimodular, $\mu(G\backslash\backslash X) < \infty$ and $G\backslash X$ is infinite. We address this case in [CR], assuming that X has more than one end, and in [CC], when X has a unique end.

Following ([BL], (3.5)) we call X *rigid* if G is discrete, and we call X *minimal* if G acts minimally on X, that is, there is no proper G-invariant subtree. If X is uniform then there is always a unique minimal G-invariant subtree $X_0 \subseteq X$ ([BL] (5.7), (5.11), (9.7)). We call X *virtually rigid* if X_0 is rigid (*cf.* ([BL], (3.6)). All lattices on virtually rigid trees must be uniform ([BL], (3.6)). Conversely:

(1.2) Non-uniform Lattices on Uniform Trees ([C1], [C2]). *If X is uniform and not virtually rigid then G contains a non-uniform X-lattice Γ, which is also (necessarily) a non-uniform G-lattice.*

We observe that the assumptions of Theorems (1.2) and (0.5) are mutually exclusive. In fact, under the conditions of Theorem (0.5), either X has a unique minimal G-invariant subtree X_0, and X_0 is not rigid, or else X is parabolic and has no rigid G-invariant subtrees.

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